Adjoint composition operators on $H^2(\mathbb{U})$
induced by strongly outer regular rational selfmaps of $\mathbb{U}$

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A Riddle

How is the problem of choosing a symbol for a composition operator similar to Hamlet’s dilemma?
To $\phi$ or not to $\phi$, that is the question.
Setting the Stage

Let $\phi$ denote a holomorphic function mapping the open unit disk $\mathbb{D}$ into itself, let $H(\mathbb{D})$ be the space of holomorphic functions on $\mathbb{D}$, and let $C_\phi : H(\mathbb{D}) \to H(\mathbb{D})$ be given by $C_\phi f = f \circ \phi$.

Let $H^2(\mathbb{D}) = \{ f \in H(\mathbb{D}) : \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty \}$. Note $H^2(\mathbb{D})$ is a Hilbert space with inner product

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n)\hat{g}(n).$$

For $\alpha \in \mathbb{D}$, let $K_\alpha(z) = \frac{1}{1-\alpha z}$ be the reproducing kernel at the point $\alpha$ for $H^2(\mathbb{D})$:

$$\langle f, K_\alpha \rangle = f(\alpha) \quad \text{for all} \quad f \in H^2(\mathbb{D})$$

For an analytic function $\phi : \mathbb{D} \to \mathbb{D}$, Littlewood showed that $C_\phi$, restricted to $H^2(\mathbb{D})$, is bounded.
Setting the Stage

- Let \( \phi \) denote a holomorphic function mapping the open unit disk \( \mathbb{U} \) into itself, let \( H(\mathbb{U}) \) be the space of holomorphic functions on \( \mathbb{U} \), and let \( C_\phi : H(\mathbb{U}) \to H(\mathbb{U}) \) be given by \( C_\phi f = f \circ \phi \).
- Let \( H^2(\mathbb{U}) = \{ f \in H(\mathbb{U}) : \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty \} \). Note \( H^2(\mathbb{U}) \) is a Hilbert space with inner product
  \[
  \langle f, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n)\hat{g}(n).
  \]

- For \( \alpha \in \mathbb{U} \), let \( K_\alpha(z) = \frac{1}{1-\overline{\alpha}z} \) be the reproducing kernel at the point \( \alpha \) for \( H^2(\mathbb{U}) \) :
  \[
  \langle f, K_\alpha \rangle = f(\alpha) \quad \text{for all} \quad f \in H^2(\mathbb{U})
  \]
- For an analytic function \( \phi : \mathbb{U} \to \mathbb{U} \), Littlewood showed that \( C_\phi \), restricted to \( H^2(\mathbb{U}) \), is bounded.
Let $\phi$ denote a holomorphic function mapping the open unit disk $\mathbb{U}$ into itself, let $H(\mathbb{U})$ be the space of holomorphic functions on $\mathbb{U}$, and let $C_\phi : H(\mathbb{U}) \rightarrow H(\mathbb{U})$ be given by $C_\phi f = f \circ \phi$.

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$$\langle f, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}.$$ 

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- Let \( H^2(\mathbb{U}) = \{ f \in H(\mathbb{U}) : \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty \} \). Note \( H^2(\mathbb{U}) \) is a Hilbert space with inner product
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Starring Players

\[ \phi_1(z) = \frac{1}{3 - z - z^2}, \quad \phi_2(z) = \frac{z^2 + z}{3 - z^2}, \quad \phi_3(z) = z^2 \]
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Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d$. If $R^{-1}(\{w\})$ contains $d$ distinct points we will say $w$ is a regular value of $R$. If $w \in \mathbb{C}$ is not a regular value, then $w$ is a critical value of $R$ (the image of a critical point—a point where $R'$ vanishes). \(^1\)

Note that the set of critical values of $R$ is finite: it contains precisely those points in the finite plane that have a preimage under $R$ at which $R'$ vanishes, and contains $\infty$ when $1/R$ has 0 as a critical value.

Examples

\(^1\) $R(\infty)$ is finite, $R'(\infty) := f'(0)$ given $f(z) = R(1/z)$
Regular Values and Critical Values

Let \( R : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be a rational map of degree \( d \). If \( R^{-1}(\{w\}) \) contains \( d \) distinct points we will say \( w \) is a regular value of \( R \). If \( w \in \mathbb{C} \) is not a regular value, then \( w \) is a critical value \( R \) (the image of a critical point—a point where \( R' \) vanishes ).

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Examples
\[ \phi_3(z) = z^2 \]

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$\phi_3(z) = z^2$ has critical values 0 and $\infty$

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Examples

$\phi_3(z) = z^2$ has critical values 0 and $\infty$

$\phi_1(z) = \frac{1}{3-z-z^2}$; $\phi_1'(z) = \frac{1+2z}{(3-z-z^2)^2}$

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Examples

\[ \phi_3(z) = z^2 \] has critical values 0 and \( \infty \)

\[ \phi_1(z) = \frac{1}{3-z-z^2} \] has critical values 0 and \( \phi_1(-1/2) = 4/13 \).

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Regular Values and Critical Values

Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d$. If $R^{-1}(\{w\})$ contains $d$ distinct points we will say $w$ is a regular value of $R$. If $w \in \mathbb{C}$ is not a regular value, then $w$ is a critical value $R$ (the image of a critical point—a point where $R'$ vanishes). \(^1\)

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Examples

- $\phi_3(z) = z^2$ has critical values 0 and $\infty$
- $\phi_1(z) = \frac{1}{3z - z^2}$ has critical values 0 and $\phi_1(-1/2) = 4/13$.
- $\phi_2(z) = \frac{z^2 + z - z^2}{3 - z^2}$; $\phi_2'(z) = \frac{z^2 + 6z + 3}{(3 - z^2)^2}$

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Examples

$\phi_3(z) = z^2$ has critical values 0 and $\infty$

$\phi_1(z) = \frac{1}{3-z-z^2}$ has critical values 0 and $\phi_1(-1/2) = 4/13$.

$\phi_2(z) = \frac{z^2+z}{3-z^2}$ has critical values $\phi_2(-3 + \sqrt{6}) \approx -0.09175$ and $\phi_2(-3 - \sqrt{6}) \approx -0.90825$.

\(^1\) $R(\infty)$ is finite, $R'(\infty) := f'(0)$ given $f(z) = R(1/z)$.
An Adjoint Calculation: $C^*_\phi$, where $\phi(z) = z^2$

\[(C^*_\phi f)(z) = \langle C^*_\phi f, K_z \rangle = \langle f, C_\phi K_z \rangle\]

\[(K_z \circ \phi)(w) = \frac{1}{1 - \bar{z}\phi(w)} = \frac{1}{(1 - \sqrt{w})(1 + \sqrt{zw})} = \frac{1/2}{1 - \sqrt{zw}} + \frac{1/2}{1 + \sqrt{zw}}\]
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\]

\[
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$(C^*_\phi f)(z) = \langle C^*_\phi f, K_z \rangle$

$= \langle f, C\phi K_z \rangle$

$(K_z \circ \phi)(w) = \frac{1}{1 - \overline{z}w^2}$

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An Adjoint Calculation: \( C_{\phi}^* \), where \( \phi(z) = z^2 \)

\[
(C_{\phi}^* f)(z) = \langle C_{\phi}^* f, K_z \rangle = \langle f, C_{\phi} K_z \rangle
\]

\[
(K_z \circ \phi)(w) = \frac{1}{1 - \bar{w} z^2}
\]

\[
= \frac{1}{(1 - \sqrt{z} w)(1 + \sqrt{z} w)}
\]

\[
= \frac{1/2}{1 - \sqrt{z} w} + \frac{1/2}{1 + \sqrt{z} w}
\]
An Adjoint Calculation: $C^*_\phi$, where $\phi(z) = z^2$

\[
(C^*_\phi f)(z) = \langle C^*_\phi f, K_z \rangle = \langle f, C\phi K_z \rangle
\]

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\]

\[
= \frac{1}{1 - \sqrt{z}w} + \frac{1}{1 - (-\sqrt{z})w}
\]
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$$= 1/2 f(\sqrt{z}) + 1/2 f(-\sqrt{z})$$

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$$= \left(1/2 C_{\sqrt{z}} + 1/2 C_{-\sqrt{z}}\right) f$$

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= 1/2 f(\sqrt{z}) + 1/2 f(-\sqrt{z}) \\
= \left(1/2 C_{\sqrt{z}} + 1/2 C_{-\sqrt{z}}\right) f \\
= 1/2 C_{\sqrt{z}} f
\]

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(K_z \circ \phi)(w) = \frac{1}{1 - \bar{z}w^2} \\
= \frac{1}{(1 - \sqrt{z}w)(1 + \sqrt{z}w)} \\
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\]
$C^*_\phi$ for $\phi(z) = \frac{1}{3-z-z^2}$

$$C^*_\phi = M_{g_1} C_{\sigma_1} + M_{g_2} C_{\sigma_2} + C_0$$

where $\sigma_1(z) = \frac{-2}{1-\sqrt{13-4z}}$, $\sigma_2(z) = \frac{-2}{1+\sqrt{13-4z}}$, $g_1(z) = \frac{z\sigma_1'(z)}{\sigma_1(z)}$, and $g_2(z) = \frac{z\sigma_2'(z)}{\sigma_2(z)}$.

Thus $C^*_\phi \equiv M_{g_1} C_{\sigma_1}$ modulo the compact operators.
C^* \phi for \phi(z) = \frac{1}{3-z-z^2}

\[ C^* \phi = M_{g_1} C_{\sigma_1} + M_{g_2} C_{\sigma_2} + C_0 \]

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and \( g_2(z) = \frac{z\sigma'_2(z)}{\sigma_2(z)}. \)

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$C^*_\phi$ for $\phi(z) = \frac{1}{3-z-z^2}$

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Thus $C^*_\phi \equiv M_{g_1} C_{\sigma_1}$ modulo the compact operators.
Cowen’s Adjoint Formula

Suppose that \( \phi(z) = \frac{az+b}{cz+d} \) is a nonconstant linear-fractional self-map of \( \mathbb{U} \). Then

\[
C^*_\phi = M_g C_\sigma M_h^*,
\]

where

\[
\sigma(z) = \frac{\bar{a}z - \bar{c}}{-bz + \bar{d}}, \quad g(z) = \frac{1}{-bz + \bar{d}}, \quad \text{and} \quad h(z) = cz + d.
\]

Note \( \sigma(z) = \frac{1}{\phi^{-1}(1/\bar{z})} \);

\[
M_h^* = \bar{c}(M_z)^* + \bar{d} = \bar{c}B + \bar{d}; \quad \text{also},
\]

\[
z\sigma'(z) = z \frac{\bar{a}\bar{d} - \bar{b}\bar{c}}{(-\bar{b}z + \bar{d})^2} - \bar{b}z + \bar{d} \quad \bar{d}(\bar{a}z - \bar{c}) + \bar{c}(-\bar{b}z + \bar{d})
\]

\[
= g(z)(\bar{d}\sigma(z) + \bar{c})
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Cowen’s Adjoint Formula

Suppose that $\phi(z) = \frac{az+b}{cz+d}$ is a nonconstant linear-fractional self-map of $\mathbb{U}$. Then

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Note $\sigma(z) = \frac{1}{\phi^{-1}(1/\bar{z})}$;

$$M^*_h = \bar{c}(M_z)^* + \bar{d} = \bar{c}B + \bar{d};$$

also,

$$z\sigma'(z) = z \frac{\bar{a}\bar{d} - \bar{b}\bar{c}}{(-bz + \bar{d})^2} = \frac{1}{-bz + \bar{d}} \left( \bar{d}(\bar{a}z - \bar{c}) + \bar{c}(-bz + \bar{d}) \right) = g(z)(\bar{d}\sigma(z) + \bar{c}).$$
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also,

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Cowen’s Adjoint Formula

Suppose that $\phi(z) = \frac{az + b}{cz + d}$ is a nonconstant linear-fractional self-map of $\mathbb{U}$. Then

$$C^*_\phi = M_g C_\sigma M_h^*,$$

where

$$\sigma(z) = \frac{\bar{a}z - \bar{c}}{-bz + d}, \quad g(z) = \frac{1}{-bz + d}, \quad \text{and } h(z) = cz + d.$$

Note $\sigma(z) = \frac{1}{\phi^{-1}(1/\bar{z})}$;

$$M_h^* = \bar{c}(M_z)^* + \bar{d} = \bar{c}B + \bar{d};$$ also,

$$z\sigma'(z) = z\frac{\bar{a}\bar{d} - \bar{b}\bar{c}}{(-bz + d)^2} = \frac{1}{-bz + d} \frac{\bar{d}(\bar{a}z - \bar{c}) + \bar{c}(-\bar{b}z + \bar{d})}{-\bar{b}z + \bar{d}} = g(z)(\bar{d}\sigma(z) + \bar{c})$$
"HMR" form: $\phi(\infty) \in \mathbb{U}$

$\phi(z) = \frac{az + b}{cz + d}$ and $\sigma(z) = \frac{-\overline{a}z - \overline{c}}{-bz + d}$ so $\phi(\infty) = a/c$ in $\mathbb{U}$ implies $\sigma$'s zero, $\overline{c}/\overline{a}$ lies outside of closure of $\mathbb{U}$; $g(z) = \frac{1}{-bz + d}$.

$$(C^*_\phi f) = (M_g C_\sigma M^*_h) f$$

$$= \bar{c}g C_\sigma (Bf) + \bar{d}g C_\sigma f$$

$$= \bar{c}g \frac{f \circ \sigma - f(0)}{\sigma} + \bar{d}g \frac{f \circ \sigma}{\sigma}$$

$$= \left(\frac{-\overline{c} + \overline{d}\sigma}{\sigma}\right) g f \circ \sigma - \overline{c}gf(0) \frac{1}{\sigma}$$

$$= (at z) \frac{z\sigma'(z)}{\sigma(z)} (C_\sigma f)(z) - \overline{c}f(0) \frac{1}{az - \overline{c}}$$

$C^*_\phi = \psi C_\sigma + \Lambda_{\infty}$, where $(\Lambda_{\infty} f)(z) = \frac{f(0)}{1 - \phi(\infty)z}$. 
“HMR” form: $\phi(\infty) \in \mathbb{U}$

$\phi(z) = \frac{az+b}{cz+d}$ and $\sigma(z) = \frac{-sz-\bar{c}}{-bz+d}$ so $\phi(\infty) = a/c$ in $\mathbb{U}$ implies $\sigma$'s zero, $\bar{c}/\bar{a}$ lies outside of closure of $\mathbb{U}$; $g(z) = \frac{1}{-bz+d}$.

\[
(C_\phi^* f) = M_g C_\sigma(\bar{c}B + \bar{d})f
\]

\[
= \bar{c}gC_\sigma(Bf) + \bar{d}gC_\sigma f
\]

\[
= \bar{c}g\left(\frac{f \circ \sigma - f(0)}{\sigma}\right) + \bar{d}g\left(\frac{f \circ \sigma}{\sigma}\right)
\]

\[
= \left(\frac{\bar{c} + \bar{d}\sigma}{\sigma}\right) f \circ \sigma - \frac{\bar{c}gf(0)}{\sigma}
\]

\[
= (at \ z) \frac{z\sigma'(z)}{\sigma(z)}(C_\sigma f)(z) - \frac{\bar{c}f(0)}{\bar{a}z - \bar{c}}
\]

$C_\phi^* = \psi C_\sigma + \Lambda_\infty$, where $(\Lambda_\infty f)(z) = \frac{f(0)}{1 - \phi(\infty)z}$. 

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Rationally Induced Adjoint Composition Operators on $H(\mathbb{U})$

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“HMR” form: $\phi(\infty) \in \mathbb{U}$

\[ \phi(z) = \frac{az+b}{cz+d} \quad \text{and} \quad \sigma(z) = \frac{-\bar{a}z-\bar{c}}{-bz+\bar{d}} \quad \text{so} \quad \phi(\infty) = \frac{a}{c} \in \mathbb{U} \]

implies $\sigma$’s zero, $\bar{c}/\bar{a}$ lies outside of closure of $\mathbb{U} ; g(z) = \frac{1}{-bz+d}$.

\[ (C^*_\phi f) = M_g C_\sigma (\bar{c}Bf + \bar{d}f) \]

\[ = \bar{c}g C_\sigma (Bf) + \bar{d}g C_\sigma f \]

\[ = \bar{c}g \left( \frac{f \circ \sigma - f(0)}{\sigma} \right) + \bar{d}g \frac{f \circ \sigma}{\sigma} \]

\[ = \left( \frac{\bar{c} + \bar{d} \sigma}{\sigma} \right) g f \circ \sigma - \frac{\bar{c}gf(0)}{\sigma} \]

\[ = (\text{at} \ z) \left( \frac{Z\sigma'(Z)}{\sigma(Z)} (C_\sigma f)(z) - \frac{\bar{c}f(0)}{\bar{a}z - \bar{c}} \right) \]

\[ C^*_\phi = \psi C_\sigma + \Lambda_\infty, \text{ where } (\Lambda_\infty f)(z) = \frac{f(0)}{1 - \phi(\infty)z}. \]
“HMR” form: \( \phi(\infty) \in \mathbb{U} \)

\[
\phi(z) = \frac{az + b}{cz + d} \quad \text{and} \quad \sigma(z) = \frac{-\bar{a}z - \bar{c}}{-bz + d}
\]

so \( \phi(\infty) = a/c \) in \( \mathbb{U} \) implies \( \sigma \)'s zero, \( \bar{c}/\bar{a} \) lies outside of closure of \( \mathbb{U} \); \( g(z) = \frac{1}{-bz + d} \).

\[
(C^*_\phi f) = M_g C_\sigma (\bar{c}Bf + \bar{d}f)
\]

\[
= \bar{c}g C_\sigma (Bf) + \bar{d}g C_\sigma f
\]

\[
= \bar{c}g \frac{f \circ \sigma - f(0)}{\sigma} + \bar{d}g \frac{\sigma f \circ \sigma}{\sigma}
\]

\[
= \left( \frac{\bar{c} + \bar{d} \sigma}{\sigma} \right) gf \circ \sigma - \frac{\bar{c}gf(0)}{\sigma}
\]

\[
= (\text{at } z) \frac{z \sigma'(z)}{\sigma(z)} (C_\sigma f)(z) - \frac{\bar{c}f(0)}{\bar{a}z - \bar{c}}
\]

\[
C^*_\phi = \psi C_\sigma + \Lambda_\infty, \quad \text{where} \quad (\Lambda_\infty f)(z) = \frac{f(0)}{1 - \phi(\infty)z}.
\]
“HMR” form: \( \phi(\infty) \in \mathbb{U} \)

\[
\phi(z) = \frac{az+b}{cz+d} \quad \text{and} \quad \sigma(z) = \frac{\bar{a}z-\bar{c}}{-bz+d} \quad \text{so} \quad \phi(\infty) = \frac{a}{c} \text{ in } \mathbb{U} \text{ implies } \\
\sigma' \text{ s zero, } \frac{\bar{c}}{\bar{a}} \text{ lies outside of closure of } \mathbb{U} ; \\
g(z) = \frac{1}{-bz+d}.
\]

\[
(C^*_\phi f) = M_g C_\sigma (\bar{c}Bf + \bar{d}f) \\
= \bar{c}gC_\sigma \left( \frac{f(z) - f(0)}{z} \right) + \bar{d}g C_\sigma f \\
= \bar{c}g \frac{f \circ \sigma - f(0)}{\sigma} + \bar{d}g \frac{\sigma f \circ \sigma}{\sigma} \\
= \left( \frac{\bar{c} + \bar{d} \sigma}{\sigma} \right) g f \circ \sigma - \frac{\bar{c}gf(0)}{\sigma} \\
= (\text{at } z) \frac{z \sigma'(z)}{\sigma(z)} (C_\sigma f)(z) - \frac{\bar{c}f(0)}{\bar{a}z - \bar{c}}
\]

\[
C^*_\phi = \psi C_\sigma + \Lambda_\infty, \text{ where } (\Lambda_\infty f)(z) = \frac{f(0)}{1 - \phi(\infty)z}.
\]
“HMR” form: $\phi(\infty) \in \mathbb{U}$

$\phi(z) = \frac{az+b}{cz+d}$ and $\sigma(z) = \frac{-\bar{a}z-\bar{c}}{-bz+d}$ so $\phi(\infty) = a/c$ in $\mathbb{U}$ implies $\sigma$’s zero, $\bar{c}/\bar{a}$ lies outside of closure of $\mathbb{U}$; $g(z) = \frac{1}{-bz+d}$.

\[
(C^*_\phi f) = M_g C_\sigma (\bar{c}Bf + \bar{d}f)
\]

\[
= \bar{c}gC_\sigma \left( \frac{f(z) - f(0)}{z} \right) + \bar{d}gC_\sigma f
\]

\[
= \bar{c}g \left( \frac{\sigma f - f(0)}{\sigma} \right) + \bar{d}g \left( \frac{\sigma f - f(0)}{\sigma} \right)
\]

\[
= (at \ z) \left( \frac{z\sigma'(z)}{\sigma(z)} \right) (C_\sigma f)(z) - \frac{\bar{c}f(0)}{\bar{a}z - \bar{c}}
\]

$C^*_\phi = \psi C_\sigma + \Lambda_\infty$, where $(\Lambda_\infty f)(z) = \frac{f(0)}{1 - \phi(\infty)z}$. 

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**Rationally Induced Adjoint Composition Operators on $H^2(\mathbb{U})$**

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**Cowen’s Adjoint Formula**

**HMR Form and Regular Form**

**Outer-Regular Self-Maps**

**Strongly Outer-Regular Self-Maps**
“HMR” form: \( \phi(\infty) \in \mathbb{U} \)

\[
\phi(z) = \frac{az+b}{cz+d} \quad \text{and} \quad \sigma(z) = \frac{\bar{a}z-\bar{c}}{-bz+d} \quad \text{so } \phi(\infty) = a/c \text{ in } \mathbb{U} \text{ implies } \\
\sigma \text{'s zero, } \bar{c}/\bar{a} \text{ lies outside of closure of } \mathbb{U} ; \ g(z) = \frac{1}{-bz+d}.
\]

\[
(C^*_\phi f) = M_g \ C_\sigma (\bar{c}Bf + \bar{d}f)
\]

\[
= \bar{c}gC_\sigma \left( \frac{f(z) - f(0)}{z} \right) + \bar{d}g \ C_\sigma f
\]

\[
= \bar{c}g \left( \frac{f \circ \sigma - f(0)}{\sigma} \right) + \bar{d}g \ \sigma f \circ \sigma
\]

\[
= \left( \bar{c} + \bar{d} \sigma \right) g f \circ \sigma - \frac{\bar{c}gf(0)}{\sigma}
\]

\[
= (at \ z) \ \frac{z \sigma'(z)}{\sigma(z)} (C_\sigma f)(z) - \frac{\bar{c}f(0)}{\bar{a}z - \bar{c}}
\]

\[
C^*_\phi = \psi C_\sigma + \Lambda_\infty, \quad \text{where } (\Lambda_\infty f)(z) = \frac{f(0)}{1 - \phi(\infty)z}.
\]
“HMR” form : \( \phi(\infty) \in \mathbb{U} \)

\[
\phi(z) = \frac{az+b}{cz+d} \quad \text{and} \quad \sigma(z) = \frac{\bar{a}z-\bar{c}}{-bz+d} \quad \text{so} \quad \phi(\infty) = \frac{a}{c} \quad \text{in} \quad \mathbb{U} \quad \text{implies}\n
\sigma’s \quad \text{zero}, \quad \bar{c}/\bar{a} \quad \text{lies outside of closure of} \quad \mathbb{U} \quad ; \quad g(z) = \frac{1}{-bz+d}.
\]

\[
(C_\phi^* f) = M_g C_\sigma (\bar{c}Bf + \bar{d}f)
\]

\[
= \bar{c}gC_\sigma \left( \frac{f(z) - f(0)}{z} \right) + \bar{d}gC_\sigma f
\]

\[
= \bar{c}g \frac{f \circ \sigma - f(0)}{\sigma} + \bar{d}g \frac{\sigma f \circ \sigma}{\sigma}
\]

\[
= \left( \frac{\bar{c} + \bar{d}\sigma}{\sigma} \right) g f \circ \sigma - \frac{\bar{c}gf(0)}{\sigma}
\]

\[
= (\text{at } z) \quad \frac{z \sigma'(z)}{\sigma(z)} (C_\sigma f)(z) - \frac{\bar{c}f(0)}{\bar{a}z - \bar{c}}
\]

\[
C_\phi^* = \psi C_\sigma + \Lambda_\infty, \quad \text{where} \quad (\Lambda_\infty f)(z) = \frac{f(0)}{1 - \phi(\infty)z}.
\]
“HMR” form: \( \phi(\infty) \in \mathbb{U} \)

\[ \phi(z) = \frac{az+b}{cz+d} \quad \text{and} \quad \sigma(z) = \frac{\bar{a}z-\bar{c}}{-bz+d}, \]

so \( \phi(\infty) = a/c \) in \( \mathbb{U} \) implies \( \sigma \)'s zero, \( \bar{c}/\bar{a} \) lies outside of closure of \( \mathbb{U} \); \( g(z) = \frac{1}{-bz+d} \).

\[
(C^*_\phi f) = M_g C_\sigma (\bar{c} B f + \bar{d} f) \\
= \bar{c} g C_\sigma \left( \frac{f(z) - f(0)}{z} \right) + \bar{d} g C_\sigma f \\
= \bar{c} g f \circ \sigma - f(0) + \bar{d} g \sigma f \circ \sigma \\
= \left( \frac{\bar{c} + \bar{d} \sigma}{\sigma} \right) g f \circ \sigma - \bar{c} g f(0) \\
= \left( \frac{z \sigma'(z)}{\sigma(z)} (C_\sigma f)(z) \right) - \bar{c} f(0) \frac{1}{\bar{a}z - \bar{c}}
\]

\[ C^*_\phi = \psi C_\sigma + \Lambda_\infty, \text{ where } (\Lambda_\infty f)(z) = \frac{f(0)}{1 - \phi(\infty)z}. \]
“HMR” form: \( \phi(\infty) \in \mathbb{U} \)

\[
\phi(z) = \frac{az + b}{cz + d} \quad \text{and} \quad \sigma(z) = \frac{-\bar{a}z - \bar{c}}{-bz + d} \quad \text{so} \quad \phi(\infty) = \frac{a}{c} \quad \text{in} \quad \mathbb{U} \quad \text{implies} \quad \\
\sigma'\text{'s zero, } \bar{c}/\bar{a} \text{ lies outside of closure of } \mathbb{U} ; \quad g(z) = \frac{1}{-bz + d}.
\]

\[
(C^*_\phi f) = Mg C_\sigma (\bar{c} B f + \bar{d} f)
\]

\[
= \bar{c} g C_\sigma \left( \frac{f(z) - f(0)}{z} \right) + \bar{d} g C_\sigma f
\]

\[
= \bar{c} g \frac{f \circ \sigma - f(0)}{\sigma} + \bar{d} g \sigma f \circ \sigma
\]

\[
= \left( \frac{\bar{c} + \bar{d} \sigma}{\sigma} \right) g f \circ \sigma - \bar{c} g f(0)
\]

\[
= (at \ z) \ \frac{Z \sigma'(Z)}{\sigma(Z)} (C_\sigma f)(z) + \frac{f(0)}{1 - \frac{\bar{a}}{c} z}
\]

\[
C^*_\phi = \psi C_\sigma + \Lambda_\infty \quad \text{where} \quad \Lambda_\infty f(z) = \frac{f(0)}{1 - \phi(\infty) z}.
\]
“HMR” form: $\phi(\infty) \in \mathbb{U}$

$\phi(z) = \frac{az+b}{cz+d}$ and $\sigma(z) = \frac{-\bar{a}z-\bar{c}}{-bz+d}$ so $\phi(\infty) = a/c$ in $\mathbb{U}$ implies $\sigma$’s zero, $\bar{c}/\bar{a}$ lies outside of closure of $\mathbb{U}$; $g(z) = \frac{1}{-bz+d}$.

\[
(C^*_\phi f) = Mg C_\sigma(\bar{c}Bf + \bar{d}f)
\]

\[
= \bar{c}gC_\sigma \left( \frac{f(z) - f(0)}{z} \right) + \bar{d}g C_\sigma f
\]

\[
= \bar{c}g \frac{f \circ \sigma - f(0)}{\sigma} + \bar{d}g \frac{f \circ \sigma}{\sigma}
\]

\[
= \left( \frac{\bar{c} + \bar{d}\sigma}{\sigma} \right) g f \circ \sigma - \bar{c}gf(0)
\]

\[
= (at Z) \frac{Z\sigma'(Z)}{\sigma(Z)}(C_\sigma f)(Z) + \frac{f(0)}{1 - \phi(\infty)Z}
\]

$C^*_\phi = \psi C_\sigma + \Lambda_\infty$, where $(\Lambda_\infty f)(Z) = \frac{f(0)}{1 - \phi(\infty)Z}$. 
“HMR” form: $\phi(\infty) \in \mathbb{U}$

$$\phi(z) = \frac{az+b}{cz+d} \quad \text{and} \quad \sigma(z) = \frac{-\bar{a}z-\bar{c}}{-bz+d} \quad \text{so} \quad \phi(\infty) = \frac{a}{c} \quad \text{in} \quad \mathbb{U} \implies \sigma' \text{‘s zero, } \bar{c}/\bar{a} \text{ lies outside of closure of } \mathbb{U}; \ g(z) = \frac{1}{-bz+d}$$

$$(C^*_\phi f) = M_g C_\sigma (\bar{c}Bf + \bar{d}f) = \bar{c}gC_\sigma \left( \frac{f(z) - f(0)}{z} \right) + \bar{d}g C_\sigma f = \bar{c}g \frac{f \circ \sigma - f(0)}{\sigma} + \bar{d}g \frac{\sigma f \circ \sigma}{\sigma} = \left( \frac{\bar{c} + \bar{d}\sigma}{\sigma} \right) g f \circ \sigma - \frac{\bar{c}g f(0)}{\sigma} = (at \ z) \frac{Z\sigma'(z)}{\sigma(z)} (C_\sigma f)(z) + \frac{f(0)}{1 - \phi(\infty)z}$$

$$C^*_\phi = \psi C_\sigma + \Lambda_\infty, \text{ where } (\Lambda_\infty f)(z) = \frac{f(0)}{1 - \phi(\infty)z}.$$
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Strongly Outer-Regular Self-Maps

"HMR" form: $\phi(\infty) \in \mathbb{U}$

$$
\psi(z) = \frac{z\sigma'(z)}{\sigma(z)}
$$

$$(C_\phi^* f) = M_g C_\sigma(\bar{c}Bf + \bar{d}f)
$$

$$
= \bar{c}gC_\sigma\left(\frac{f(z) - f(0)}{z}\right) + \bar{d}g C_\sigma f
$$

$$
= \bar{c}g\frac{f \circ \sigma - f(0)}{\sigma} + \bar{d}g \sigma f \circ \sigma
$$

$$
= \left(\frac{\bar{c} + \bar{d}\sigma}{\sigma}\right)gf \circ \sigma - \frac{\bar{c}gf(0)}{\sigma}
$$

$$
= (\text{at } z) \frac{z\sigma'(z)}{\sigma(z)}(C_\sigma f)(z) + \frac{f(0)}{1 - \phi(\infty)z}
$$

$$
C_\phi^* = \psi C_\sigma + \Lambda_\infty, \text{ where } (\Lambda_\infty f)(z) = \frac{f(0)}{1 - \phi(\infty)z}. \text{HMR - Form}
$$
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Regular Form

Recall $\phi(z) = \frac{az+b}{cz+d}$, $\sigma(z) = \frac{\bar{a}z-\bar{c}}{-bz+d}$, $g(z) = \frac{1}{-bz+d}$, $z\sigma'(z) = g(\bar{c} + \bar{d}\sigma)$.

\[(C^*_\phi f) = (M_g C_\sigma M^*_h) f\]

\begin{align*}
= & \quad \bar{c} g C_\sigma (Bf) + \bar{d} g C_\sigma f \\
= & \quad \bar{c} g C_\sigma (Bf) + \bar{d} g (f \circ \sigma - f(0) + f(0)) \\
= & \quad \bar{c} g C_\sigma (Bf) + \bar{d} g \frac{(f \circ \sigma - f(0))}{\sigma} + \bar{d} g f(0) \\
= & \quad g(\bar{c} + \bar{d}\sigma) C_\sigma (Bf) + \bar{d} g f(0) \\
= & \quad (at \ z) \ z\sigma'(z)(C_\sigma Bf)(z) + \frac{\bar{d} g f(0)}{-bz+d}
\end{align*}

\[
C^*_\phi = \nu C_\sigma B + \Lambda_0, \quad \text{where} \quad (\Lambda_0 f)(z) = \frac{f(0)}{1 - \phi(0)z}.
\]
Regular Form

Recall $\phi(z) = \frac{az+b}{cz+d}$, $\sigma(z) = \frac{\bar{a}z-\bar{c}}{-bz+d}$, $g(z) = \frac{1}{-bz+d}$,

$z\sigma'(z) = g(\bar{c} + \bar{d}\sigma)$.

$$ (C^*_\phi f) = Mg \ C_\sigma (\bar{c}B + \bar{d})f $$

$$ = \bar{c}gC_\sigma (Bf) + \bar{d}gC_\sigma f $$

$$ = \bar{c}gC_\sigma (Bf) + \bar{d}g(\sigma \circ f - f(0) + f(0)) $$

$$ = \bar{c}gC_\sigma (Bf) + \bar{d}g\sigma \frac{(f \circ \sigma - f(0))}{\sigma} + \bar{d}gf(0) $$

$$ = g(\bar{c} + \bar{d}\sigma)C_\sigma (Bf) + \bar{d}gf(0) $$

$$ = (at \ z) \ z\sigma'(z)(C_\sigma Bf)(z) + \frac{\bar{d}f(0)}{-bz+d} $$

$$ C^*_\phi = \nu C_\sigma B + \Lambda_0, \quad \text{where} \quad (\Lambda_0 f)(z) = \frac{f(0)}{1 - \phi(0)z}. $$
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Recall $\phi(z) = \frac{az+b}{cz+d}$, $\sigma(z) = \frac{\bar{a}z-\bar{c}}{-bz+d}$, $g(z) = \frac{1}{-bz+d}$,

$z\sigma'(z) = g(\bar{c} + \bar{d}\sigma)$.

$$(C^*_\phi f) = MgC_\sigma(\bar{c}Bf + \bar{d}f)$$

$$= \bar{c}gC_\sigma(Bf) + \bar{d}gC_\sigma f$$

$$= \bar{c}gC_\sigma(Bf) + \bar{d}g(f \circ \sigma - f(0) + f(0))$$

$$= \bar{c}gC_\sigma(Bf) + \bar{d}g\sigma \left( \frac{f \circ \sigma - f(0)}{\sigma} \right) + \bar{d}gf(0)$$

$$= g(\bar{c} + \bar{d}\sigma)C_\sigma(Bf) + \bar{d}gf(0)$$

$$= (at z) \ z\sigma'(z)(C_\sigma Bf)(z) + \frac{\bar{d}f(0)}{-bz + d}$$

$$C^*_\phi = \nu C_\sigma B + \Lambda_0$$, where $(\Lambda_0 f)(z) = \frac{f(0)}{1 - \phi(0)z}$. 
Regular Form

Recall $\phi(z) = \frac{az+b}{cz+d}$, $\sigma(z) = \frac{\bar{a}z-\bar{c}}{-bz+d}$, $g(z) = \frac{1}{-bz+d}$, $z\sigma'(z) = g(\bar{c} + \bar{d}\sigma)$.

\[
(C_\phi^*f) = Mg\ C_\sigma(\bar{c}Bf + \bar{d}f)
\]

\[
= \bar{c}gC_\sigma(Bf) + \bar{d}g\ C_\sigma f
\]

\[
= \bar{c}gC_\sigma(Bf) + \bar{d}g(f \circ \sigma - f(0) + f(0))
\]

\[
= \bar{c}gC_\sigma(Bf) + \bar{d}g\sigma(f \circ \sigma - f(0)) + \bar{d}gf(0)
\]

\[
= g(\bar{c} + \bar{d}\sigma)C_\sigma(Bf) + \bar{d}gf(0)
\]

\[
= (at \ z) \ z\sigma'(z)(C_\sigma Bf)(z) + \frac{\bar{d}f(0)}{-bz+d}
\]

\[\]

\[
C_\phi^* = \nu C_\sigma B + \Lambda_0, \quad \text{where} \quad (\Lambda_0 f)(z) = \frac{f(0)}{1 - \phi(0)z}
\]
Regular Form

Recall \( \phi(z) = \frac{az+b}{cz+d} \), \( \sigma(z) = \frac{\bar{a}z-\bar{c}}{-bz+d} \), \( g(z) = \frac{1}{-bz+d} \),

\[ z\sigma'(z) = g(\bar{c} + \bar{d}\sigma). \]

\[ (C^*_\phi f) = Mg C_\sigma (\bar{c}Bf + \bar{d}f) \]

\[ = \bar{c}gC_\sigma (Bf) + \bar{d}g f \circ \sigma \]

\[ = \bar{c}gC_\sigma (Bf) + \bar{d}g(f \circ \sigma - f(0) + f(0)) \]

\[ = \bar{c}gC_\sigma (Bf) + \bar{d}g\sigma \frac{(f \circ \sigma - f(0))}{\sigma} + \bar{d}gf(0) \]

\[ = g(\bar{c} + \bar{d}\sigma) C_\sigma (Bf) + \bar{d}gf(0) \]

\[ = (at \ z) \ z\sigma'(z)(C_\sigma Bf)(z) + \frac{\bar{d}f(0)}{-bz+d} \]

\[ C^*_\phi = \nu C_\sigma B + \Lambda_0, \quad \text{where} \quad (\Lambda_0 f)(z) = \frac{f(0)}{1 - \phi(0)z}. \]
Regular Form

Recall $\phi(z) = \frac{az+b}{cz+d}$, $\sigma(z) = \frac{\bar{a}z-\bar{c}}{-bz+d}$, $g(z) = \frac{1}{-bz+d}$, $z\sigma'(z) = g(\bar{c} + \bar{d}\sigma)$.

\[(C^*_\phi f) = \quad M_g \, C_\sigma (\bar{c} B f + \bar{d} f)\]

\[= \quad \bar{c} g C_\sigma (B f) + \bar{d} g f \circ \sigma\]

\[= \quad \bar{c} g C_\sigma (B f) + \bar{d} g (f \circ \sigma - f(0) + f(0))\]

\[= \quad \bar{c} g C_\sigma (B f) + \bar{d} g \sigma \left( \frac{f \circ \sigma - f(0)}{\sigma} \right) + \bar{d} g f(0)\]

\[= \quad g(\bar{c} + \bar{d}\sigma) C_\sigma (B f) + \bar{d} g f(0)\]

\[= \quad (at \ z) \quad z\sigma'(z)(C_\sigma B f)(z) + \frac{\bar{d} f(0)}{-bz+d}\]

\[C^*_\phi = \nu C_\sigma B + \Lambda_0, \quad \text{where} \quad (\Lambda_0 f)(z) = \frac{f(0)}{1 - \phi(0)z}.\]
Regular Form

Recall $\phi(z) = \frac{az+b}{cz+d}$, $\sigma(z) = \frac{\bar{az}-\bar{c}}{-bz+d}$, $g(z) = \frac{1}{-bz+d}$,

$z\sigma'(z) = g(\bar{c} + \bar{d}\sigma)$.

$$(C^*_\phi f) = Mg C_{\sigma}(\bar{c}Bf + \bar{d}f)$$

$$= \bar{c}gC_{\sigma}(Bf) + \bar{d}g f \circ \sigma$$

$$= \bar{c}gC_{\sigma}(Bf) + \bar{d}g(f \circ \sigma - f(0) + f(0))$$

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$C^*_\phi = \nu C_{\sigma}B + \Lambda_0$, where $(\Lambda_0 f)(z) = \frac{f(0)}{1 - \phi(0)z}$. 
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Recall $\phi(z) = \frac{az + b}{cz + d}$, $\sigma(z) = \frac{a\bar{z} - \bar{c}}{-bz + d}$, $g(z) = \frac{1}{-bz + d}$,

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Recall \( \phi(z) = \frac{az+b}{cz+d} \), \( \sigma(z) = \frac{-\bar{a}z-\bar{c}}{-bz+d} \), \( g(z) = \frac{1}{-bz+d} \), 
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(C^*_\phi f) = Mg C_\sigma(\bar{c}Bf + \bar{d}f)
\]

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= \bar{c}gC_\sigma(Bf) + \bar{d}g f \circ \sigma
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$C_\phi^* = \nu C_\sigma B + \Lambda_0$, \hspace{1em} where \hspace{1em} $(\Lambda_0 f)(z) = \frac{f(0)}{1 - \phi(0)z}$.
Regular Form

Recall \( \phi(z) = \frac{az+b}{cz+d} \), \( \sigma(z) = \frac{-az-c}{-bz+d} \), \( g(z) = \frac{1}{-bz+d} \),
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$$(C_{\phi}^* f) = M_g C_{\sigma}(\bar{c}Bf + \bar{d}f)$$

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Rationally Induced Adjoint Composition Operators on $H(U)$

Setting the Stage

Regular and Critical Values

A Nice Example

Cowen’s Adjoint Formula

HMR Form and Regular Form

Outer-Regular Self-Maps

Strongly Outer-Regular Self-Maps

**Regular Form**

Recall $\phi(z) = \frac{az+b}{cz+d}$, $\sigma(z) = \frac{-\bar{a}z-\bar{c}}{-bz+d}$, $g(z) = \frac{1}{-bz+d}$, $z\sigma'(z) = g(\bar{c} + \bar{d}\sigma)$.

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\[ C^*_\phi = \nu C_\sigma B + \Lambda_0, \quad \text{where} \quad (\Lambda_0 f)(z) = \frac{f(0)}{1 - \phi(0)z}. \]
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$C^*_\phi = \nu C_\sigma B + \Lambda_0$, where $(\Lambda_0 f)(z) = \frac{f(0)}{1 - \phi(0)z}$. Regular - Form
HMR Form and Regular Form

Let $\phi$ be a rational self-map of $\mathbb{U}$ of degree $d$. For all but at most finitely many points $z_0$ in $\mathbb{U}$, $\phi^{-1}(\{1/\bar{z}_0\})$ will contain $d$ distinct elements in $\hat{\mathbb{C}} \setminus \mathbb{U}^-$: $p_1, \ldots, p_d$. Since $\phi'(p_j) \neq 0$, $\phi|_{N(p_j)}$ will be invertible. Thus, on a sufficiently small open disk $D$ about $z_0$, the following functions will be holomorphic:

$$\sigma_j(z) = \frac{1}{\left(\phi|_{N(p_j)}\right)^{-1}(1/\bar{z})}, \quad j = 1, 2, \ldots, d. \quad (\sigma(D) \subseteq \mathbb{U})$$

**Local HMR Form ($\phi(\infty) \in \mathbb{U}$):**

$$(C^*_\phi f)(z) = \sum_{j=1}^{d} \frac{z \sigma'_j(z)}{\sigma_j(z)} (C_{\sigma_j} f)(z) + \frac{f(0)}{1 - \phi(\infty)z}$$

**Local Regular Form:**

$$(C^*_\phi f)(z) = \sum_{j=1}^{d} z \sigma'_j(z) \left( C_{\sigma_j} (Bf) \right)(z) + \frac{f(0)}{1 - \phi(0)z}$$
HMR Form and Regular Form

Let \( \phi \) be a rational self-map of \( \mathbb{U} \) of degree \( d \). For all but at most finitely many points \( z_0 \) in \( \mathbb{U} \), \( \phi^{-1}(\{1/\bar{z}_0\}) \) will contain \( d \) distinct elements in \( \mathbb{C} \setminus \mathbb{U}^- : p_1, \ldots, p_d \). Since \( \phi'(p_j) \neq 0 \), \( \phi|_{N(p_j)} \) will be invertible. Thus, on a sufficiently small open disk \( D \) about \( z_0 \), the following functions will be holomorphic:

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(C^*_\phi f)(z) = \sum_{j=1}^{d} \frac{z\sigma'_j(z)}{\sigma_j(z)} (C_{\sigma_j} f)(z) + \frac{f(0)}{1 - \phi(\infty)z}
\]

Local Regular Form:

\[
(C^*_\phi f)(z) = \sum_{j=1}^{d} z\sigma'_j(z) \left( C_{\sigma_j}(Bf) \right)(z) + \frac{f(0)}{1 - \phi(0)z}
\]
HMR Form and Regular Form

Let \( \phi \) be a rational self-map of \( \mathbb{U} \) of degree \( d \). For all but at most finitely many points \( z_0 \) in \( \mathbb{U} \), \( \phi^{-1}(\{1/\bar{z}_0\}) \) will contain \( d \) distinct elements in \( \hat{\mathbb{C}} \setminus \mathbb{U}^{-} : p_1, \ldots, p_d \). Since \( \phi'(p_j) \neq 0 \), \( \phi|_{N(p_j)} \) will be invertible. Thus, on a sufficiently small open disk \( D \) about \( z_0 \), the following functions will be holomorphic:

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Local HMR Form (\( \phi(\infty) \in \mathbb{U} \)):

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\]
Definition. Call a rational self-map $\phi$ of $\mathbb{U}$ outer regular provided every critical value of $\phi$ lies in $\mathbb{U}$; hence, every point in $\hat{\mathbb{C}} \setminus \mathbb{U}$ is regular.

Observation: Since $\phi$ has only finitely many critical values, if $\phi$ is outer regular, then there is an $r < 1$ such that every point of $\hat{\mathbb{C}} \setminus (r\mathbb{U})$ is regular. Thus by the Monodromy Theorem, any local inverse $\phi^{-1}$ defined near a point $1/\bar{z} \in \{|z| > 1\}$ has a holomorphic continuation to a branch of $\phi^{-1}$ defined on the simply connected domain $\hat{\mathbb{C}} \setminus (r\mathbb{U})^{-}$.

Examples. $\phi_1(z) = 1/(3 - z - z^2)$ with critical values $0$ and $4/13$ is outer regular.

$\phi_2(z) = (z^2 + z)/(3 - z^2)$ with critical values $\approx -0.09175$ and $\approx -0.90825$ is outer regular.

$\phi_3(z) = z^2$ with critical values $0$ and $\infty$ is not outer regular.
Outer Regular Rational Self-Maps

*Definition.* Call a rational self-map $\phi$ of $\mathbb{U}$outer regularprovided every critical value of $\phi$ lies in $\mathbb{U}$; hence, every point in $\hat{\mathbb{C}} \setminus \mathbb{U}$ is regular.

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**Definition.** Call a rational self-map $\phi$ of $\mathbb{U}$ *outer regular* provided every critical value of $\phi$ lies in $\mathbb{U}$; hence, every point in $\hat{\mathbb{C}} \setminus \mathbb{U}$ is regular.

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Outer Regular Rational Self-Maps

**Definition.** Call a rational self-map $\phi$ of $U$ *outer regular* provided every critical value of $\phi$ lies in $U$; hence, every point in $\hat{C} \setminus U$ is regular.

**Observation:** Since $\phi$ has only finitely many critical values, if $\phi$ is outer regular, then there is an $r < 1$ such that every point of $\hat{C} \setminus (rU)$ is regular. Thus by the Monodromy Theorem, any local inverse $\phi^{-1}$ defined near a point $1/\bar{z} \in \{|z| > 1\}$ has a holomorphic continuation to a branch of $\phi^{-1}$ defined on the simply connected domain $\hat{C} \setminus (rU)^-$.  

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Outer Regular Rational Self-Maps

**Definition.** Call a rational self-map $\phi$ of $\mathbb{U}$ *outer regular* provided every critical value of $\phi$ lies in $\mathbb{U}$; hence, every point in $\hat{\mathbb{C}} \setminus \mathbb{U}$ is regular.

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Theorem (BS)

If \( \phi \) has degree \( d \) and is outer regular, there exist self-maps of \( \mathbb{U} \), \( \{\sigma_j\}_{j=1}^d \), holomorphic on the closure of \( \mathbb{U} \) such that

\[
C^*_\phi = \sum_{j=1}^{d} M_{h_j} C_{\sigma_j} B + \Lambda_0,
\]

where \( h_j(z) = z\sigma'_j(z) \) and \((\Lambda_0 f)(z) = f(0)/(1 - \overline{\phi(0)}z)\).

Theorem (BS)

If \( \phi \) is outer regular and maps exactly one point of \( \partial \mathbb{U} \) to \( \partial \mathbb{U} \), then \( C^*_\phi \) is a compact perturbation of an operator of the form \( M_{h} C_{\sigma} B \) (backward shift followed by a weighted composition operator).

Examples: \( \phi_1(z) = (z^2 + z)/(3 - z^2) \) and \( \phi_2(z) = 1/(3 - z - z^2) \) satisfy the hypotheses of both theorems.
Theorem (BS)

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Here \( \sigma_j(z) = \frac{1}{\phi_j^{-1}(1/\overline{z})} \) where \( \phi_j^{-1} \) are branches of \( \phi^{-1} \).

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If \( \phi \) is outer regular and maps exactly one point of \( \partial \mathbb{U} \) to \( \partial \mathbb{U} \), then \( C^*_\phi \) is a compact perturbation of an operator of the form \( M_h C_{\sigma} B \) (backward shift followed by a weighted composition operator).

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Theorem (BS)

If $\phi$ has degree $d$ and is outer regular, there there exist self-maps of $\mathbb{U}$, $\{\sigma_j\}_{j=1}^d$, holomorphic on the closure of $\mathbb{U}$ such that

$$C_\phi^* = \sum_{j=1}^d M_h C_{\sigma_j} B + \Lambda_0,$$

where $h_j(z) = z\sigma_j'(z)$ and $(\Lambda_0 f)(z) = f(0)/(1 - \overline{\phi(0)}z)$.

Theorem (BS)

If $\phi$ is outer regular and maps exactly one point of $\partial \mathbb{U}$ to $\partial \mathbb{U}$, then $C_\phi^*$ is a compact perturbation of an operator of the form $M_h C_{\sigma} B$ (backward shift followed by a weighted composition operator).

Examples: $\phi_1(z) = (z^2 + z)/(3 - z^2)$ and $\phi_2(z) = 1/(3 - z - z^2)$ satisfy the hypotheses of both theorems.
Theorem (BS)

If $\phi$ has degree $d$ and is outer regular, there there exist self-maps of $\mathbb{U}$, $\{\sigma_j\}_{j=1}^d$, holomorphic on the closure of $\mathbb{U}$ such that

$$C^*_\phi = \sum_{j=1}^d M_{h_j} C_{\sigma_j} B + \Lambda_0,$$

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Examples: $\phi_1(z) = (z^2 + z)/(3 - z^2)$ and $\phi_2(z) = 1/(3 - z - z^2)$ satisfy the hypotheses of both theorems.
**Definition.** If a rational self-map $\phi$ of $U$ is outer regular and $\phi(\infty) \in U$, the $\phi$ is strongly outer regular

**Observation :** $\phi(\infty) \in U$ makes $\Lambda_\infty$ defined by 
$$(\Lambda_\infty f)(z) = \frac{1}{1 - \phi(\infty)z}$$
a legitimate rank-one operator on $H^2(U)$.

Also if $\sigma_j(z) = \frac{1}{\phi_j^{-1}(1/\bar{z})}$, will not vanish on the closed disk:

$\sigma_j(z) = 0 \implies \phi_j^{-1}(1/\bar{z}) = \infty \implies \phi(\infty) = 1/\bar{z} \implies 1/\bar{z} \in U \implies |z| > 1.$
Theorem (BS)

If $\phi$ has degree $d$ and is strongly outer regular, there exist self-maps of $\mathbb{U}$, $\{\sigma_j\}_{j=1}^d$, holomorphic and nonzero the closure of $\mathbb{U}$ such that

$$C_\phi^* = \sum_{j=1}^d M_{g_j} C_{\sigma_j} + \Lambda_\infty,$$

where $g_j(z) = z\sigma'_j(z)/\sigma_j(z)$ and

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Hence, $C_\phi^*$ is a rank-one perturbation of a sum of weighed composition operators.

Theorem (BS)

If $\phi$ is strongly outer regular and maps exactly one point of $\partial \mathbb{U}$ to $\partial \mathbb{U}$, then $C_\phi^*$ is a compact perturbation of a weighted composition operator.

Examples, Both apply to $\phi(z) = \frac{1}{3 - z - z^2}$ and the first to $\phi(z) = z/(3 - z^n)$ for all $n \geq 2$. 
Theorem (BS)

If \( \phi \) has degree \( d \) and is strongly outer regular, there there exist self-maps of \( \mathbb{U} \), \( \{\sigma_j\}_{j=1}^d \), holomorphic and nonzero the closure of \( \mathbb{U} \) such that

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\]

where \( g_j(z) = z\sigma'_j(z)/\sigma_j(z) \) and

\[
(\Lambda_{\infty} f)(z) = f(0)/(1 - \phi(\infty)z).
\]

Hence, \( C_{\phi}^* \) is a rank-one perturbation of a sum of weighed composition operators.

Here \( \sigma_j(z) = \frac{1}{\phi_j^{-1}(1/\bar{z})} \) where \( \phi_j^{-1} \) are branches of \( \phi^{-1} \).

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The End