Interpolating Measures for Subnormal Operators

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If \( \mu \in M^+(K) \) is a positive regular Borel measure supported on a compact set \( K \) in the complex plane, then let \( R^2(K, \mu) \) denote the closure of \( \text{Rat}(K) \), the rational functions with poles off \( K \), in \( L^2(\mu) \). If we define \( S_{K, \mu} = M_z \) on \( R^2(K, \mu) \), then \( S_{K, \mu} \) is a typical rationally cyclic subnormal operator. When \( K \) is polynomially convex, then \( R^2(K, \mu) = P^2(\mu) \), the closure of the analytic polynomials in \( L^2(\mu) \), and \( S_{\mu} := S_{K, \mu} \) will be a cyclic subnormal operator.

If \( \lambda \in K \), then \( \lambda \) is a bounded point evaluation (b.p.e.) for \( S_{K, \mu} = M_z \) on \( R^2(K, \mu) \) if there is a constant \( C > 0 \) such that \( |f(\lambda)| \leq C \| f \|_{L^2(\mu)} \) for all \( f \in \text{Rat}(K) \). This is equivalent to requiring that the densely defined linear operator \( A : \text{Rat}(K) \rightarrow \mathbb{C} \) given by \( A(f) = f(\lambda) \) extends to an (onto) bounded linear operator \( A : R^2(K, \mu) \rightarrow \mathbb{C} \) (the extension is also called \( A \)).

Thomson’s Theorem [1] says that if \( S_{\mu} = M_z \) on \( P^2(\mu) \) is pure, then b.p.e.’s always exist for \( S_{\mu} \). However it is known (see [2]) that b.p.e.’s need not exist for \( R^2(K, \mu) \) spaces. We are looking to generalize the idea of a b.p.e. for a \( R^2(K, \mu) \) space to the notion of an interpolating measure for any subnormal operator.

For a measure \( \nu \in M^+(K), \nu \) will be an interpolating measure for \( S_{K, \mu} = M_z \) on \( R^2(K, \mu) \) if the densely defined map \( A : \text{Rat}(K) \rightarrow L^2(\nu) \) defined by \( A(f) = f \) extends to be an (into and) onto bounded linear operator \( A : R^2(K, \mu) \rightarrow L^2(\nu) \).

**Question:** If \( K \) is a compact set in the complex plane and \( \mu \) a measure on \( K \), then does \( S_{K, \mu} = M_z \) on \( R^2(K, \mu) \) have an interpolating measure?

For an arbitrary operator \( S \) on a Hilbert space \( H \), a measure \( \nu \) is said to be an interpolating measure for \( S \) if there exists an (into and) onto bounded linear operator \( A : H \rightarrow L^2(\nu) \) such that \( AS = N_\nu A \), where \( N_\nu = M_z \) on \( L^2(\nu) \).

**Question:** If \( S \) is a subnormal operator, then does \( S \) have an interpolating measure? If not, which subnormal operators have interpolating measures?

**Theorems:**

(a) If \( S_{\mu} = M_z \) on \( P^2(\mu) \) is pure and \( G \) is the set of b.p.e.’s for \( S_{\mu} \), then a measure \( \nu \) is an interpolating measure for \( S_{\mu} \) if and only if \( \nu \) is a discrete measure carried by \( G \) whose atoms form a \( P^2(\mu) \) interpolating sequence.

(b) If \( S = M_z \) on \( H^2(G) \) where \( G = \mathbb{D} \setminus [0, 1] \), then Lebesgue measure on \([0, 1]\) is an interpolating measure for \( S \).

(c) If \( S = M_z \) on \( L^2(\mathbb{D})_+ \) is the dual of the Bergman operator, then for any compact set \( K \subseteq \mathbb{D} \), \( \nu = \text{area measure on } K \) is an interpolating measure for \( S \).

**Question:** Are there bounded regions \( G \) in \( \mathbb{C} \) such that \( S = M_z \) on the Bergman space \( L^2_a(G) \) has a continuous interpolating measure that is supported on \( \partial G \) ?

**References**
