Arithmetic Functions Evaluated at Factorials!

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(F. Luca)
Fix \( r \in \mathbb{Q} \), there are a finite number of positive integers \( n \) and \( m \) for which

\[
f(n!) = r \cdot m!
\]

where \( f \) is one of \( \tau \), \( \phi \), or \( \sigma \).
**Theorem 1.** Let \( f \) denote one of the arithmetic functions \( \tau \), \( \phi \) or \( \sigma \), and let \( k \) be a fixed positive integer. Then there are finitely many positive integers \( n, m, a \) and \( b \) such that

\[
b \cdot f(n!) = a \cdot m!, \quad \gcd(a, b) = 1 \quad \text{and} \quad \omega(ab) \leq k.
\]

i.e. the total number of distinct primes dividing the numerator and denominator of the fraction obtained by reducing the quotient \( f(n!)/m! \) tends to infinity as the product \( nm \) tends to infinity.
Theorem 2. There are finitely many positive integers $a$, $b$, $n$ and $m$ such that

$$b \cdot \tau(n!) = a \cdot m!, \quad \gcd(a, b) = 1, \quad \omega(b) \leq m^{1/4}$$

and $P_0(a) \leq \frac{\log n}{22},$

where $P_0(a)$ denotes the least prime not dividing $a$.

Theorem 3. and for $n > 1$

$$b \cdot \phi(n!) = a \cdot m!, \quad \gcd(a, b) = 1 \quad \text{and} \quad \max\{\omega(a), \omega(b)\} \leq \frac{n}{7 \log n}.$$
Theorem 4. Fix $\varepsilon > 0$. Then there are finitely many positive integers $a$, $b$, $n$ and $m$ such that

\[ b \cdot \sigma(n!) = a \cdot m!, \quad \gcd(a, b) = 1, \quad \omega(ab) \leq n^{0.2-\varepsilon}. \]
Let $q$ be a prime and $\nu_q(N)$ denote the exponent of $q$ in the prime factorization of $N$.

\[
\sigma(n!) = \prod_{p \leq n} \sigma(p^{\nu_p(n!)}) = \prod_{p \leq n} \frac{p^{\nu_p(n!)} + 1 - 1}{p - 1}
\]

**Lemma 1.** If $0 < \epsilon < 1/5$ and $q$ is a prime $\leq n^{1/5 - \epsilon}$, then

(i.) $\nu_q(\sigma(n!)) \ll \frac{n \log \log(q + 1)}{q \log n}$

If $0 < \delta < 1/3$ and $q$ is a prime, then

(ii.) $\nu_q\left(\prod_{n^{1-\delta} \leq p \leq n} \sigma(p^{\nu_p(n!)})\right) \ll \frac{n \log \log(q + 1)}{q} + \frac{n^{3\delta} \log n}{\log q}$. 
Proof of Theorem for $\sigma$:

\[ b \cdot \sigma(n!) = a \cdot m!, \quad \gcd(a, b) = 1 \quad \text{and} \quad \omega(ab) \leq n^{0.2-\varepsilon} \]

The proof:

• Show there is a $N$ such that $n \leq N$ holds.

• then we can deduce that $b \cdot \sigma(n!)$ has a bounded number of distinct prime factors (depending only on $N$).

• implying that $m$ is bounded and, hence, that there are only a finite number of possibilities for the value of $a/b = f(n!)/m!$.

• Given that $\gcd(a, b) = 1$, we can then deduce that there are a finite number of possibilities for the quadruple $(n, m, a, b)$. 
\[ b \cdot \sigma(n!) = a \cdot m! \]

Set \( c = 1/5 - 2\epsilon \) where \( 0 < \epsilon < 1/10 \).

Assume \( n \) is sufficiently large and \( \omega(ab) \leq n^c \).

First, we consider the case that \( \omega(\sigma(n!)) \geq 2n^c \).

Then there exists \( \geq n^c \) distinct primes \( p \) dividing \( \sigma(n!) \) and not dividing \( ab \). Among the first \( 2n^c \) primes, \( \exists \) prime \( q \) s.t. \( q \mid \sigma(n!) \) but \( q \nmid ab \).

Moreover, \( q \leq n^{c+\epsilon} \leq n^{1/5-\epsilon} \). Since \( q \) does not divide \( ab \), we have

\[ \nu_q(\sigma(n!)) = \nu_q(m!) \geq \frac{m}{q} - 1. \]
\[ b \cdot \sigma(n!) = a \cdot m! \]

Lemma 1 (i) now implies

\[ m \ll \frac{n \log \log n}{\log n}. \]

The case when \( \omega(\sigma(n!)) < 2n^c \) also gives this. Indeed,

\[ \frac{m}{\log m} \ll \pi(m) = \omega(m!) \leq \omega(b \cdot \sigma(n!)) \leq \omega(b) + \omega(\sigma(n!)) \ll n^c \]

implying that \( m \ll n^c \log n \).
\[ b \cdot \sigma(n!) = a \cdot m! \]

Observe that
\[ \log \sigma(n!) \geq \log(n!) \sim n \log n \]

and now
\[ \log(m!) \sim m \log m \ll n \log \log n. \]

Hence,
\[ \log a = \log(b/m!) + \log \sigma(n!) \gtrsim n \log n. \]
\[ b \cdot \sigma(n!) = a \cdot m! \]

Fix \( 0 < \delta < 1/3 \). Let

\[
a' = \prod_{p \leq n^{1-\delta}} \sigma(p^{\nu_p(n!)} \text{ and } a'' = \gcd(a, \sigma(n!)/a').
\]

Clearly, \( a \leq a'a'' \). Notice that

\[
\nu_p(n!) = \sum_{u=1}^{\infty} \left\lfloor \frac{n}{p^u} \right\rfloor < \sum_{u=1}^{\infty} \frac{n}{p^u} = \frac{n}{p-1}
\]

\[
\frac{n}{p} - 1 \leq \nu_p(n!) < \frac{n}{p - 1} \text{ so that}
\]

\[
\log a' \lesssim \sum_{p \leq n^{1-\delta}} \nu_p(n!) \log p \sim (1 - \delta)n \log n.
\]
\[ b \cdot \sigma(n!) = a \cdot m! \]

\[ n \log n \lesssim \log a \leq \log a' + \log a'' \lesssim (1-\delta)n \log n + \sum_{q \mid a''} \nu_q(a'') \log q. \]

From Lemma 1, \( \sum_{q \mid a''} \nu_q(a'') \log q \) is

\[ \ll \sum_{q \mid a''} \frac{n \log \log n}{q \log n} \log q + \sum_{q \mid a''} \left( \frac{n \log \log(q + 1)}{q} \log q + n^{3\delta} \log n \right). \]
So, $\delta n \log n$ is

$$\ll \sum_{q|a'' \atop q \leq n^{c+\epsilon}} \frac{n \log \log n}{q \log n} \log q + \sum_{q|a'' \atop q > n^{c+\epsilon}} \left(\frac{n \log \log(q+1)}{q} \log q + n^{3\delta} \log n\right).$$

For the first sum on the right, we have

$$\sum_{q|a'' \atop q \leq n^{c+\epsilon}} \frac{n \log \log n}{q \log n} \log q \leq \frac{n \log \log n}{\log n} \sum_{q \leq n^{c+\epsilon}} \frac{\log q}{q} \ll n \log \log n.$$
For the second sum, we use that the number of terms is bounded by $\omega(a)$.

$$\sum_{q \mid a'' \atop q > n^{c+\epsilon}} \frac{n \log \log (q + 1)}{q} \log q \leq \omega(a) \cdot \frac{n \log \log (n^{c+\epsilon} + 1)}{n^{c+\epsilon}} \log n^{c+\epsilon}$$

$$\ll n^{c} \cdot \frac{n \log \log n}{n^{c+\epsilon}} \log n \ll n$$

and

$$\sum_{q \mid a'' \atop q > n^{c+\epsilon}} n^{3\delta} \log n \ll \omega(a) n^{3\delta} \log n.$$
\[ b \cdot \sigma(n!) = a \cdot m! \]

\[ \delta n \log n \ll n \log \log n + n + \omega(a)n^{3\delta} \log n. \]

Consequently,

\[ \omega(a)n^{3\delta} \log n \gg \delta n \log n. \]

Taking \( \delta = 4/15 < 1/3 \), the left-hand side is \( \ll n \) and we reach the desired contradiction. Hence,...
the proof is complete...

“uhh... third dimension”
Preliminaries for the function $\sigma$

Recall: $\nu_q(N)$ denotes the exponent of $q$ in the prime factorization of $N$.

$$\sigma(n!) = \prod_{p \leq n} \frac{p^{\nu_p(n!)+1} - 1}{p - 1}$$

Let $\Phi_N(x)$ denote the $N$th cyclotomic polynomial.

$$x^N - 1 = \prod_{d|N} \Phi_d(x)$$

**GOAL:** approximate $\nu_q(\sigma(n!))$

**HOW:** analyze the highest power of a given prime $q$ that can divide an expression of the form $a^N - 1$
Ideas for Lemma 6

\[ m \ll \frac{n \log \log n}{\log n}. \]

The case when \( \omega(\sigma(n!)) < 2n^c \) also gives this. Indeed,

\[ \frac{m}{\log m} \ll \pi(m) = \omega(m!) \leq \omega(b \cdot \sigma(n!)) \leq \omega(b) + \omega(\sigma(n!)) \ll n^c \]

implying that \( m \ll n^c \log n \).
Ideas for Lemma 6

Proof. $e(p) = \nu_p(n!), \ N(p) = e(p) + 1,$ and $L = q^2 \log_q n.$

\[
\sigma(n!) = \prod_{p \leq n/L} \sigma(p^{e(p)}) \cdot \prod_{n/L < p \leq n} \sigma(p^{e(p)})
\]

\[
= \prod_{p \leq n/L} \frac{p^{N(p)} - 1}{p - 1} \cdot \prod_{n/L < p \leq n} \frac{p^{N(p)} - 1}{p - 1},
\]

Estimate the contribution of factors of $q$ arising from $\sigma(p^{e(p)})$ separately depending on whether $p \leq n/L$ or $p > n/L.$
Smaller primes $p$, i.e. $p \leq n/L$

Lemma 2. $a, N \in \mathbb{Z}, N = q^r M, r \geq 0$
$q | \Phi_N(a)$ if and only if $M = \text{ord}_q(a)$

Lemma 3. $\nu_q(a^N - 1) \ll \frac{\log N + \text{ord}_q(a) \log a}{\log q}$
Larger primes $p$, i.e. $p > n/L$

For each positive integer $\ell < L$, we consider the contribution of $q$’s from $\sigma(p^{e(p)})$ with $p \in I_\ell = (n/(\ell + 1), n/\ell]$. Fix such an $\ell$ and a prime $p \in I_\ell$. The definition of $L$ implies $p > \sqrt{n}$. Since $p \in I_\ell$, we obtain

$$N(p) = \lceil n/p \rceil + 1 = \ell + 1.$$  

Let $f_\ell(x) = x^\ell + x^{\ell-1} + \cdots + x^2 + x + 1$. Then $\sigma(p^{e(p)}) = f_\ell(p)$. Observe that this polynomial defining $\sigma(p^{e(p)})$ does not change as $p$ varies over the primes in $I_\ell$. 
\[ \nu_q \left( \sigma \left( \prod_{p \in I_{\ell}} p^{e(p)} \right) \right) = \sum_{p \in I_{\ell}} \nu_q(f_{\ell}(p)) = \sum_{j \geq 1} \sum_{p \in I_{\ell}} f_{\ell}(p) \equiv 0 \pmod{q^j} \]

Brun-Titchmarsh inequality that as \(|I_{\ell}|/q^j \to \infty\), we have

\[ \pi \left( n/\ell; q^j, a \right) - \pi \left( n/((\ell+1); q^j, a \right) \leq \left( 2 + o(1) \right) \frac{|I_{\ell}|}{\phi(q^j) \log \left( |I_{\ell}|/q^j \right)}. \]
\[ \rho_{j,\ell} = \rho_{j,\ell}(q) = \left| \{ t \in \mathbb{Z} : 0 \leq t \leq q^j - 1, f_\ell(t) \equiv 0 \pmod{q^j} \} \right|. \]

With some love and tender care:

\[ \nu_q \left( \prod_{n/L < p \leq n} \frac{p^{N(p)} - 1}{p - 1} \right) = \sum_{\ell < L} \sum_{p \in I_\ell} \nu_q(f_\ell(p)) \]

\[ \ll \sum_{\ell < L} \left( \sum_{1 \leq j \leq J} \frac{2|I_\ell|\rho_{j,\ell}}{\phi(q^j) \log \left( |I_\ell|/q^j \right)} + \sum_{J < j < K_L} \rho_{j,\ell} \left( \frac{|I_\ell|}{q^j} + 1 \right) \right) \]

which gives the result.
The second part of the lemma? It’s similar but less involved.

Take $L = n^\delta$. Partition the interval $I_\ell$ into congruence classes of length $q^j$.

\[
\sum_{\ell < L} \sum_{p \in I_\ell} \nu_q(f_\ell(p)) \leq \sum_{\ell < L} \sum_{1 \leq j < K_\ell} \rho_{j,\ell} \left( \frac{|I_\ell|}{q^j} + 1 \right)
\leq \sum_{1 \leq j < K_L} \sum_{\ell < L} \rho_{j,\ell} \frac{|I_\ell|}{q^j} + \sum_{\ell < L} \sum_{1 \leq j < K_L} \rho_{j,\ell}.
\]

**Lemma 4.** $\rho_{j,\ell} \leq 2 \gcd \left( \phi(q^j), \ell + 1 \right)$ distinct roots modulo $q^j$. 
Lemma 5. \[ \sum_{\ell=1}^{\infty} \frac{\gcd(\phi(q^j), \ell)}{\ell^2} \ll \log \log(q + 1) \]

Applying Lemma 4 and Lemma 5 to the first double sum on the right-hand side above and using that \( \rho_{j,\ell} \ll \ell \) to the latter,

\[
\nu_q \left( \prod_{n^{1-\delta} < p \leq n} \frac{p^{N(p)} - 1}{p - 1} \right) = \sum_{\ell < n^\delta} \sum_{p \in I_\ell} \nu_q(f_\ell(p)) \ll \frac{n \log \log(q + 1)}{q} + \frac{n^{3\delta} \log n}{\log q}.
\]

The Main Lemma follows. \( \square \)
Thank you.
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