Lemma 1. Let \( R = \prod_{p \leq e^{2\sqrt{\log n}}} p \). Then \( \exists c' > 0 \) such that
\[
\sum_{\substack{1 \leq d \leq n \\
d \mid R}} 1 \ll n \exp \left\{ -c' \sqrt{\log n \log \log n} \right\}.
\]

Lemma 2. Let \( \theta \in \mathfrak{m} \) and \( ||x|| = \min_{\beta \in \mathbb{Z}} |x - \beta| \)
\[
\sum_{1 \leq j \leq X} \sum_{u \leq k \leq U} e(jk\theta) \ll \frac{XU}{q} + (X + q) \log q.
\]

Lemma 3. \( c_1, c_2, \ldots \subset \mathbb{C} \) and \( F \in C^\infty[0, X] \)
\[
\sum_{j \leq X} c_j F(j) = F(X) \sum_{j \leq X} c_j - \int_0^X F'(\gamma) \sum_{j \leq \gamma} c_j d\gamma.
\]

Lemma 4. (Summation by parts)
\[
\sum_{i=0}^{N} a_i b_i = \sum_{i=0}^{N-1} A_i (b_i - b_{i+1}) + A_N b_N
\]
Applications of the Hardy-Littlewood Circle Method

by Daniel Baczkowski
Problem 1. (Waring’s) For every natural number $k \geq 2$ there exists a positive integer $s$ such that every natural number is the sum of at most $s$ $k^{th}$ powers of natural number (for example, every natural number is the sum of at most 4 squares, or 9 cubes, or 19 fourth powers, etc.).

The affirmative answer, known as the Hilbert-Waring theorem, was provided by Hilbert in 1909.
Let $\mathcal{A} = (a_m)$ denote a strictly increasing sequence of nonnegative integers. Consider

$$F(z) = \sum_{m=1}^{\infty} z^{a_m} \quad (|z| < 1)$$

and its $s^{th}$ power

$$F(z)^s = \sum_{m_1=1}^{\infty} \cdots \sum_{m_s=1}^{\infty} z^{a_{m_1}+\cdots+a_{m_s}} = \sum_{n=0}^{\infty} R_s(n) z^n$$

where $R_s(n)$ is the number of representations of $n$ as the sum of $s$ elements from $\mathcal{A}$. 
Hardy and Ramanujan

By the Cauchy Integral Formula,

\[ R_s(n) = \frac{1}{2\pi i} \int_C \frac{F(z)^s}{z^{n+1}} \, dz \]

where \( C = \{ z \in \mathbb{C} : |z| < \rho \} \) and \( 0 < \rho < 1 \).
Vinogradov replaced $F(z) = \sum_{m=1}^{\infty} z^{a_m}$ by

$$f(\alpha) = \sum_{m=1}^{N} e(\alpha m^k)$$

where $e(z) = e^{2\pi i z}$ and $N = \lfloor n^{1/k} \rfloor$. 

$$f(\alpha)^s = \sum_{m=1}^{sn} R_s(m) e(\alpha m)$$
\[ f(\alpha)^s = \sum_{m=1}^{sn} R_s(m)e(\alpha m) \]

and then a trivial case of the Cauchy Integral Formula yields

\[ \int_0^1 e(k\alpha) \, d\alpha = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases} \]

which gives

\[ \int_0^1 f(\alpha)^s e(-\alpha n) \, d\alpha = R_s(n). \]
For $k \geq 2$, let $g(k)$ denote the minimum $s$ such that every natural number is the sum of at most $s$ $k^{th}$ powers of natural numbers. More precisely for $k \geq 2$, $g(k)$ is the minimum $s$ so that for every $n \in \mathbb{N}$

$$n = m_1^k + m_2^k + \cdots + m_s^k$$

for some $m_1, m_2, \ldots, m_s \geq 0$

M. Filaseta has commented before, it is known that

$$g(k) = 2^k + \left(\frac{3}{2}\right)^k - 2,$$

but no one has proved it yet.
In fact, $g(k) \geq 2^k + \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 2$ since the integer

$$n = 2^k \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 1 \leq 3^k$$

it can only be the sum of $k^{th}$ powers of 1 and 2.

$$n = \left(\left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 1\right) \cdot 2^k + (2^k - 1) \cdot 1$$

Close upper bounds are known, too.
\[ G(k) \] is the minimum \( s \) such that every sufficiently large natural number is the sum of at most \( s \) \( k^{th} \) powers of natural numbers.

\[ n = m_1^k + m_2^k + \cdots + m_s^k \quad \text{for some } m_1, m_2, \ldots, m_s \geq 0 \]

\( G(k) \) is much smaller than \( g(k) \)

\( G(k) \) is more difficult to determine

\( G(2) = 4 \) and \( G(4) = 16 \)
From 1920-1928, Hardy and Littlewood showed that

\[ G(k) \leq (k - 2)2^{k-1} + 5. \]

Also, they conjectured that

\[ G(k) < \begin{cases} 2k + 1 & \text{for } k \text{ not a power of 2} \\ 4k & \text{for } k \text{ a power of 2}. \end{cases} \]

In 1936, Heilbronn improved results by Vinogradov to obtain

\[ G(k) \leq 6k \log k + \left[ 4 + 3 \log \left( 3 + \frac{2}{k} \right) \right] k + 3. \]
In 1985, Karatsuba showed that if $k > 224791$, then

$$G(k) < 2k \log k + 2k \log \log k + 6k.$$ 

Currently, the best known was proven by Wooley in 1991 exclaiming that for large $k$

$$G(k) < (1 + c)k \log k$$

for any $c > 0$. 
Goldbach’s Conjecture, Vinogradov’s Theorem, and the Hardy-Littlewood Method

**Conjecture 1.** *(the Goldbach conjecture)*
Every even integer \( \geq 4 \) is the sum of two primes.

**Conjecture 2.** *(the ternary Goldbach conjecture)*
Every odd integer \( \geq 7 \) is the sum of three primes.

**Theorem 1.** *(Vinogradov)*
Every sufficiently large odd integer is the sum of three primes.

Which is the strongest?
Vinogradov’s significant refinements

\[ S_n(\theta) = \sum_{p \leq n} e(p\theta) \]

\[ (S_n(\theta))^3 = \sum_{p_1 \leq n} \sum_{p_2 \leq n} \sum_{p_3 \leq n} e((p_1 + p_2 + p_3)\theta) \]

\[ r(n) = \# \text{ of ways to write } n \text{ as the sum of three primes} \]

\[ \sum \sum \sum \int_0^1 e((p_1+p_2+p_3-n)\theta) d\theta = \int_0^1 (S_n(\theta))^3 e(-n\theta) d\theta \]

Thus, our function of interest is an integral over \([0, 1]\).
Idea of Hardy and Littlewood

Analyze $\int_0^1 (S_n(\theta))^3 e(-n\theta) d\theta$ via cleverly dissecting the interval $[0, 1]$. For $1 \leq a < q \leq (\log n)^u$ and $\gcd(a, q) = 1$, a typical major arc will be denoted by

$$\mathcal{M}(q, a) = \left\{ \theta : \left| \theta - \frac{a}{q} \right| \leq \frac{(\log n)^u}{n} \right\}$$

and let $\mathcal{M}$ denote the union of all such sets. So that we do not have any wrap-around intervals such as $[0, n^{-1}(\log n)^u)$ and $(1 - n^{-1}(\log n)^u, 1]$, instead of working on $[0, 1]$ we work on the unit interval $\mathcal{U} = [n^{-1}(\log n)^u, 1 + n^{-1}(\log n)^u]$. 
\[
\int_0^1 (S_n(\theta))^3 e(-n\theta) d\theta = (\int_{\mathcal{M}} + \int_m) (S_n(\theta))^3 e(-n\theta) d\theta
\]

The minor arcs are \( m = \mathcal{U} \setminus \mathcal{M} \). Recall from class that \( \mathcal{M} \), the major arcs, consists of disjoint intervals.

Notice that as \( n \to \infty \), the size of \( \mathcal{M} \) tends to 0. So, the term “major” is insinuating major contribution to the size of the integral and not the size of the interval.

The hope is that by staying away from rationals with small denominator, we will be able to obtain significant cancelation in the integral.
Now, Dirichlet’s approximation by rationals gives us that for every \( \theta \in \mathbb{m} \),

\[
\theta = \frac{a}{q} + \frac{\epsilon}{q^2}
\]

with \( \gcd(a, q) = 1 \) and \( |\epsilon| \leq 1 \)

where most importantly \( (\log n)^u \leq q \leq n(\log n)^{-u} \).

Letting

\[
I_1 = \int_{\mathbb{m}} \left( S_n(\theta) \right)^3 e(-n\theta) d\theta \quad \text{and} \quad I_2 = \int_{\mathbb{m}} \left( S_n(\theta) \right)^3 e(-n\theta) d\theta
\]

clearly \( r(n) = I_1 + I_2 \). We proceed on an adventure to show that for odd integers \( n \), \( I_1 \gg \frac{n^2}{(\log n)^3} \) and \( I_2 = O\left(\frac{n^2}{(\log n)^{u/2}}\right) \) which establishes Theorem 1.
An Estimate for $S_n(\theta)$ for $\theta \in m$

We wish to show that if $\theta \in m$, then there exists a $c > 0$ such that

$$|S_n(\theta)| \ll (\log n)^3 \left( \frac{n}{\sqrt{q}} + \frac{n}{e^{\sqrt{\log n}}} + \sqrt{nq} \right).$$ \hspace{1cm} (1)

Notice that this gives the desired estimate for $I_2$. 
Recall that the Möbius function is defined by

\[ \mu(n) = \begin{cases} 
1 & \text{if } n = 1 \\
(-1)^r & \text{if } n = p_1p_2 \cdots p_r \text{ where } p_i \text{'s are distinct} \\
0 & \text{otherwise} 
\end{cases} \]

Notice that if \( n \) has any repeated prime divisors it is 0. Note

\[ \sum_{d \mid a} \mu(d) = \begin{cases} 
1 & \text{if } a = 1 \\
0 & \text{if } a > 1 
\end{cases} \]
Let $Q = \prod_{p \leq \sqrt{n}} p$. Then, this identity yields $S_n(\theta) =$

\[
= \sum_{\sqrt{n} < p \leq n} e(p \theta) + O(\sqrt{n}) = \sum_{\sqrt{n} < m \leq n \atop (Q, m) = 1} e(m \theta) + O(\sqrt{n})
\]

\[
= \sum_{1 \leq m \leq n \atop (Q, m) = 1} e(m \theta) + O(\sqrt{n}) = \sum_{1 \leq m \leq n} \sum_{d \mid (Q, m)} \mu(d) e(m \theta) + O(\sqrt{n})
\]

\[
= \sum_{d \mid Q} \mu(d) \sum_{1 \leq m \leq n \atop d \mid m} e(m \theta) + O(\sqrt{n})
\]

\[
= \sum_{d \mid Q} \mu(d) \sum_{1 \leq r \leq n/d} e(dr \theta) + O(\sqrt{n})
\]
\[ S_n(\theta) = \sum_{d \mid Q} \mu(d) \sum_{1 \leq r \leq n/d} e(dr\theta) + O(\sqrt{n}) \]

Now, the last double sum needs much special attention.

\[ \sum_{d \mid Q} \mu(d) \sum_{1 \leq r \leq n/d} e(dr\theta) = \sum_{1 \leq r \leq n} \sum_{1 \leq d \leq n/r} \mu(d) e(dr\theta) = T' + T'' \]

where

\[ T' = \sum_{1 \leq r \leq n} \sum_{1 \leq d \leq n/r \atop d \mid Q} \mu(d) e(dr\theta) \quad \text{and} \quad T'' = \sum_{1 \leq r \leq n} \sum_{1 \leq d \leq n/r \atop d \mid Q} \mu(d) e(dr\theta). \]
By Lemma 1, it follows that

$$|T''| \leq \sum_{1 \leq r \leq n} \sum_{1 \leq d \leq \frac{n}{r}} 1 \ll \sum_{1 \leq r \leq n} \frac{n}{r e^{c' \sqrt{\log(n/r)}}} \ll \frac{n}{e^{c \sqrt{\log n}}}$$

for some fixed $c > 0$. So far, we have derived that $S_n(\theta) = T' + O(n e^{-c \sqrt{\log n}})$. 
\[ T' = \sum_{1 \leq r \leq n} \sum_{1 \leq d \leq n/r \atop d \mid Q} \mu(d) e(dr\theta) \]

Write each \( d \) as \( p_j \) where \( p \) is one of the large prime divisors and simply multiply by the number of ways \( d \) may be written as \( p_j \) which is \( \ll \log n \). Notice that

\[ |T'| \ll \log n \cdot \left| \sum_{1 \leq r \leq n} \sum_{e^{2\sqrt{\log n}} \leq p \leq \sqrt{n}} \sum_{j \mid Q} \mu(p_j) e(pjr\theta) \right| \]

REMEMBER: the goal is to find an upper bound for \( |S_n(\theta)| \)
\[ |T'| \ll \log n \cdot \sum_{1 \leq r \leq n} e^{2 \sqrt{\log n}} \sum_{e \leq p \leq \sqrt{n}} \sum_{1 \leq p_j \leq n/r} \mu(p_j) e(p_j r \theta) \]

Let \( k = pr \). If \( j = p \), then the inner most sum is 0. Otherwise, we have that \( \mu(pj) = -\mu(j) \); hence, it suffices to consider

\[
U = \sum_{e^{2 \sqrt{\log n}} \leq k \leq n} d_1(k) \sum_{j | Q} \mu(j) e(jk \theta)
\]

where \( d_1(k) \) denotes the number of ways to write \( k \) as \( pr \) which, of course, \( d_1(k) \leq d(k) \). To estimate \( U \), we dissect the double sum into dyadic blocks.
\[
U = \sum_{e^{2\sqrt{\log n} \leq k \leq n}} d_1(k) \sum_{1 \leq j \leq n/k \atop j \mid Q} \mu(j)e(jk\theta)
\]

We let \( y = e^{2\sqrt{\log n}} \) and

\[
U(z) = \sum_{z \leq k \leq 2z} d_1(k) \sum_{1 \leq j \leq n/k \atop j \mid Q} \mu(j)e(jk\theta).
\]

\[|U| \leq \sum_{z \in A} |U(z)| \text{ where } A = \{y, 2y, 2^2y, \ldots, 2^K y\} \text{ and } K < \log n.\]
\[
|U(z)|^2 \leq \sum_{z \leq k \leq 2z} |d(k)|^2 \sum_{z \leq k \leq 2z} \left| \sum_{j | Q} \mu(j) e(j k \theta) \right|^2 \\
\ll z (\log z)^3 \sum_{z \leq k \leq 2z} \sum_{1 \leq j_1 \leq n/z} \sum_{1 \leq j_2 \leq n/z} \sum_{j_1 | Q} \sum_{j_2 | Q} \mu(j_1) \mu(j_2) e((j_1 - j_2) k \theta) \\
\ll z (\log z)^3 \sum_{1 \leq j_1 \leq n/z} \sum_{1 \leq j_2 \leq n/z} \sum_{z \leq k \leq 2z} e((j_1 - j_2) k \theta)
\]

Notice that \(0 \leq j_1 - j_2 < n/z\) as \(j_1\) and \(j_2\) vary; and moreover there are \(\leq n/z\) choices for each difference.
Altogether, we have that

\[ |U(z)|^2 \ll n(\log n)^3 \sum_{1 \leq j \leq n/z} \sum_{z \leq k \leq 2z} e(jk\theta) \]

Applying Lemma 2,

\[ |U(z)|^2 \ll n(\log n)^4 (n/q + n/z + q). \]

Using that \( \sqrt{A + B} \leq \sqrt{A} + \sqrt{B} \) for nonnegative real numbers \( A \) and \( B \), we have that

\[ |U(z)| \ll \left( \frac{n}{\sqrt{q}} + \frac{n}{\sqrt{z}} + \sqrt{qn} \right) (\log n)^2. \]
\[|U(z)| \ll \left( \frac{n}{\sqrt{q}} + \frac{n}{\sqrt{z}} + \sqrt{nq} \right) (\log n)^2\]

Thus, as \( A = \{y, 2y, 2^2y, \ldots 2^K y\} \) and \( K < \log n \)

\[|U| \leq \sum_{z \in A} |U(z)| \ll (\log n)^2 \sum_{1 \leq i \leq K} \left( \frac{n}{\sqrt{q}} + \frac{n}{\sqrt{2^{i-1}y}} + \sqrt{nq} \right)\]

\[\ll (\log n)^3 \left( \frac{n}{\sqrt{q}} + \frac{n}{e^{\sqrt{c \log n}}} + \sqrt{nq} \right).\]

which is the desired result!
Estimating $I_2$

Recall that $I_2 = \int_m (S_n(\theta))^3 e(-n\theta) \, d\theta$. So, applying Parseval's identity

$$|I_2| \leq \sup_{\theta \in m} |S_n(\theta)| \int_0^1 |S_n(\theta)|^2 \, d\theta = \pi(n) \sup_{\theta \in m} |S_n(\theta)|.$$  

Then a Chebyshev estimate for $\pi(n)$ and the estimate from the previous section gives

$$|I_2| \ll \left( \log n \right)^2 \left( \frac{n^2}{\sqrt{q}} + \frac{n^2}{e \sqrt{\log n}} + n^{3/2} \sqrt{q} \right) \ll \frac{n^2}{(\log n)^{u/2}}.$$
Estimating $I_1$

The important tool to bound $I_1$ is a result from analytic number theory:

**Theorem 2.** Let $\pi(x; q, a) = \#\{p \leq x : p \text{ is prime}, p \equiv a \pmod{q}\}$. If $1 \leq q \leq (\log n)^u$ and $\gcd(a, q) = 1$, then

$$\pi(n; q, a) = \frac{1}{\phi(q)} \int_2^n \frac{dt}{\log t} + O(ne^{-c\sqrt{\log n}})$$

where $c > 0$ and $\phi(q)$ is the number of positive integer $\leq q$ that are relatively prime to $q$ (Euler’s phi function).
\[ \theta \in \mathcal{M}(q, a) = \{ \alpha : |\alpha - a/q| \leq (\log n)^u / n \} \]

**Lemma 5.** Utilizing the above fact, for \(1 \leq a \leq q \leq (\log n)^u\) with \(\gcd(a, q) = 1\) we have that

\[ S_n(a/q + \beta) = \frac{\mu(q)}{\phi(q)} \sum_{j=2}^{n} \frac{e(j\beta)}{\log j} + O(ne^{-c\sqrt{\log n}}). \]

**Proof.** For such a \(\theta \in \mathcal{M}(q, a)\), we will let \(\beta = \theta - a/q\). Notice that

\[ S_n(a/q) = \frac{\mu(q)}{\phi(q)} \int_{2}^{n} \frac{dt}{\log t} + O(ne^{-c\sqrt{\log n}}) \]

where \(\mu(\cdot)\) denotes the Möbius function.
\[
S_n(a/q + \beta) = \frac{\mu(q)}{\phi(q)} \sum_{j=2}^{n} \frac{e(j\beta)}{\log j} + O(ne^{-c\sqrt{\log n}})
\]

Next, \( \chi_j = e(ja/q) \) when \( j \) is prime and 0 otherwise

\[
S_n(a/q + \beta) = \sum_{j=1}^{n} e(j\beta)\chi_j
\]

Use Lemma 3: \( F(j) = e(j\beta), \ c_j = \chi_j - \frac{\mu(q)}{(\phi(q) \log j)} \)

\[
S_n(a/q + \beta) - \frac{\mu(q)}{\phi(q)} \sum_{j=2}^{n} \frac{e(j\beta)}{\log j} \ll ne^{-c\sqrt{\log n}}.
\]
For $\beta = \theta - a/q$, we let $\tau(\beta) = \sum_{j=2}^{n} e(j\beta) \log j$. From Lemma 5,

$$\left| S_n(\theta)^3 - \frac{\mu(q)}{\phi(q)^3} \tau(\beta)^3 \right| \ll n^2 \left| S_n(\theta) - \frac{\mu(q)}{\phi(q)} \tau(\beta) \right| \ll n^3 e^{-c\sqrt{\log n}}.$$ 

Now, integrating over $\mathcal{M}$ gives

$$\sum_{q \leq (\log n)^u} \sum_{a=1}^{q} \sum_{(a,q)=1} \int_{\mathcal{M}(q,a)} \left( S_n(\theta)^3 - \frac{\mu(q)}{\phi(q)^3} \tau(\beta)^3 \right) e(-n\theta) d\theta \ll \frac{n^2 (\log n)^{3u}}{e^{c\sqrt{\log n}}}.$$
Let \( g(n) = (\log n)^u/n \), by definition of \( M \)

\[
\int_{M} (S_n(\theta))^3 e(-n\theta) d\theta = \mathcal{S}(n, u) \int_{-g(n)}^{g(n)} \tau(\beta)^3 e(-n\beta) d\beta + O\left(\frac{n^2 (\log n)^{3u}}{e^{c\sqrt{\log n}}}\right)
\]

where

\[
\mathcal{S}(n, u) = \sum_{q \leq (\log n)^u} \sum_{a=1}^{q} \frac{\mu(q)}{\phi(q)^3} e(-an/q).
\]

\[
\tau(\beta) = \sum_{j=2}^{n} \frac{e(j\beta)}{\log j}, \text{ so } \tau(\beta) \ll ||\beta||^{-1}\text{ when } \beta \text{ is not an integer.}
\]
Hence, replace $[-g(n), g(n)]$ with $[-1/2, 1/2]$ with error

\[ \ll \sum_{q \leq (\log n)^u} \frac{n^2}{\phi(q)^2 (\log n)^{2u}} \]

Letting $J(n) = \int_{-1/2}^{1/2} \tau(\beta)^3 e(-n\beta) d\beta$, we have that

\[ \int_{\mathcal{M}} S_n(\theta) e(-n\theta) d\theta = \mathcal{G}(n, u) J(n) + O\left(\frac{n^2}{(\log n)^{2u}}\right). \]
Claim: $J(n) \cdot (\log n)^3$ is roughly the number solutions to $j_1 + j_2 + j_3 = n$ with $1 \leq j_i \leq n$ ($i = 1, 2, 3$) which is $(n - 1)(n - 2)/2$. Indeed, by Lemma 3 or 4

$$
\tau(\beta)^3 = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \sum_{j_3=1}^{n} \frac{e((j_1 + j_2 + j_3)\beta)}{(\log n)^3} + O(n^{3/2}).
$$

So,

$$
J(n) = \frac{1}{2} \frac{n^2}{(\log n)^3} + O(n^{3/2}).
$$
\[ J(n) = \frac{1}{2} \frac{n^2}{(\log n)^3} + O(n^{3/2}) \]

Next, we let

\[ G(n) = \sum_{q=1}^{\infty} \sum_{a=1}^{q} \frac{\mu(q)}{\phi(q)^3} e\left(-\frac{an}{q}\right), \]

so that we have that

\[ G(n, u) = G(n) - \sum_{q>(\log n)^u} \sum_{\substack{a=1 \atop (a,q)=1}}^{q} \frac{\mu(q)}{\phi(q)^3} e\left(-\frac{an}{q}\right). \]
\( \mathcal{G}(n, u) = \mathcal{G}(n) + O\left( \sum_{q > (\log n)^u} \frac{1}{\phi(q)^2} \right) = \mathcal{G}(n) + O\left( \frac{1}{(\log n)^{u/2}} \right). \)

Thus,

\[ I_1 = \int_{\mathbb{R}} \left( S_n(\theta) \right)^3 e(-n\theta) d\theta = \mathcal{G}(n) J(n) + O\left( \frac{n^2}{(\log n)^{u/2}} \right), \]

where \( J(n) = \frac{n^2}{(\log n)^3} + O(n^{3/2}) \)

\[ \mathcal{G}(n) = \sum_{q=1}^{\infty} \sum_{a=1}^{\phi(q)} \frac{\mu(q)}{\phi(q)^3} e\left( -an/q \right). \]
\[ g(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{\mu(q)}{\phi(q)^3} e\left( -\frac{an}{q} \right) = \sum_{q=1}^{\infty} \frac{\mu(q)c_q(n)}{\phi(q)^3} \]

Ramanujan's sum

\[ c_q(n) = \sum_{a=1 \atop (a,q)=1}^{q} e\left( -\frac{an}{q} \right), \]

is a multiplicative function of \( q \) which satisfies

\[ c_q(n) = \frac{\mu\left(\frac{q}{(q,n)}\right) \phi(q)}{\phi\left(\frac{q}{(q,n)}\right)}. \]
Hence,

\[ \mathcal{G}(n) = \prod_{p | n} \left( 1 + (p - 1)^{-3} \right) \left( 1 - (p - 1)^{-2} \right) \]

Observe that \( \mathcal{G}(n) \gg 1 \) when \( n \) is odd and is 0 when \( n \) is even. Combining the estimates for \( I_1 \) and \( I_2 \) gives

**Theorem 3.** Define \( r(n) \) to be the number of ways to write \( n \) as the sum of three primes. Then for odd \( n \)

\[
r(n) = \frac{1}{2} \mathcal{G}(n) \frac{n^2}{(\log n)^3} + O \left( \frac{n^2}{(\log n)^{u/2}} \right) \gg \frac{n^2}{(\log n)^3}
\]

provided that \( u \geq 7 \). As a Cor. we obtain Vinogradov’s Thm.
Progress on the odd Goldbach conjecture

- (1923) assuming the Riemann Hypothesis, Hardy and Littlewood showed that it follows for all sufficiently large integers.

- (1937) as discussed above, Vinogradov removed the dependence on the Riemann Hypothesis, and proved that it is true for all sufficiently large odd integers $n$ (but did not quantify “sufficiently large”).

- (1956) Borodzkin found a bound that worked, $n \geq 3^{14348907}$. 
Progress on the odd Goldbach conjecture

• (1989) Chen and Wang reduced this bound to $10^{43000}$

• (1996) Chen and Wang, via their investigation of the zero free region of Dirichlet $L$ series and the mean-value estimate of the distribution of primes, reduced the bound to $10^{7194}$.

The exponent needs to be reduced before computers are able to aid in the finitely many cases leftover; yet, this appears to be within reach.
Noteworthy Notes

- Theorem 2 is ineffective.
- Other weaker, yet effective, statements must be applied.
- Vinogradov’s proof was more involved due to the lack of Theorem 2.
- Unfortunately, these ideas cannot be applied to Goldbach’s problem.
"The problems for the exam will be similar to those discussed in the class. Of course, the numbers will be different. But not all of them. Pi will still be 3.14159..."