Complex Analysis: Things to Know for Test 1

1. Basic operations on complex numbers: addition, multiplication, inverses, conjugation, absolute value, Re, Im, arg, $\arg_{y_0}$, Arg. (Note Arg = $\arg_{-\pi}$)
2. Properties of these basic operations: The complex numbers form a field (pp. 5 and 6); see also Propositions 1.2.1, 1.2.4, and 1.2.5.
3. The Triangle Inequality: 1.2.5 (v) and (vi). **Be able to prove 1.2.5 (v).**
4. Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$ (θ assume to be real here)
5. Polar form: $z = re^{i(\theta+2k\pi)}$, where $r = |z|$, $\theta = \text{Arg } z$ (for example), and k is an integer.
6. DeMoivre's Theorem: $\left(re^{i\theta}\right)^n = r^n e^{in\theta}$ and applications (e.g.,finding the n-th power of or the n distinct nth roots of a nonzero complex number)
7. Geometric interpretation of addition, mulitplication, conjugation, absolute value
8. Elementary functions of a complex variable: polynomials in z, $f(z) = e^z$, $g(z) = \log_{y_0} z$, $g(z) = \text{Log}z$, sine and cosine functions. You should understand the (inverse function) relationship between the exponential function and branches of the log: $e^{log_{y_0} z} = z$ for any nonzero z and $\log_{y_0} e^z = z$ if $y_0 \leq Imz < y_0 + 2\pi$. Of course, we've been using "Log" to denote the principal branch of the log:
$$\text{Log}(z) = \log_{-\pi}(z) = \ln|z| + iarg_{-\pi}(z) = \ln|z| + iArg(z)$$
9. The "MVO" z^a ($z^a = e^{a\log(z)}$) and the *principal power function* $z^a = e^{a\text{Log}(z)}$; thus, e.g., the principal (branch of the) square root function is given by $\sqrt{z} = e^{(1/2)\text{Log}(z)}$.
10. Open sets, closed sets, compact sets, limits, continuous functions, connected open sets
11. Definitions of differentiability and analyticity
12. Differentiation Rules (such as the Chain rule)
13. Differentiability at $z = z_0$ implies continuity at $z = z_0$. **Be able to prove this.**
14. Infinite Differentiability Theorem: If f is analtyic on the open set A, then f' is analytic on A.
15. Definition of curve in $\mathbb{C}$ and differentiable curve. (Curve is a funtion from an interval in $\mathbb{R}$ into $\mathbb{C}$; e.g., $\gamma : [-1, 1] \to \mathbb{C}$ given by $\gamma(t) = t + it^2$.
16. Proposition 1.5.5 "ZDC" (Derivative zero on a region implies constant)
17. Cauchy-Riemann Theorem (CRT), including representations for f' in terms of the partials of its real and imaginary parts.
18. Cauchy-Riemann Corollary (CRC): If partials of u and v are continuous on the open set A and satisfy the C-R equations on A, then $u + iv$ is analytic on A (and its derivative is $\partial u/\partial x + i\partial v/\partial x$).
19. Definition of Harmonic Function.
20. Harmonic Parts Theorem (Proposition 1.5.12): If f is analytic on the open set A then both its real and imaginary parts are harmonic on A. **Be able to prove this.**
21. Harmonic conjugates
22. Proposition: Harmonic conjugates exist on disks, the whole plane, and, in general, or open sets without "holes" (and one can often find harmonic conjugates via a "partial integration" process using the "CRES" (Cauchy-Riemann Equatons))
23. Result of HW problem 25 on page 80 (this fact is behind many applications of complex analysis): Suppose that A and B are open subsets of $\mathbb{C}$, that $f : A \to \mathbb{C}$ is analytic while $w : B \to \mathbb{R}$ is harmonic, and that $f(A) \subset B$. Then $w \circ f : A \to \mathbb{R}$ is harmonic.
24. Definition of Conformality (You may use this one: a mapping is conformal at a point z_0 in an open set provided that it preserves angles between curves intersecting at z_0 and preserves the sense/orientation of those angles.)
25. Conformality Theorem 1.5.7: Suppose that $A \subset \mathbb{C}$ is open, that $z_0 \in A$, and that $f : A \to \mathbb{C}$ is analytic. If $f'(z_0) \neq 0$, then f is conformal at z_0.
26. Definition of *entire* function.
27. Derivaties of exp, Log, sin, and cos.
28. Mapping properties of $f(z) = e^z$ and $g(z) = \text{Log}(z)$. For example, what does f do to vertical lines? horizontal lines? What does g do to rays emanating from the origin?