DIFFERENT PARTNERS, DIFFERENT PLACES : MATHEMATICS APPLIED TO THE CONSTRUCTION OF FOUR-COUPLE FOLK DANCES

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Abstract. We describe the solution to a mathematical question that arises in the context of constructing four-couple dances satisfying certain constraints. Our description makes use of various properties of permutations and cycle notation which are well known to mathematicians but probably less so more broadly. An implementation of the mathematical solution as an actual dance is discussed. We also consider generalizations of the original problem and explain a connection with the theory of orthogonal Latin squares.

1. Introduction

This paper examines a mathematical question that arises in the context of four-couple social dancing. Four-couple dances can be found in many folk dancing traditions. Square dancing is probably the best known of these, but there are also four-couple dances in Scottish dancing (where the majority of the dances are in four couple sets), English dancing, Irish dancing, Polish dancing, German dancing, and many other traditions.

In four-couple dances, regardless of the source tradition, the couples usually start out arranged in either a square or linear ("longways") formation. The dances also usually have certain characteristics in common:

1. The dance is done 4 times through. Each time through is called a *progression*.
2. Each progression consists of a series of moves (called *figures*) carried out by the dancers. In square dancing, for example, the figures might include "forward and back", "right-hand star", "promenade" and "swing with corner". Usually the same progression is carried out each time through the dance.
3. At the end of each progression, the spatial positions (or *places*) of the dancers (which we refer to collectively as *an arrangement*) are transformed in some way from their starting positions at the beginning of the progression. This has two effects:

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(i) Some or all of the dancers may move to new places;
(ii) Some or all of the dancers are paired with new partners.

4. When the dance ends, everyone is back in their original place with their original partner.

There are, of course, many variations on this plan. In square dancing, the caller may insert a "break" between the progressions. Breaks are simply extra figures that move everyone around temporarily and then bring them back to where they were when the break began (essentially, they add up to an identity transformation). Even with these occasional insertions, however, the underlying structure of the dance remains the same.

Focusing simply on the rearrangements that occur as one moves through the progressions, most dances fall into one of three categories:

1. *Same Partners, Different Places:* Each couple dances the figures starting from each of the four places, staying together throughout. For example, the couples might rotate cyclically through the four places from one progression to the next: 1 to 3 to 4 to 2 to 1.
2. *Different Partners, Some Different Places:* The men "stay home" as the women travel through the 4 places; or the opposite, the women stay home while the men travel. For example, the women might rotate through the places: 1 to 2 to 3 to 4 to 1 while the men stay in their original places throughout.
3. *Some Different Partners, Different Places:* Everyone gets to all 4 places during the sequence of progressions but they encounter their original partner at least once in the intermediate stages. The most common form here is to encounter the original partner halfway through the dance and one other partner the other two times, never encountering the other two partners. For example, the men might move 1 to 3 to 4 to 2 to 1, while the women move 1 to 2 to 4 to 3 to 1.

A fourth possible category in the taxonomy above does not seem to have arisen in practice as far as the authors have been able to determine.

4. *Different Partners, Different Places (DPDP):* Every dancer moves to a different place and meets a different partner at the end of each progression until finally returning to their original places and original partners at the end of the dance.

The mathematical question of whether or not it is possible to construct dances of this fourth type turns out to have a positive answer. In this paper, we will demonstrate this fact by explicitly constructing sequences of permutations which can be employed to achieve the desired effect. The second author has recently created several dances based on this construction. We will also show that there is essentially only one approach to constructing such dances and that this involves interleaving two *different* progressions.

The analysis employed bears some pleasing connections with parts of abstract algebra and combinatorics, although a knowledge of neither field is required to follow the arguments we give. We claim no originality and in fact will show that our problem is directly related to the problem of constructing mutually orthogonal Latin squares which has a long history. However, some of the ideas and notational conventions used in our analysis, may be of use more broadly to those attempting to solve similar problems in dance choreography, and we have written the paper with this broader audience in mind.

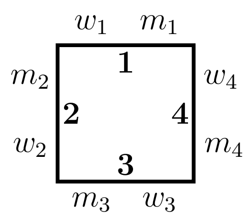
We now give a brief outline of the contents of the paper. In Section 2 we formalize the notions of arrangement and progression and review various properties of functions and permutations (including cycle notation). In Section 3 we consider the restrictions imposed by the Different Places property on the movements of the dancers. In particular, in Theorem 1 we show that the possible movements of the male or female partners in a dance is constrained to be one of four basic types. Our main results then appear in Section 4 where we consider the additional implications of the Different Partners property. In Theorems 2 and 3 we completely characterize those sequences of progressions which have the DPDP property. In particular, we show that the permutations involved in such progressions must always swap two pairs of places and cannot involve 4-cycles. In Section 5, we establish the connection between DPDP sequences and orthogonal Latin squares of order 4. The reader who is primarily interested in seeing a solution to the problem may wish to jump ahead to this section since it is largely self-contained. Finally, in Section 6 we discuss issues related to dance construction and give an example of a dance with the DPDP property.

2. Arrangements, progressions and cycle notation

In order to carry out a mathematical analysis we need to formalize certain notions using sets and functions. Recall that a set is just another name for a collection of objects. If A and B are sets, then a function f from A to B (also written $f : A \to B$) is simply a rule that assigns to each element a in A, a single element b in B. Functions can be specified in various ways. When the set A is finite, one can simply list the assignments. We will use the notation $f(a) = b$ or $f : a \mapsto b$ to indicate that f assigns b to a.

At the completion of each progression, the male partners and female partners find themselves in various spatial positions or places. If we let $M = \{m_1, m_2, m_3, m_4\}$ denote the set of four male partners, $W = \{w_1, w_2, w_3, w_4\}$ denote the set of four female partners and $P = \{1, 2, 3, 4\}$ denote the set of four places, then we can encode this information with a pair functions $\mu : M \to P$ and $\omega : W \to P$ where $\mu(m_i)$ is the place of m_i and $\omega(w_i)$ is the place of w_i for $i = 1, 2, 3, 4$.

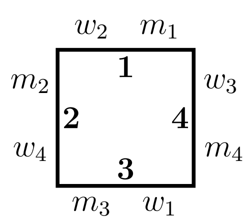
Example 1. We adopt the convention that the partners m_i and w_i in the ith couple always start a dance in place i (for $i = 1, 2, 3, 4$). This arrangement is specified by the functions $\mu(m_i) = i$ and $\omega(w_i) = i$ and will be referred to as *home position*. In terms of the actual physical positions of the dancers, there is more than one choice. In this paper we will focus on square formations in which the four couples line up along the sides of a square as illustrated below. The places are then the sides of the square and we will agree to always number them counterclockwise as displayed.



Example 2. If the men were staying home and the women were cycling through the places 1 to 3 to 4 to 2 to 1, then after one progression (starting from home position) the functions describing their places would be:

$$\mu = \left\{ \begin{array}{ccc} m_1 & \mapsto & 1 \\ m_2 & \mapsto & 2 \\ m_3 & \mapsto & 3 \\ m_4 & \mapsto & 4 \end{array} \right\} \quad \text{and} \quad \omega = \left\{ \begin{array}{ccc} w_1 & \mapsto & 3 \\ w_2 & \mapsto & 1 \\ w_3 & \mapsto & 4 \\ w_4 & \mapsto & 2 \end{array} \right\},$$

and the physical positions of the dancers would be as displayed below.



It is implicitly assumed that two men cannot occupy the same place at the conclusion of a progression (and similarly for the women). Since there are the same number of places as couples, this implies that each function must pair up elements of M (or W) with those of P so that every element on each side belongs to exactly one pair. Mathematicians call a function with this property a *bijection* or *one-to-one correspondence*. We can now give our first definition.

Definition 1. An *arrangement of couples* (or simply an *arrangement*) is an ordered pair of functions (μ, ω) with both functions $\mu : M \to P$ and $\omega : W \to P$ bijections.

Remark 1. We have been using gender (male/female) to distinguish the members of a couple. As the definition makes clear however, the key point is the ordering. One could simply refer to the first member of a couple and the second member of a couple rather than using gender.

In a similar manner, we now formalize the notion of progression. In practice, a progression refers to the sequence of figures carried out by the dancers. However, for the mathematical problem we are considering, we are primarily interested in the net effect this sequence has on the places of both the male and female partners. This information can be encoded using two functions from P to P that specify which places male and female partners end up in as a function of their starting places.

Definition 2. A *progression* is an ordered pair of functions (σ, τ) with both functions $\sigma : P \to P$ and $\tau : P \to P$ bijections. If $\sigma(i) = j$ then the progression moves the male partner in place i to place j. Similarly, if $\tau(i) = j$ then the progression moves the female partner in place i to place j.

Example 3. The progression that moves the dancers from home position to the arrangement specified in Example 2 above consists of the functions:

$$\sigma = \left\{ \begin{array}{ccc} 1 & \mapsto & 1 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 3 \\ 4 & \mapsto & 4 \end{array} \right\} \quad \text{and} \quad \tau = \left\{ \begin{array}{ccc} 1 & \mapsto & 3 \\ 2 & \mapsto & 1 \\ 3 & \mapsto & 4 \\ 4 & \mapsto & 2 \end{array} \right\}.$$

Remark 2. It is important to keep the indexing of the dancers separate, in one's mind, from the places that they occupy. For instance, at any given stage of the dance it is impossible to determine where m_i (or w_i) will end up under a given progression without further information about their current place. In particular, $\sigma(i) = j$ does not mean that m_i will end up in place j *unless* m_i happens to currently occupy place i (as for instance occurs at the beginning when the dancers are in home position).

In general, given an arrangement and a progression, the functions in the new arrangement are obtained using function composition. Recall that if $f : A \to B$ and $g : B \to C$ are functions, then the *composition of f followed by g* (denoted $g \circ f$) is the function $A \to C$ defined by $(g \circ f)(a) = g(f(a))$ for all $a \in A$. That is, one applies the function f to a to get $f(a) \in B$, and then applies the function g to get $g(f(a)) \in C$. Given an arrangement (μ, ω) and progression (σ, τ), the new arrangement is simply $(\sigma \circ \mu, \tau \circ \omega)$.

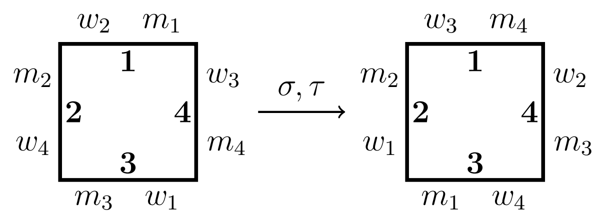
Example 4. If one starts with the arrangement (μ, ω) given in Example 2 and applies the progression (σ, τ) with

$$\sigma = \left\{ \begin{array}{ccc} 1 & \mapsto & 3 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 4 \\ 4 & \mapsto & 1 \end{array} \right\} \quad \text{and} \quad \tau = \left\{ \begin{array}{ccc} 1 & \mapsto & 4 \\ 2 & \mapsto & 3 \\ 3 & \mapsto & 2 \\ 4 & \mapsto & 1 \end{array} \right\}$$

then the new arrangement (μ', ω') consists of the functions

$$\mu' = \sigma \circ \mu = \left\{ \begin{array}{ccc} m_1 & \mapsto & 3 \\ m_2 & \mapsto & 2 \\ m_3 & \mapsto & 4 \\ m_4 & \mapsto & 1 \end{array} \right\} \quad \text{and} \quad \omega' = \tau \circ \omega = \left\{ \begin{array}{ccc} w_1 & \mapsto & 2 \\ w_2 & \mapsto & 4 \\ w_3 & \mapsto & 1 \\ w_4 & \mapsto & 3 \end{array} \right\}.$$

The change in the physical positions of the dancers would be represented by the following diagram.



The functions involved in progressions start and end from the same set P. In general, a bijection from a set to itself is called a *permutation*. Although permutations can be specified as functions in the usual way by listing element assignments, there is a more compact notation called *cycle notation* which will be used throughout the rest of the paper and which we now briefly explain with an example.

Example 5. Suppose $\sigma : P \to P$ is the permutation defined by:

$$\begin{array}{ccc} 1 & \mapsto & 3 \\ 2 & \mapsto & 2 \\ 3 & \mapsto & 4 \\ 4 & \mapsto & 1 \end{array}.$$

Then selecting an element (say 1) we can follow where it goes as the function σ is applied iteratively. We have $\sigma(1) = 3$, $\sigma(3) = 4$ and $\sigma(4) = 1$ at which point we have come back to the starting value 1. We can represent this information compactly with a cycle: (134) which one reads from left to right and imagines wrapping around at the end. Only one element in P has still not appeared at this point, namely 2. We see that $\sigma(2) = 2$, this fact can be represented with the cycle: (2). Thus in cycle notation we represent σ by writing (134)(2).

In general, it can be shown that every permutation decomposes into a collection of disjoint cycles where the appearance of the cycle $(a_1 a_2 \ldots a_n)$ means that the permutation sends $a_1 \mapsto a_2$, $a_2 \mapsto a_3, \ldots, a_n \mapsto a_1$. This decomposition is not unique in general. The order in which the cycles are listed and the starting value appearing on the left within each cycle can be varied. For example, (341)(2) and (2)(413) would be different but equally valid cycle decompositions of the permutation σ in Example 5 above.

A single cycle $(a_1, \ldots, a_n)$ of length n is called an *n-cycle*. The number of n-cycles appearing in a permutation for each $n \geq 1$ is referred to as the *cycle type* of the permutation. In the example above, $\sigma = (134)(2)$ consists of one 3-cycle and one 1-cycle so we say it has cycle type $[3^1, 1^1]$ in which the notation n^k indicates there are k n-cycles. If one considers all 24 permutations of the set P one sees that there are only 5 distinct cycle types: $[4^1]$, $[3^1, 1^1]$, $[2^2]$, $[2^1, 1^2]$ and $[1^4]$.

We are interested in finding sequences of progressions which satisfy the DPDP condition. To satisfy the Different Places part of the condition, it is necessary that each individual progression leaves none of the dancers fixed in place and so we introduce the following terminology.

Definition 3. A permutation $\sigma : P \to P$ is said to be *fixed point free* if $\sigma(i) \neq i$ for all i in P. A progression (σ, τ) is said to be *allowable* if both of the permutations σ and τ are fixed point free.

Fixed points correspond to 1-cycles in the cycle decomposition of a permutation. The possible cycle types of permutations in allowable progressions are therefore restricted to those without any 1-cycles. Looking at the 5 possible types, there are only two that qualify:

1. Cycle type: $[4^1]$. There are 6 permutations of this type:
$$(1234), (1243), (1324), (1342), (1423), (1432).$$
2. Cycle type : $[2^2]$. There are 3 permutations of this type:
$$(12)(34), (13)(24), (14)(23).$$

It follows that the 9 permutations above are the only ones that we need to consider in attempting to construct sequences of progressions with the DPDP property.

Remark 3. When listing permutations of a given cycle type, it is helpful to fix the position of one of the numerals. In the list of 4-cycles above, we have chosen to start each cycle with a 1 on the extreme left. The cycle is then of the form $(1abc)$ and is completely determined by the assignment of the remaining numerals 2, 3 and 4 to the variables a, b and c. Similarly, a permutation consisting of two 2-cycles can always be written in the form $(1a)(bc)$. There are only 3 possibilities rather than 6 in this case since there are 3 choices for a and the order of the remaining two numerals in the second 2-cycle doesn't matter, eg. $(12)(34) = (12)(43)$.

In the sections which follow we will begin to consider sequences of progressions and will use indices to distinguish the permutations in different progressions. Thus, a four couple dance consisting of four progressions will be denoted (σ_1, τ_1), (σ_2, τ_2), (σ_3, τ_3), (σ_4, τ_4). It will also be helpful to think of this as consisting of two separate sequences of permutations σ_1, σ_2, σ_3, σ_4 and τ_1, τ_2, τ_3, τ_4 where the first describes how the male partners change places from one stage of the dance to the next and the second does the same for the female partners.

We will find it convenient to drop the $\circ$ notation and use multiplicative notation when composing functions. So for instance, instead of writing $g \circ f$ to denote the composition of the functions f and g, we will simply write gf. If f and g are bijections then it is not hard to check that the same is true for the composition gf. In particular, given two permutations σ_1 and σ_2 from P to P, the composition $\sigma_2\sigma_1$ is also a permutation of P and we will sometimes refer to it as the *product* of σ_2 and σ_1. From the dancer's perspective, $\sigma_2\sigma_1$ describes the effect of the *first two* progressions on the places of the male partners. In particular, if $\sigma_2\sigma_1(i) = j$, then the male partner in place i at the start of the dance will have moved to place j after the first two progressions have taken place.

When composing functions, the order of composition is important. In general, $f \circ g \neq g \circ f$ and this is the case even if one restricts to functions which are permutations. The reader who has not encountered this phenomenon before might like to check that if $\sigma_1 = (1234)$ and $\sigma_2 = (12)(34)$ then $\sigma_2\sigma_1 = (1)(24)(3)$ while $\sigma_1\sigma_2 = (13)(2)(4)$.

The permutation $\epsilon : P \to P$ defined by $\epsilon(i) = i$ for all i, is called the *identity permutation.* It has the property that $\epsilon\sigma = \sigma\epsilon = \sigma$ for all permutations σ of P. For every permutation σ, there is a special permutation denoted σ^{-1} called the *inverse permutation.* One defines $\sigma^{-1}(i) = j$ if and only if $\sigma(j) = i$. It has the property that $\sigma\sigma^{-1} = \sigma^{-1}\sigma = \epsilon$. If σ is expressed in cycle notation, then σ^{-1} can be obtained simply by reversing the order of the numerals in each cycle. For example, if $\sigma = (1234)$ then $\sigma^{-1} = (4321)$ which could also be written (1432) if we adhere to the convention of writing the numeral 1 as far to the left as possible. More generally, if $\sigma = (1abc)$ then $\sigma^{-1} = (1cba)$. Similarly, if we consider the three permutations of the form $\sigma = (1a)(bc)$, then $\sigma^{-1} = (a1)(cb) = (1a)(bc) = \sigma$ so, in these cases, σ is its own inverse.

Remark 4. The basic properties of permutations discussed so far form the starting point for the theory of permutation groups and group theory more generally. This subject is beyond the scope of this paper and apart from some brief remarks at the end of the next section, we will not assume any familiarity with it on the part of the reader. Those interested in learning more should consult an introductory text on abstract algebra such as [2] or [3].

3. Compatible sequences of permutations

Suppose that (σ_1, τ_1) and (σ_2, τ_2) are consecutive progressions forming part of a sequence of progressions satisfying the DPDP condition. As pointed out in Section 2, the permutations involved must be fixed point free which restricts the possible cycle types. There is, however, an additional restriction. If (μ, ω) is the starting arrangement, then after both progressions have been applied in succession we reach the arrangement $(\sigma_2\sigma_1\mu, \tau_2\tau_1\omega)$ obtained by composing the functions involved in the correct order. The Different Places condition implies that none of the dancers can return to the same place after *two progressions* which means that the permutations

$\sigma_2\sigma_1$ and $\tau_2\tau_1$ must also be fixed point free. As the next lemma demonstrates, this is quite restrictive.

Lemma 1. *Let α and β be fixed point free permutations of P and suppose that $\beta\alpha$ is also fixed point free. There are two cases depending on the cycle type of α.*

1. *If $\alpha = (1abc)$ then there are 2 possible choices for β:*

$$\beta = \alpha \text{ or } \alpha^2.$$

Note: α^2 is short for $\alpha \circ \alpha = (1b)(ac)$.

2. *If $\alpha = (1a)(bc)$ then there are 4 possible choices for β:*

$$\beta = (1bac),\ (1cab),\ (1b)(ac) \text{ or } (1c)(ab).$$

Proof. Since α is fixed point free we can write $\alpha = (1abc)$ or $(1a)(bc)$ for some choice of a, b, c from $\{2, 3, 4\}$. Consider first the case where α has the form $(1abc)$. Since β is fixed point free it must be one of the following 9 permutations:

$$(1abc), (1acb), (1bac), (1bca), (1cab), (1cba), (1a)(bc), (1b)(ac), (1c)(ab).$$

It is now a tedious but straightforward process to try each of these values for β and check to see if $\beta\alpha$ is fixed point free. The process can be speeded up by observing that if β includes a 4-cycle of the form $(...a1...)$ or the 2-cycle $(a1)$ $(= (1a))$ indicating that $\beta(a) = 1$, then $\beta\alpha$ fixes 1 since $\beta\alpha(1) = \beta(a) = 1$. Hence $\beta\alpha$ is not fixed point free. Similar statements hold for the pairs of symbols: ba, cb and $1c$ in place of $a1$. In this manner, one eliminates all of the permutations except $\beta = (1abc) = \alpha$ for which one has $\beta\alpha = \alpha^2 = (1b)(ac)$; and $\beta = (1b)(ac) = \alpha^2$ for which one has $\beta\alpha = \alpha^3 = (1cba)$.

A similar argument applies in the case where α has the form $(1a)(bc)$ except now one must avoid choices for β with cycles containing the symbol strings: $1a$, $a1$, bc and cb. □

We are interested in sequences of 4 permutations that move each of the dancers from place to place without repetition until the end when they all return to their original starting positions. The two parts of the following definition encapsulate these conditions mathematically.

Definition 4. A sequence of 4 permutations σ_1, σ_2, σ_3, σ_4 is said to be *compatible* if

(i) For all i with $1 \leq i \leq 4$, the subproducts σ_i, $\sigma_{i+1}\sigma_i$ and $\sigma_{i+2}\sigma_{i+1}\sigma_i$ are fixed point free (when defined).

(ii) $\sigma_4\sigma_3\sigma_2\sigma_1 = \epsilon$.

As part (i) of the definition indicates, it is necessary for each permutation σ_i in a compatible sequence σ_1, σ_2, σ_3, σ_4 to be fixed point free, but far from sufficient. With the help of Lemma 1 we will see that the additional restrictions imposed by the definition reduce the number of possible compatible sequences to only a few different types.

Theorem 1. *If σ_1, σ_2, σ_3, σ_4 is a compatible sequence of permutations then it must be of one of the following types:*

1. $\sigma_1 = (1efg) = \sigma_2 = \sigma_3 = \sigma_4$.
2. $\sigma_1 = (1efg)$, $\sigma_2 = \sigma_4 = (1f)(eg)$ *and* $\sigma_3 = (1gfe)$.
3. $\sigma_1 = \sigma_3 = (1f)(eg)$, $\sigma_2 = (1efg)$ *and* $\sigma_4 = (1gfe)$.
4. $\sigma_1 = \sigma_3 = (1f)(eg)$ *and* $\sigma_2 = \sigma_4 = (1e)(fg)$.

Proof. One can easily verify that each of the sequences above is compatible (much of this verification is in fact carried out below). Our main task then is to show that a compatible sequence has to be of one of these four types. Since σ_1 is fixed point free it is either of the form $(1efg)$ or $(1f)(eg)$ where the numerals 2, 3 and 4 are assigned to e, f and g in some order. We consider each case in turn:

(a) If $\sigma_1 = (1efg)$ then by Lemma 1 with $\alpha = \sigma_1$ and $\beta = \sigma_2$ we see that $\sigma_2 = (1efg)$ or $(1f)(eg)$.
  (i) If $\sigma_2 = (1efg)$ then $\sigma_2\sigma_1 = (1f)(eg)$. We now apply Lemma 1 twice to make deductions about the possible values for $\beta = \sigma_3$. For the first application, let $\alpha = \sigma_2$. By the lemma we see that $\beta = \sigma_3 = (1efg)$ or $(1f)(eg)$. For the second application, let $\alpha = \sigma_2\sigma_1 = (1f)(eg)$. By the lemma we see that $\beta = \sigma_3 = (1efg)$, $(1gfe)$, $(1e)(fg)$ or $(1g)(ef)$. There is only one permutation $(1efg)$ that simultaneously satisfies the constraints coming from both applications of the lemma. It follows that we must have $\sigma_3 = (1efg)$ in this case.
  (ii) If $\sigma_2 = (1f)(eg)$ then $\sigma_2\sigma_1 = (1gfe)$. Applying Lemma 1 twice as in part (i), firstly with $\alpha = \sigma_2$ and secondly with $\alpha = \sigma_2\sigma_1$, we deduce in a similar fashion that $\beta = \sigma_3 = (1gfe)$ is the only permutation satisfying the constraints coming from both applications of the lemma.

  Given σ_1, σ_2, σ_3 we see that σ_4 is uniquely determined by the compatibility condition $\sigma_4\sigma_3\sigma_2\sigma_1 = \epsilon$. Indeed we have $\sigma_4 = (\sigma_3\sigma_2\sigma_1)^{-1}$. Parts (i) and (ii) above then give rise to parts 1 and 2 respectively in the statement of the theorem.

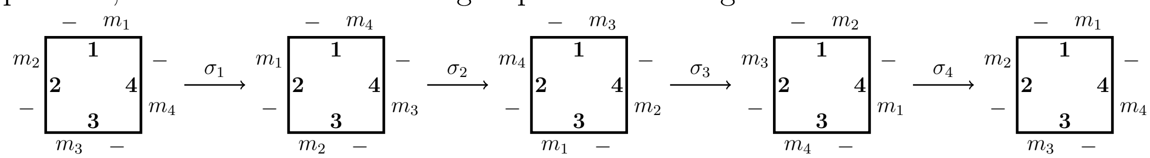
(b) If $\sigma_1 = (1f)(eg)$ then by Lemma 1 with $\alpha = \sigma_1$ and $\beta = \sigma_2$ we see that $\sigma_2 = (1efg)$, $(1gfe)$, $(1e)(fg)$ or $(1g)(ef)$.
  (i) If $\sigma_2 = (1efg)$ then $\sigma_2\sigma_1 = (1gfe)$. Applying Lemma 1 twice, firstly with $\alpha = \sigma_2$ and secondly with $\alpha = \sigma_2\sigma_1$, we deduce that $\beta = \sigma_3 = (1f)(eg)$.
  (ii) If $\sigma_2 = (1gfe)$ then arguing as in part (i) or using the symmetry that arises by switching the symbols $e \leftrightarrow g$, we see that $\sigma_3 = (1f)(eg)$.
  (iii) If $\sigma_2 = (1e)(fg)$ then $\sigma_2\sigma_1 = (1g)(ef)$. Applying Lemma 1 twice, firstly with $\alpha = \sigma_2$ and secondly with $\alpha = \sigma_2\sigma_1$, we deduce that $\beta = \sigma_3 = (1f)(eg)$.
  (iv) If $\sigma_2 = (1g)(ef)$ then arguing as in part (iii) or using the symmetry that arises by switching the symbols $e \leftrightarrow g$, we see that $\sigma_3 = (1f)(eg)$.

As in part (a), the condition $\sigma_4\sigma_3\sigma_2\sigma_1 = \epsilon$ uniquely determines σ_4. Under the symmetry induced by switching the symbols $e \leftrightarrow g$, parts (i) and (ii) are equivalent and give rise to part 3 in the theorem. Similarly, parts (iii) and (iv) give rise to part 4 of the theorem.

□

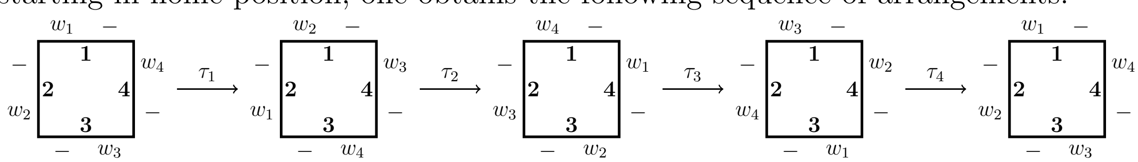
We now consider two specific examples of compatible sequences and illustrate their effects with diagrams showing the places of the dancers at each intermediate stage.

Example 6. Consider the sequence $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ with $\sigma_1 = (1234) = \sigma_2 = \sigma_3 = \sigma_4$ which is of the first type in Theorem 1. Applied to the male partners starting in home position, one obtains the following sequence of arrangements.



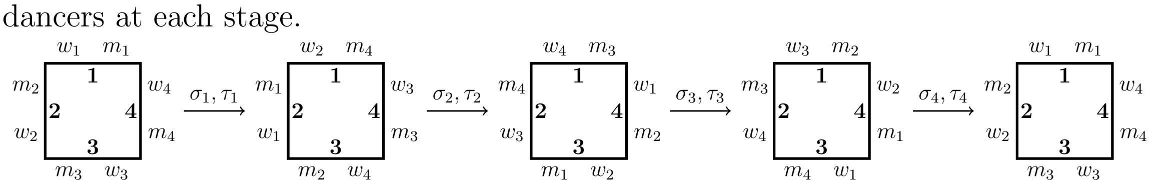
Observe that all the dancers move in the same cyclical fashion through the places. They only differ in their starting positions. This is the simplest method for moving all of the dancers through all of the places.

Example 7. Consider the sequence $\tau_1,\ \tau_2,\ \tau_3,\ \tau_4$ with $\tau_1 = \tau_3 = (12)(34)$ and $\tau_2 = \tau_4 = (13)(24)$ which is of the fourth type in Theorem 1. Applied to the female partners starting in home position, one obtains the following sequence of arrangements.



Observe that the dancers w_1 and w_4 start in different positions but move through the places in the same cyclical order $1 \to 2 \to 4 \to 3$. Meanwhile, w_2 and w_3 start in different positions but move in the same order $3 \to 4 \to 2 \to 1$ which is the reverse of the previous cyclic pattern. This type of sequence is a more complicated method for moving all of the dancers through all of the places but as we will see in the next section, it is essential in the construction of sequences of progressions with the DPDP property.

Example 8. We now combine the sequences in the two previous examples to form a sequence of progressions. Here is a diagram illustrating the arrangements of all the dancers at each stage.



Although the Different Places property is satisfied, it is not hard to see that the Different Partners property fails to hold. Indeed, after the first progression is applied, the partners in the first and third couples are still together. We will see in the next section that the source of the problem is the presence of a 4-cycle in one of the sequences of permutations.

Remark 5. For those readers with a knowledge of group theory, the results in this section can be described in a more succinct form. If we let S_4 denote the symmetric group consisting of all permutations of $P = \{1, 2, 3, 4\}$ then inside S_4 we can consider the three cyclic subgroups of order 4 which we denote $H_1 = \langle(1234)\rangle$, $H_2 = \langle(1243)\rangle$ and $H_3 = \langle(1324)\rangle$. We also consider the transitive subgroup $V = \{\epsilon, (12)(34), (13)(24), (14)(23)\} \cong C_2 \times C_2$. These four subgroups are exactly the subgroups of order 4 whose non-identity elements are all fixed point free. If α and β satisfy the conditions of Lemma 1 then parts 1 and 2 are both asserting that β must lie in one of these four subgroups that also contains α. If α is a 4-cycle then there is only one such subgroup (one of the cyclic subgroups H_i) and within this subgroup there are only two choices for β (since one must avoid both the identity element and α^{-1}). If, on the other hand, α is of cycle type $[2^2]$ then it lies in both V and one of the cyclic subgroups (since it is the square of a 4-cycle). This accounts for the larger number of choices for β in the second part.

If σ_1, σ_2, σ_3, σ_4 is a compatible sequence then the lemma implies that σ_1 and σ_2 lie in one of these four subgroups (which is uniquely determined). It follows that $\sigma_2\sigma_1$ also lies in this subgroup and by the lemma we see that σ_3 (and similarly σ_4) also lie inside this same subgroup. One can thus restrict attention to sequences of permutations all lying inside the same subgroup of order 4. For the cyclic groups H_i one finds sequences of the first three types in Theorem 1, and for V the sequence must be of the fourth type.

4. Different partners, different places

Suppose now that we have a sequence of 4 progressions (σ_1, τ_1), (σ_2, τ_2), (σ_3, τ_3), (σ_4, τ_4), satisfying DPDP. Necessarily, the sequences of permutations σ_1, σ_2, σ_3, σ_4 and τ_1, τ_2, τ_3, τ_4 must be compatible and hence are restricted to those listed in Theorem 1. Our task in this section will be to see which of these can be paired together to give a full solution to the DPDP problem. In particular, we will impose the Different Partners constraint which has played no role up until this point.

Before proceeding, let us consider exactly how the Different Partners property constrains the permutations arising in a sequence of progressions with this property. At the start of the dance, the dancers are in home position which means that m_i and w_i are both in place i. In order for each person to meet a different partner after the first progression, it is necessary that $\sigma_1(i) \neq \tau_1(i)$ for $i = 1, 2, 3, 4$. Applying the inverse permutation τ_1^{-1} to both sides, this is equivalent to $\tau_1^{-1}\sigma_1(i) \neq \tau_1^{-1}\tau_1(i) = i$ for $i = 1, 2, 3, 4$. That is, $\tau_1^{-1}\sigma_1$ must be a fixed point free permutation. More generally, we must have $\tau_j^{-1}\sigma_j$ fixed point free for $j = 1, 2, 3, 4$, since the two people in position i in *any* intermediate arrangement must be moved to different positions by the next progression in the sequence. In fact, since the the Different Partners property also requires one to meet different partners after two or three consecutive progressions, it is equivalent to the more general condition that all products of the form $\tau_j^{-1}\sigma_j$, $(\tau_{j+1}\tau_j)^{-1}\sigma_{j+1}\sigma_j$ and $(\tau_{j+2}\tau_{j+1}\tau_j)^{-1}\sigma_{j+2}\sigma_{j+1}\sigma_j$ are fixed point free. We now use these observations to prove the following negative result.

Theorem 2. *Let σ_1, σ_2, σ_3, σ_4 and τ_1, τ_2, τ_3, τ_4 be two sequences of permutations. If either sequence contains a 4-cycle then the sequence of progressions (σ_1, τ_1), (σ_2, τ_2), (σ_3, τ_3), (σ_4, τ_4) does not have the DPDP property.*

Proof. If either sequence of permutations fails to have the Different Places property then the corresponding sequence of progressions will not be DPDP. We thus assume at the outset that both sequences of permutations are compatible. To prove the theorem we must show that the presence of a 4-cycle in either sequence causes failure of the Different Partners property for the corresponding sequence of progressions.

Suppose then that σ_1, σ_2, σ_3, σ_4 is a compatible sequence containing a 4-cycle. (An identical argument applies if the other sequence contains a 4-cycle. Just reverse the roles of the two sequences.) By Theorem 1, the sequence must be of one of the first three types (since the fourth does not contain any 4-cycles). Note that in the first two types the first permutation σ_1 is a 4-cycle while in the third type, σ_2 is the first 4-cycle to appear.

Suppose that $\sigma_1 = (1xyz)$ is a 4-cycle. By Theorem 1 there are two possibilities for the sequence σ_1, σ_2, σ_3, σ_4: (i) $\sigma_1 = (1xyz) = \sigma_2 = \sigma_3 = \sigma_4$ and (ii) $\sigma_1 = (1xyz)$, $\sigma_2 = \sigma_4 = (1y)(xz)$ and $\sigma_3 = (1zyx)$. We consider each in turn.

(i) To satisfy the Different Partners property $\tau_j^{-1}\sigma_j$ must be fixed point free for $j = 1, 2, 3, 4$. We can apply Lemma 1 with $\alpha = \sigma_j = (1xyz)$ and $\beta = \tau_j^{-1}$ to deduce that $\tau_j^{-1} = (1xyz)$ or $(1y)(xz)$ for each $j = 1, 2, 3, 4$. Thus $\tau_j = (1xyz)^{-1} = (1zyx)$ or $(1y)(xz)^{-1} = (1y)(xz)$ for $j = 1, 2, 3, 4$. By Theorem 1, the only compatible sequence that can be formed out of these two permutations is $\tau_1 = (1zyx) = \tau_2 = \tau_3 = \tau_4$ but then $\tau_2\tau_1 = (1y)(xz) = \sigma_2\sigma_1$ (equivalently $(\tau_2\tau_1)^{-1}\sigma_2\sigma_1 = \epsilon$) which means that everyone meets their original partner after two progressions. Thus the Different Partners property fails to hold.

(ii) Applying Lemma 1 as in part (i) we deduce that $\tau_1 = (1zyx)$ or $(1y)(xz)$ and $\tau_3 = (1xyz)$ or $(1y)(xz)$, while τ_2 and τ_4 must be among the following 4 permutations: $(1xyz)$, $(1zyx)$, $(1x)(yz)$ or $(1z)(xy)$. Focusing on τ_1 and τ_3 and using Theorem 1, we see that there are only four possible compatible sequences τ_1, τ_2, τ_3, τ_4 consistent with the restrictions imposed by the lemma. In all four cases $\tau_1 = \tau_3 = (1y)(xz)$.

(a) $(1y)(xz)$, $(1xyz)$, $(1y)(xz)$, $(1zyx)$.

(b) $(1y)(xz)$, $(1zyx)$, $(1y)(xz)$, $(1xyz)$.
(c) $(1y)(xz)$, $(1x)(yz)$, $(1y)(xz)$, $(1x)(yz)$.
(d) $(1y)(xz)$, $(1z)(xy)$, $(1y)(xz)$, $(1z)(xy)$.

In part (a), $\tau_2\tau_1 = (1zyx) = \sigma_2\sigma_1$ indicating that everyone meets their original partner after two progressions and so the Different Partners property does not hold. In part (b), $\tau_3\tau_2 = (1xyz) = \sigma_3\sigma_2$ indicating that dancers partnered after the first progression, meet each other after two further progressions thus the Different Partners property does not hold. Finally in part (c), $\tau_2\tau_1(1) = z = \sigma_2\sigma_1(1)$ and in part (d), $\tau_2\tau_1(x) = 1 = \sigma_2\sigma_1(x)$, indicating failure of the Different Partners property after two progressions for at least one couple in each case.

To finish, we consider the case where $\sigma_2 = (1xyz)$ is a 4-cycle. By Theorem 1, σ_1, σ_2, σ_3, σ_4 must be of the third type with $\sigma_1 = \sigma_3 = (1y)(xz)$, $\sigma_2 = (1xyz)$ and $\sigma_4 = (1zyx)$. One now proceeds to argue as in part (ii) above. Using Lemma 1 there are 4 possible values for τ_1 and τ_3 and 2 possible values each for τ_2 and τ_4. Focusing on τ_2 and τ_4 and using Theorem 1, we see that there are only four possible compatible sequences τ_1, τ_2, τ_3, τ_4 consistent with the restrictions imposed by the lemma. In all four cases $\tau_2 = \tau_4 = (1y)(xz)$.

(a) $(1xyz)$, $(1y)(xz)$, $(1zyx)$, $(1y)(xz)$.
(b) $(1zyx)$, $(1y)(xz)$, $(1xyz)$, $(1y)(xz)$.
(c) $(1x)(yz)$, $(1y)(xz)$, $(1x)(yz)$, $(1y)(xz)$.
(d) $(1z)(xy)$, $(1y)(xz)$, $(1z)(xy)$, $(1y)(xz)$.

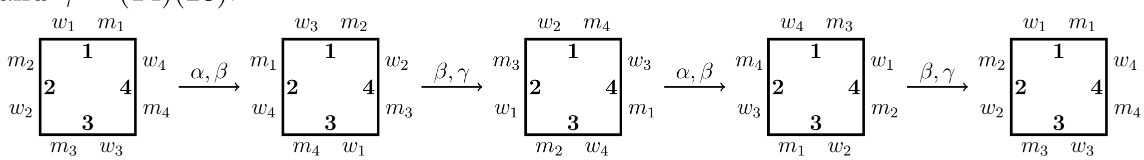
Similar arguments to those in part (ii) above, show that the Different Partners property fails to hold for the corresponding sequences of progressions. Having exhausted all the possibilities, we conclude that it is impossible for two compatible sequences to form a sequence of progressions satisfying DPDP if one of the sequences contains a 4-cycle. □

Although it is natural to make use of sequences of permutations involving 4-cycles when one wants a dance to satisfy the Different Places constraint, it is an immediate consequence of the theorem that the two compatible sequences of permutations involved in any sequence of progressions satisfying DPDP cannot include such cycles and must necessarily both be of the fourth type in Theorem 1. Knowing where to look, it is now straightforward to verify:

Theorem 3. *Sequences of progressions satisfying DPDP exist. They can be constructed using compatible sequences of the fourth type in Theorem 1.*

Indeed, if we let α, β and γ represent the three permutations $(12)(34)$, $(13)(24)$ and $(14)(23)$ in some order, then a type 4 sequence α, β, α, β can be paired with either of the type 4 sequences β, γ, β, γ or γ, α, γ, α to give such a sequence of progressions. Verification of the Different Partners property in each case can be carried out using arguments identical to those in the proof of Theorem 2. Indeed one observes that these are the only sequences that can be paired with the first. We omit the details but note that the computations involved are made considerably easier by the observation that the product of any two distinct permutations among (12)(34), (13)(24) and (14)(23) always gives the third, and that each of these three permutations is its own inverse.

Example 9. Here is a diagram illustrating what happens to the dancers for the DPDP sequence of progressions (α, β), (β, γ), (α, β), (β, γ) where $\alpha = (12)(34)$, $\beta = (13)(24)$ and $\gamma = (14)(23)$.



5. Latin squares

Another way to keep track of the dancers is to use grids, one for the men and one for the women, to chart their positions at each stage of the dance. The four columns in each grid represent the four places, numbered 1 to 4 from left to right. By filling in the symbols m_1, m_2, m_3 and m_4 into each row of the first grid and w_1, w_2, w_3 and w_4 into each row of the second grid, we can indicate the dancer's positions at different stages of the dance. The first row will indicate their initial home positions with m_i and w_i both in position i for $i = 1, 2, 3, 4$. The second row will indicate their positions after the first progression, the third row, their positions after the second progression, and so on. After the fourth progression the dancers return to their initial positions so we do not bother to include a fifth row as it would simply be a repeat of the first.

For example, let's consider one of the DPDP sequences of progressions discussed in Section 4. If the men are permuted via the sequence of permutations α, β, α, β with $\alpha = (12)(34)$ and $\beta = (13)(24)$ then the corresponding grid representation would be:

Place	1	2	3	4
Initial arrangement	m_1	m_2	m_3	m_4
Arrangement after 1st progression	m_2	m_1	m_4	m_3
Arrangement after 2nd progression	m_4	m_3	m_2	m_1
Arrangement after 3rd progression	m_3	m_4	m_1	m_2

Similarly, if the women are permuted via the sequence of permutations β, γ, β, γ with β as above and $\gamma = (14)(23)$ then the corresponding grid representation would be:

Place	1	2	3	4
Initial arrangement	w_1	w_2	w_3	w_4
Arrangement after 1st progression	w_3	w_4	w_1	w_2
Arrangement after 2nd progression	w_2	w_1	w_4	w_3
Arrangement after 3rd progression	w_4	w_3	w_2	w_1

To obtain a more compact representation we strip away the row and column labels to get:

m_1	m_2	m_3	m_4
m_2	m_1	m_4	m_3
m_4	m_3	m_2	m_1
m_3	m_4	m_1	m_2

w_1	w_2	w_3	w_4
w_3	w_4	w_1	w_2
w_2	w_1	w_4	w_3
w_4	w_3	w_2	w_1

Observe that in both grids, each symbol occurs exactly once in every row since no dancer can be in two positions at once. The Different Places property, satisfied by both of the given sequences, translates into each symbol occurring exactly once in every column (since each dancer visits each place/column exactly once during the intermediate stages of the dance). Grids with such properties often arise in mathematics and have been extensively studied.

Definition 5. Given n distinct symbols, an $n \times n$ grid in which each symbol occurs exactly once in every row and every column is called a *Latin square of order* n.

The grids above are both examples of Latin squares of order 4. If we now overlap or merge these grids into one single 4×4 grid we can more easily see the different partnerships formed during the intermediate stages of the dance.

m_1, w_1	m_2, w_2	m_3, w_3	m_4, w_4
m_2, w_3	m_1, w_4	m_4, w_1	m_3, w_2
m_4, w_2	m_3, w_1	m_2, w_4	m_1, w_3
m_3, w_4	m_4, w_3	m_1, w_2	m_2, w_1

We see that there is no repetition in the partnerships. Indeed, each of the $4^2 = 16$ possible symbol pairings (m_i, w_j) can be found in exactly one of the 16 positions in the 4×4 grid. This independently verifies that these sequences have the Different Partners property and so form a DPDP sequence. More generally we have the following definition.

Definition 6. Let L_1 be a Latin square of order n with symbols $m_1, \ldots, m_n$. Let L_2 be a Latin square of order n with symbols $w_1, \ldots, w_n$. We say that L_1 and L_2 are *orthogonal* if when one overlaps the two squares, each of the n^2 possible symbol pairings (m_i, w_j) arises exactly once.

A DPDP sequence gives rise to a pair of orthogonal Latin squares of order 4 as illustrated above. Conversely, one can go backwards and use an orthogonal pair of Latin squares of order 4 to construct a DPDP sequence. Examples of such pairs have been known since at least the eighteenth century so the solution to our initial problem is, in this sense, well known.

Mathematically, it is natural to ask about the existence of pairs of orthogonal Latin squares of order n for values of n other than 4. From the perspective of dance construction, this corresponds to asking whether or not there are dances involving n couples and places, consisting of n progressions, and satisfying the DPDP property.

The mathematician Leonhard Euler showed that orthogonal pairs do exist for all odd values of n and multiples of 4. It is easy to see that a pair does not exist when $n = 2$ and based on his inability to construct pairs when $n = 6$ or 10, he conjectured in 1782 that pairs do not exist whenever n is an integer of the form $2(2k + 1)$, ie. n is even, but not a multiple of 4. In 1900, Tarry rigorously verified Euler's conjecture for $n = 6$ by systematically enumerating various cases. However, in 1959 and 1960 counterexamples to the conjecture were discovered for $n = 22$ by Bose and Shrikhande and $n = 10$ by Parker. Shortly thereafter in [1], they together established that the conjecture is false for all values of n except $n = 2$ and $n = 6$. Thus a DPDP sequence of progressions for n couples exists for all choices of n except these two values. Further references and additional discussion can be found in [4].

The problem of constructing collections of more than two Latin squares of some fixed order n in which every pair of squares is orthogonal has also been considered. It can be shown that the maximum size for such a collection is $n - 1$ and that this upper bound is attained when n is a prime power. In particular, for $n = 4 = 2^2$ one can find three such squares. Indeed, if we take the two compatible sequences α, β , α, β and β, γ, β, γ whose squares were displayed above, then it is not hard to see that the sequence γ, α, γ, α can be paired with either to give a DPDP sequence of progressions. Taken together then, these sequences give rise to three squares, each of which is orthogonal to the other two:

m_1	m_2	m_3	m_4
m_2	m_1	m_4	m_3
m_4	m_3	m_2	m_1
m_3	m_4	m_1	m_2

w_1	w_2	w_3	w_4
w_3	w_4	w_1	w_2
w_2	w_1	w_4	w_3
w_4	w_3	w_2	w_1

z_1	z_2	z_3	z_4
z_4	z_3	z_2	z_1
z_3	z_4	z_1	z_2
z_2	z_1	z_4	z_3

Overlapping the squares, one obtains:

m_1, w_1, z_1	m_2, w_2, z_2	m_3, w_3, z_3	m_4, w_4, z_4
m_2, w_3, z_4	m_1, w_4, z_3	m_4, w_1, z_2	m_3, w_2, z_1
m_4, w_2, z_3	m_3, w_1, z_4	m_2, w_4, z_1	m_1, w_3, z_2
m_3, w_4, z_2	m_4, w_3, z_1	m_1, w_2, z_4	m_2, w_1, z_3

In theory, this could be used to construct a dance for 12 dancers based on triples (rather than couples) satisfying the analogous DPDP property, ie. each dancer m_i moves through all four of the positions and meets all of the other partners w_j and z_j $(j = 1, 2, 3, 4)$ during the course of the dance, with similar statements holding for each of the dancers w_i and z_i.

6. Constructing a DPDP dance

We have established now that it is *theoretically* possible to devise a four-couple dance that has the DPDP property. Once the required sequences of progressions are understood, it will almost always be possible for the choreographer to find *some* set of figures that will cause the progressions to occur. But finding those figures is only half the challenge for the choreographer who is striving to write dances that people will actually want to do. He or she will be concerned about whether the figures make sense to dancers. Will dancers be able to remember them? Will they be fun?

In all good dance writing, the challenge is to hide from dancers the complexities of what is really going on, so that everything seems logical and natural once it is understood. Nothing must seem contrived. This is difficult enough to achieve in writing ordinary dances, and it is more difficult to achieve in a DPDP dance because, as we have seen, different progressions (and hence different sequences of figures) must be carried out alternately, each time through the dance.

THE INVITATION
Square Formation
Historical English Style

This is a different-partners/different-places dance. It has *alternating B parts* which are similar, but not quite the same. With this property, it is possible for each person to dance from a different side of the set with a different partner each time through.

A	1-4	Taking hands-eight, circle halfway around.
	5-8	With Ptr, 2HT once around.
	9-12	Head couples pass through and courtesy turn.
	13-16	Side couples pass through and courtesy turn.

		1st and 3rd Times	2nd and 4th Times
B	1-2	Head couples lead out to the *right* and face side couples squarely.	Head couples lead out to the *left* and face side couples squarely.
	3-4	With opposite, *RHT* halfway.	With opposite, *LHT* halfway.
	5-8	Start circling hands four *left* and original head *men* lead your lines back out into a square (*men* are home, *women* have changed places on the corners of the square).	Start circling hands four *right* and original head *women* lead your lines back out into a square (*women* are home, *men* have changed places on the corners of the square).
	9-10	Men *LHA* halfway and meet new Ptr (quick).	Women *RHA* halfway and meet new Ptr (quick).
	11-12	With new Ptr, set R&L.	With new Ptr, set R&L.
	13-16	With new Ptr, back-to-back.	With new Ptr, back-to-back.

Music: "The Invitation" by Peter Barnes

Forthcoming in *A Group of Calculated Figures* by Gary Roodman

The ultimate judgement about whether DPDP dances can be *good* dances will have to come from the dancers who do them. Nevertheless, it is worth noting that the second author has written several DPDP dances that are in circulation (which is to say that someone other than the author has chosen to teach them). One of these dances, “The Invitation”, is displayed above. Each pass through the dance corresponds to 32 bars of the accompanying music and gives rise to one progression. The figures in the first and second parts of the dance are the same each time through; the third and fourth parts alternate between two similar, but slightly different, sets of figures which gives rise to an alternating sequence of progressions. The underlying sequences of permutations are as follows:

- Men’s sequence: (13)(24), (14)(23), (13)(24), (14)(23).
- Women’s sequence: (12)(34), (13)(24), (12)(34), (13)(24).

Other DPDP dances, including two that are in four-couple longways formation, may be found on the second author’s web site [5].

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