FLAT FOLDED RIBBONS

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1. Introduction

The properties of knots composed of flat ribbons are considered within knot theory. In his paper, *Minimal Flat Knotted Ribbons*, Louis Kauffman developed a model of flat ribbon knots considered in the plane [7]. He examined the question of radial width to length for a given knot in this new context. With respect to flat ribbons, this becomes a question of minimizing the length of ribbon needed to make a given knot. Kauffman finds a minimum length of ribbon for the trefoil knot and the figure eight knot.

Following the work of Kauffman and others, we continue to investigate the minimum length of ribbon needed to construct flat folded ribbon knots for different types of families of knots. In addition, we explore the construction and structure of these folded ribbons. Our paper has the following organization. In Section 2, we introduce standard definitions in knot theory that will be used throughout the paper. In Section 3, we define a flat folded ribbon knot and explain its construction. We then describe the structure of these folded ribbons by looking at linking number and folding information, and defining ribbon Reidemeister moves. In Sections 5, 6, 7, and 8, we define ribbonlength and return to the question of determining upper bounds on the least length of ribbon required to form flattened versions of particular knots. We show that the structure of the ribbon as described in Section 4 affects minimizations. In Section 9, we summarize our main results and articulate open questions.

2. Knot Theory Background Information

The following standard definitions may be found in a number of references including [1], [4], [5], and [6].

**Definition 1.** A knot \( (K) \) is a simple closed polygonal curve in \( \mathbb{R}^3 \).

Saying the curve is simple means that it never intersects itself. Closed polygonal means that given an ordered, finite set of vertices \( \{v_1, ..., v_n\} \), these vertices are joined by \( n \) straight line segments, \( e_i \), such that \( e_i = [v_i, v_{i+1}] \) where \( 1 \leq i \leq n - 1 \) and \( e_n = [v_n, v_1] \). There are a number of equivalent definitions of a knot. While knots are usually depicted as smooth curves, one can approximate a smooth representation using a polygonal curve by increasing the number, and decreasing the size of the line segments. The polygonal definition eliminates the possibility of wild knots and is best suited for modeling the geometry of flat folded ribbon knots.
Figure 1. Three examples of non-regular knot projections. On the left, a two-to-one mapping along a non-transversal intersection. In the middle, a three-to-one mapping. On the right, a vertex projecting to the same point as another point on the knot.

Figure 2. The knot diagram of a 3-stick unknot is depicted on the left. The knot diagram of the trefoil, the simplest non-trivial knot, is shown on the right.

**Definition 2.** Two knots, $K_1$ and $K_2$, are equivalent if one can be deformed into the other, that is, if there exists an ambient isotopy of $\mathbb{R}^3$ taking $K_1$ to $K_2$. Knots are classified by knot type based on this equivalence relation.

**Definition 3.** An ambient isotopy between $K_1$ and $K_2$ is a homotopy $h_t: \mathbb{R}^3 \to \mathbb{R}^3$ that satisfies the following:

1. $h_0$ is the identity
2. every $h_t$ is a homeomorphism and
3. $h_1 = h$ where $h: \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism such that $h(K_1) = K_2$.

**Definition 4.** A knot is trivial or the unknot if it is equivalent to a simple, closed polygonal curve with three vertices connected by three line segments in the shape of a triangle.

As explained above, the number of line segments may be increased to obtain the depiction of the unknot as a smooth circle, but the minimum number of segments is three. The minimum number of segments needed to form $K$ is known as the stick number. In this paper, we investigate $n$-stick unknots where $n \geq 3$. 
Definition 5. The projection of a knot $K$ is the image of $K$ under a mapping from $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ which takes every ordered triple $(x, y, z)$ to $(x, y)$. A regular projection is a two-to-one or one-to-one mapping. In a regular projection, the two-to-one mapping only occurs when the interior of two edges intersect transversally. A non-regular projection results when

1. a two-to-one mapping occurs along a non-transversal intersection,
2. more than two points project to the same point, or
3. a vertex projects to the same point as another point on the knot.

These types of non-regular projections are illustrated in Figure 1.

Definition 6. A knot diagram adds gaps to the regular projection of a knot in order to retain over- and undercrossing information.

Knot diagrams for the 3-stick unknot and the trefoil are depicted in Figure 2.

Depending on how a given knot is situated in space, multiple knot diagrams may represent the same knot. To minimize ribbonlength of a given knot, we look at both regular and non-regular knot diagrams and find that the latter may result in smaller upper bounds on ribbonlength.

Definition 7. A link is a finite union of disjoint knots.

3. Modeling Flat Folded Ribbon Knots

Kauffman defines a ribbon as “a space homeomorphic to a rectangle” \[7\]. One can imagine creating a ribbon knot in $\mathbb{R}^3$ by taking a long, rectangular piece of paper, tying a knot in it, and connecting the two ends. Flattening the ribbon knot into the plane while retaining over- and undercrossing information, creates a flat folded ribbon knot.

Definition 8. A flat folded ribbon knot, henceforth referred to as a folded ribbon, is a framed polygonal knot diagram, as illustrated in Figure 3, that is constructed as follows:

1. Choose an arbitrary point on a line segment of the polygonal knot diagram.
2. Extend a perpendicular of equal length on either side of this point to create two parallel lines equidistant from the knot diagram. The width, $w$, of the ribbon is the distance between these two parallel lines. Hence, by construction, the distance between each parallel line and the knot diagram is $w/2$. 

Figure 3. Modeling the construction of a flat folded ribbon knot.
(3) Repeat this process for each line segment of the knot diagram, maintaining $w$. This establishes the boundary of the ribbon.

(4) At each vertex of the knot diagram, extend the length of the two line segments of the knot diagram incident to the vertex until each segment crosses the outside boundary of the ribbon corresponding to the adjacent line segment. Construct the ribbon around this additional length as outlined above. This creates an overlapping rhombus of ribbon whose center is the vertex. To prevent the introduction of new crossing information into the ribbon, create a fold at each vertex of the knot diagram by cutting along the rhombus’ diagonal which is tangent to the vertex. Connect the once separate ribbons as one along this same diagonal. Depending on how the two pieces of ribbon are connected in relation to each other, an overfold or an underfold is created.

**Remark 1.** We orient the knot diagram by assigning a direction to one segment and maintaining that direction through the ordered set of vertices of the knot (see Definition 1.1). To orient a folded ribbon, we ascribe the orientation of the knot diagram to its boundaries.

**Definition 9.** Given an oriented knot diagram, as depicted in Figure 4, in which vertex $v_i$ is traversed from segment $e_i$ to segment $e_{i+1}$, an overfold is created at $v_i$ if the ribbon of segment $e_{i+1}$ is over the ribbon of segment $e_i$. An underfold is created if the ribbon corresponding to $e_{i+1}$ is under the ribbon of $e_i$.

**Remark 2.** Since there are two choices of folding direction (over or under) at each vertex, a knot diagram with $k$ vertices has $2^k$ possible choices for folding information. An example of two knots with the same knot diagram but different folding information can be seen in Figure 28. These two knots are studied further in Section 6.2 where we show that folding information affects the minimization of ribbonlength for a given knot diagram.

**Remark 3.** A folded ribbon has the same over/undercrossing information of the nearby knot diagram. This over/under information is preserved by weaving of the ribbon. Any folding of the folds would introduce additional crossing information. Thus, folded folds...
are disallowed in the folded ribbon in order to preserve consistency of the over/under information from the original knot diagram.

**Remark 4.** We note that our definition of a flat folded ribbon knot is still a work in progress. While we have established how folds are created at each vertex of a knot diagram and disallowed folding of the folds, we have not addressed whether or not there may be extra folds that do not affect knot type but do affect the width of the ribbon. For a given knot, we observe that while the introduction of extra folds would decrease the width along that portion of the ribbon, it would increase the overall ribbonlength. However, in the case of a link, we observe that by reducing the width of the ribbon of one component, $C_1$, we can reduce the length of another component, $C_2$, by minimizing $C_2$ around the section of $C_1$ that has reduced width. But, by reducing the width of this section of $C_1$, we have increased the overall ribbonlength of $C_1$. Thus, any reduction in ribbonlength of $C_2$ is negated by an increase in ribbonlength of $C_1$. Our intuition tells us that, globally, the ribbonlength of the link with these additional folds will not be less than the ribbonlength of the link without these folds. Therefore, we conclude that any extra folds introduced into a folded ribbon knot or link in the manner described above will not lead to minimized ribbonlength.

**Remark 5.** While the width of the ribbon for a particular knot diagram may vary, it is important to note that it is constrained by the folds and weaving pattern. Kauffman asserts that there exists a unique flat folded ribbon knot of greatest width for each knot diagram [7]. In subsequent sections, we fix the width of the ribbon in order to analyze the minimal length of ribbon required to form particular types of folded ribbons.

**Definition 10.** The folding angle, $\alpha$, is the angle formed between the fold and the boundary of the ribbon (see Figure 5). By construction of the fold, $\alpha = \angle FAB = \angle FBA$. 

![Figure 5. Folding angle, $\alpha$, of the folded ribbon [2].](image-url)
We observe that in the case of a regular knot diagram, \( 0 < \alpha < \frac{\pi}{2} \). For a knot diagram corresponding to an non-regular projection, \( 0 < \alpha \leq \frac{\pi}{2} \).

Following our construction of a folded ribbon, lines \( e_i \) and \( e_{i+1} \) in Figure 5 represent the knot diagram, while \( a_1, a_2, b_1, \) and \( b_2 \) form the boundary for the segments, \( e_i \) and \( e_{i+1} \), respectively. By construction, \( a_1, e_i, \) and \( a_2 \) are pairwise parallel lines as are \( b_1, e_{i+1}, \) and \( b_2 \). Point \( C \) corresponds to the vertex, \( v_i \), incident to \( e_i \) and \( e_{i+1} \), and is the midpoint of the folded edge \( AB \). Hence, \( |AC| = |CB| \).

We use Figures 5, 6, and 7 to observe the following geometric relations which are useful in describing properties of a folded ribbon.

**Lemma 1.** Given a ribbon of width \( w \) and an acute folding angle \( \alpha \) along fold \( AB \), as shown in Figure 5, \( \angle FAB = \angle ECB = \angle DCA = \angle FBA = \alpha \).

**Proof.** Since \( a_1, e_i, \) and \( a_2 \) are pairwise parallel, \( \angle DCA \) and \( \angle FBA \) are corresponding angles and therefore, \( \alpha = \angle DCA = \angle FBA \). Similarly, since \( b_1, e_{i+1}, \) and \( b_2 \) are pairwise parallel, \( \angle FAB = \angle ECB = \alpha \). Furthermore, we recall that by construction, the fold, \( AB \), bisects \( \angle GAF \) and therefore, \( \angle GAB = \angle FAB = \alpha \). Since \( \angle GAB \) and \( \angle FBA \) are alternate interior angles, \( \angle GAB = \angle FBA \). Hence, \( \angle FAB = \angle FBA = \alpha \) in accordance with our definition of \( \alpha \). \( \square \)

**Corollary 2.** Given a ribbon of width \( w \) and an acute folding angle \( \alpha \) along fold \( AB \), as shown in Figure 6, the knot diagram (lines \( e_i \) and \( e_{i+1} \)) and boundary of the ribbon (\( b_1 \) and \( a_2 \)) form the rhombus \( CDFE \). In addition, the over-/underfold \( AB \), with the knot diagram and boundary of the ribbon, forms three, similar isosceles triangles: \( \triangle ABF \), \( \triangle ACD \) and \( \triangle BCE \). Moreover, \( \triangle AC'D \) and \( \triangle BCE \) are congruent.

**Proof.** Lemma 1 states that \( \angle FAB = \angle ECB = \angle DCA = \angle FBA = \alpha \). This implies \( |AD| = |DC|, |CE| = |EB| \) and \( |AF| = |FB| \). Hence, \( \triangle ABF \), \( \triangle ACD \) and \( \triangle BCE \) are
similar isosceles triangles. By construction, $C$ is the midpoint of $AB$ so, $|AC| = |CB|$. By angle-side-angle, △ACD and △BCE are congruent, isosceles triangles. Lastly, since $|AC| = \frac{|AB|}{2}$ and △ABF ~ △ACD, this implies $|AD| = \frac{|AF|}{2}$. Similarly, $|CB| = \frac{|AB|}{2}$ and △ABF ~ △BCE implies $|EB| = \frac{|FB|}{2}$. Hence, $|CD| = |DF| = |FE| = |EC|$, and CDFE is a rhombus.

Corollary 3. Given a ribbon of width $w$ and an acute folding angle $\alpha$ along fold $AB$, as shown in Figure 7, the isosceles triangle, △ABF, of the over-/underfold contains four, congruent isosceles triangles that each have area equal to $\frac{1}{4}$ the area of △ABF.

Proof. Rhombus CDFE may be split into two, congruent triangles by drawing a diagonal from $D$ to $E$ which bisects $\angle CDF$ and $\angle CEF$. Since $\angle CDF = \angle CEF = 2\alpha$, diagonal $DE$ creates two, congruent, isosceles triangles with base angles equal to $\alpha$ and corresponding sides $|CD| = |DF| = |FE| = |EC|$. Because △ACD and △BCE are also isosceles triangles with base angles equal to $\alpha$ and corresponding sides $|CD| = |EC|$, by angle-angle-side, △ADC ≅ △BEC ≅ △DCE ≅ △DFE. Since each of these four, congruent, isosceles triangles subdivide △ABF, each triangle must have area equal to $\frac{1}{4}$Area(△ABF). □

Having observed the above geometric relations determined by the fold, acute folding angle, and boundary of the ribbon, we make note of specific trigonometric relations that will be useful in calculating ribbonlength in subsequent sections.

Lemma 4. Given a ribbon of width $w$ and an acute folding angle $\alpha$ along fold $AB$, as shown in Figure 8,

1. The length of the fold is $|AB| = \frac{w\sin \alpha}{\sin^2 \alpha}$.
2. The depth of the fold is $|FC| = \frac{w}{\sin \alpha}$.
3. The length of each transversal of the ribbon, FA and FB, is equivalent to the length of the knot diagram inside the fold and is $|AF| = |FB| = |DC| + |CE| = \frac{w}{\sin 2\alpha}$.
Figure 8. Depiction of heights and angles used in calculating lengths of the ribbon [2].

Proof. (1) Dropping a perpendicular, \( AH_1 \), from the corner of the fold at \( A \) to the boundary of the ribbon, \( a_2 \), we create a right triangle, \( \triangle AH_1B \), whose hypotenuse is fold \( AB \). We observe that \( |AH_1| = w \). Therefore, \( |AB| = \frac{w}{\sin \alpha} \).

(2) Having established that \( C \) is the midpoint of fold \( AB \), we know that \( |AC| = |CB| = \frac{|AB|}{2} = \frac{w}{2\sin \alpha} \). Dropping a perpendicular, \( FC \), which bisects \( \angle AFB \) and intersects fold \( AB \) at \( C \), we create another right triangle \( \triangle FCB \) within \( \triangle AFB \). Side \( FC \) of \( \triangle FCB \) represents the depth of the fold. We observe that \( \tan \alpha = \frac{|FC|}{|CB|} \). Therefore, \( |FC| = \frac{w}{2\cos \alpha} \).

(3) Recalling that \( |AF| = |FB| \), \( |AD| = |DC| = \frac{|AF|}{2} \), and \( |CE| = |EB| = \frac{|FB|}{2} \) as determined in Corollary 2 and observing that in right triangle \( \triangle FCB \), \( \cos \alpha = \frac{|CB|}{|FB|} \), we find \( |DC| + |CE| = |AF| = |FB| = \frac{|CB|}{\cos \alpha} = \frac{w}{2\sin \alpha \cos \alpha} = \frac{w}{\sin 2\alpha} \).

\( \square \)

Corollary 5. Given a ribbon of width \( w \) and an acute folding angle \( \alpha = \frac{\pi}{4} \) along fold \( AB \), \( \triangle ABF \) is a right, isosceles triangle so, \( |AF| = |FB| = |DC| + |CE| = w \) and \( |AD| = |DF| = |DC| = |CE| = |EB| = |EF| = \frac{w}{2} \).

Proof. This result follows immediately from Lemma 4 part 3. \( \square \)

Corollary 6. Given a ribbon of width \( w \) and an acute folding angle \( \alpha \) along fold \( AB \), as shown in Figure 8, let \( AI \) be a perpendicular dropped from the corner of the fold at \( A \) to the knot diagram. Then the length of the knot diagram, \( |CI| \), between the vertex, \( v_i = C \), associated with fold \( AB \) and the perpendicular \( AI \), is \( \frac{w}{2} \tan(\frac{\pi}{2} - \alpha) \).

Proof. We observe that \( \angle IAC = \frac{\pi}{2} - \alpha \) and \( \tan(\frac{\pi}{2} - \alpha) = \frac{|CI|}{|AI|} \). Since \( |AI| \) is the distance between the boundary of the ribbon \( a_1 \) and the knot diagram \( e_i \), by construction of the ribbon (see Figure 3), \( |AI| = \frac{w}{2} \). Therefore, \( |CI| = |AI| \tan(\frac{\pi}{2} - \alpha) = \frac{w}{2} \tan(\frac{\pi}{2} - \alpha) \). \( \square \)
Remark 6. We note that when $\alpha = \frac{\pi}{2}$, the ribbon is folded over itself. By Lemma 4:

$$|AB| = \frac{w}{\sin \alpha} = \frac{w}{\sin \frac{\pi}{2}} = w$$

4. Structure of Folded Ribbons

4.1. Linking Number. Linking number is used to determine the degree to which components of a link are joined together. The disjoint knots which form links are called components. The intersections of the components in an oriented link diagram determine crossing value which in turn determines linking number.

Definition 11. A positive crossing has the form show on the left of Figure 9. A negative crossing has the form show on the right of Figure 9.

![Figure 9](image)

**Figure 9.** The crossing on the right has value +1, the crossing on the left has value -1.

Definition 12 ([5]). Given an oriented link diagram $D$, with components $C_1, ..., C_m$, the linking number of $D$, $Lk(D)$, is one-half the sum of the sign of the crossings of $C_i$ with $C_j$.

![Figure 10](image)

**Figure 10.** On the left the four-stick unknot has two boundaries, while the three-stick unknot on the right has one boundary.

Linking number provides information about the number of twists that must be introduced into the ribbon in order to construct a given folded ribbon. Kauffman examined properties of ribbon knots in $\mathbb{R}^3$ related to linking number [8]. Here, we develop measures of twisting related to linking number with respect to folded ribbons.

In order to calculate the linking number of a folded ribbon, the knot diagram and a boundary from the model of the folded ribbon are considered as components of a link. For a folded ribbon topologically equivalent to a Mobius strip, there is only one boundary. For
folded ribbons topologically equivalent to the annulus, we can arbitrarily choose one of the
two boundaries since all components have the same orientation and contribute the same
information about the twisting properties of the folded ribbon. The two possible variations
for link diagrams are pictured in Figure 10.

![Figure 10](image10.png)

**Figure 11.** On the left, the boundary and knot diagram form a negative
crossing; on the right, the boundary and knot diagram form a positive
crossing.

**Definition 13.** The linking number of a folded ribbon, $Lk(D,F)$, with oriented link dia-
gram $D$ and folding information $F$, is one-half the sum of the crossing values at intersections
of the knot diagram and boundary.

Figure 12 contains an example of three folded ribbons with equivalent knot diagrams
but different folding information. To determine the linking number for the leftmost folded
ribbon we observe that there are two positive crossings and two negative crossings of
the knot diagram and boundary. Then the linking number is zero. The middle folded
ribbon has three negative crossings and one positive crossing of the knot diagram and
boundary. Then the linking number is negative one. Finally, with a similar observation, we
determine that the rightmost folded ribbon has linking number negative two. If we change
the orientation of the link diagram, the absolute value of the linking number remains the
same but the sign is reversed, i.e. the second folded ribbon would have linking number
positive one.

![Figure 12](image12.png)

**Figure 12.** Here, the crossing value is indicated at each fold. The leftmost
folded ribbon has linking number zero. The middle folded ribbon has linking
number negative one. The rightmost folded ribbon has linking number
negative two.
Figure 12 demonstrates that multiple folded ribbon knots can be formed from a knot diagram that corresponds to only one three-dimensional knot. Here, linking number distinguishes between five folded ribbons constructed on this knot diagram, since there are two additional cases not pictured where linking number is one and two. Thus, linking number captures information with respect to the properties that folding information has introduced to the knot. For the four-stick unknot, we observe that there is a direct correspondence between the folding information and the linking number. One-half the difference between the number of underfolds and overfolds is equal to the linking number. We formalize this property noting the existence of an analogous property for ribbon knots in $\mathbb{R}^3$.

**Definition 14.** The twist, $T(K, F)$, of a folded ribbon with oriented link diagram $D$ and folding information $F$ is one-half the sum of the crossing values of the boundary curve and knot diagram at folds. The values of crossings established for Definition 12 are again used.

We now consider another set of folded ribbons with equivalent knot diagrams that adds to our understanding of the relationship between linking number and twist.

**Definition 15.** We define the $X$-Unknot, denoted $UX_n$, to be an unknot with linking number $n$ in which the knot diagram has one self-intersection.

The leftmost folded ribbon in Figure 13 is the $X$-Unknot with linking number negative three and twist negative two. The next folded ribbon shows the $X$-Unknot with linking number negative two and twist negative one. The third folded ribbon is the $X$-Unknot with linking number negative one and twist zero. Finally, the rightmost folded ribbon is the $X$-Unknot with linking number zero and with twist negative one.

![Figure 13](image)

**Figure 13.** The crossing values at each fold are given. The leftmost folded ribbon is $UX_{-3}$ and has $T(UX_{-3}) = -2$. The next folded ribbon shows $UX_{-2}$ with $T(UX_{-2}) = -1$. The next folded ribbon is $UX_{-1}$ with $T(UX_{-2}) = -2$. Finally, the rightmost folded ribbon is $UX_0$ with $T(UX_0) = -1$.

Here, the linking number cannot be explained by twist alone. Linking number expresses the degree to which components are joined. For a link with two components, linking...
number is equivalent to the number of times that the two components wind around each other, in this case the boundary and knot diagram. Twist captures the local twisting that contributes to the number of times the components wind around each other. However, here there is also global twisting of the folded ribbon, i.e. self-intersection of the knot diagram. The self-intersections of the knot diagram contribute to the total twisting of the ribbon when considered in $\mathbb{R}^3$. The global twisting of a knot is defined to be the writhe of a knot. Since folding information does not effect writhe, here we apply the standard definition to folded ribbons.

**Definition 16.** The writhe, $W(K)$, of the oriented knot diagram of a given folded ribbon, is the sum of the values of all self-intersections of the knot diagram with respect to values determined for Definition 12.

An expression involving linking number, writhe, and twist has been derived from the phenomenon colloquially known as the belt trick. For ribbons in three-space, this relationship is formalized with the following equality.

**Theorem 7.** Given a ribbon knot, the linking number of the corresponding oriented link diagram is determined by the following equality: $Lk(D) = T(D) + W(K)$.

For folded ribbons we have an analogous equality.

**Theorem 8.** Given a folded ribbon, the linking number of the corresponding oriented link diagram is determined by the following equality: $Lk(D,F) = T(D,F) + W(K)$.

Figure 14 demonstrates the interplay between the twist and writhe. Here, twisting introduced by folding information is exchanged for twisting introduced by writhe. In Figure 14 the two folded ribbons have linking number negative one. However, in one case, global twisting accounts for the linking number while in the other it is the orientation of the folds. In Figure 15 both figures have linking number negative two. Again, global twisting accounts for a portion of the linking number with respect to the $X$-Unknot while in the leftmost figure it is exclusively the orientation of the folds. There are three other similar pairs of $X$-Uncnots and 4-stick unknots with the same linking number. However, there is no 4-stick unknot counterpart to the $X$-Unknot with linking number plus or minus three.

4.2. **Reidemeister moves and defining equivalent folded ribbons.** As defined previously, two knots are said to be equivalent if one can be deformed into the other. Since different knot diagrams can represent the same knot, specific moves have been defined on the knot diagram that allow the equivalence to be demonstrated [5]. Of course, planar isotopies or deformations of the plane containing the knot diagram are permitted since this corresponds to stretching the segments of the knot in $\mathbb{R}^3$. Additionally, the three Reidemeister moves describe more significant allowable modifications to the knot diagram used to demonstrate knot equivalence. Each of these moves is an action performed on some section of the knot diagram. When a Reidemeister move is performed, note that the part of the knot not included in the section is unaffected by the move. (For illustrations of these moves, see [1].)
Figure 14. Both the crossing value at folds and the writhe are shown on the diagrams. Both figures have linking number -1.

Figure 15. Both the crossing value at folds and and the writhe are shown on the diagrams. Here both figures have linking number negative two.

Theorem 9. Two knots are equivalent if and only if their diagrams are related by a finite series of Reidemeister moves and planar isotopies.

We use the existing Reidemeister moves and descriptions of the planar isotopies to identify similar moves for folded ribbons. Note that since folded ribbons have width and contain folding information in addition to the information in the knot diagram, we define different kinds of folded ribbon equivalence. After identifying these types of folded ribbon equivalence, we define moves that correspond to each type of equivalence. To better understand the types of equivalence, note that although we have described the construction of folded ribbon diagrams in $\mathbb{R}^2$, we can consider the ribbons that these diagrams represent in $\mathbb{R}^3$. Also, note that each type of folded ribbon equivalence requires that the knot diagrams of the folded ribbons be equivalent. We begin by defining the most restrictive type of folded ribbon equivalence.

Definition 17. Two folded ribbon diagrams, $D_1$ and $D_2$, are ribbon equivalent if and only if, when considered as ribbons $R_1$ and $R_2$ in $\mathbb{R}^3$, there exists an ambient isotopy of $\mathbb{R}^3$ that takes $R_1$ to $R_2$. 
Definition 18. Two folded ribbon diagrams are topologically equivalent if and only if they have equivalent knot diagrams and, when considered as ribbons in $\mathbb{R}^3$, both ribbons are topologically equivalent to a Mobius strip or both ribbons are topologically equivalent to an annulus.

Definition 19. Two folded ribbon diagrams are knot diagram equivalent if and only if they had equivalent knot diagrams.

We see that these definitions describe a hierarchy of folded ribbon equivalence. Any two folded ribbons which are ribbon equivalent are necessarily also topologically and knot diagram equivalent.

4.2.1. Ribbon equivalence moves. We define each of the moves used to demonstrate ribbon equivalence using existing moves on knot diagrams. First consider the R0 move. Although this move is a planar isotopy and is not one of the three Reidemeiseter moves, some texts label this the R0 move [5].

Definition 20. Two knot diagrams, $D_1$ and $D_2$, are defined to be equivalent by an R0 move if there exists a continuous deformation of the plane that transforms $D_1$ to $D_2$ without changing any crossings.

Similarly, for folded ribbons, we allow shifting of the edges and the vertices attached to those edges as long as the position of the vertices with respect to the boundary of the ribbon is preserved (see Figure 16).

Definition 21. Two knot diagrams, $D_1$ and $D_2$, are equivalent by a ribbon R0 move if, by shifting a vertex of the knot diagram, and the edges attached to that vertex, while preserving the position of the vertices with respect to the boundary of the ribbon, $D_1$ can be transformed to $D_2$.

In this move, as in all of the ribbon equivalence moves, the folding information at each vertex of the knot diagram is preserved. All overfolds remain overfolds and all underfolds remain underfolds. However, note that moving a vertex of the knot diagram in this way usually changes the folding angle at the vertex that was moved and also at the two adjacent vertices.

Next, we describe the $\triangle$-move and define a similar folded ribbon move. When considering polygonal knots, since the 5-stick unknot is surely equivalent to the 4-stick unknot, the $\triangle$-move can be used to add or remove a vertex and edge from the knot as shown in Figure 17.

Definition 22. Let $K$ be a polygonal link in $\mathbb{R}^3$ and $\triangle$ be a triangle such that:

1. $K$ does not meet the interior of $\triangle$,
2. $K$ meets one or two sides of $\partial \triangle$, and
3. the vertices of $K$ in $K \cup \triangle$ are also vertices of $\triangle$.

Then a $\triangle$-move is a move that replaces $K$ with $(K - (K \cap \triangle)) \cup (\partial \triangle - K)$ [6].
Figure 16. Ribbon R0 allows the vertices of the knot diagram to be shifted as long as the position of the vertices with respect to the boundary of the ribbon is preserved.

Figure 17. The $\triangle$-move allows a vertex and an edge to be added to the knot diagram.

Since this does not change the crossings of the knot, the triangle-move is an R0 move. When considering folded ribbons, we must similarly allow vertices and therefore folds to be added and removed from the knot diagram. Given a folded ribbon, two folds must be simultaneously added or removed to ensure that as new vertices and edges are added, the linking number of the knot is preserved. For this reason, we refer to our modification of the polygonal $\triangle$-move as the $\square$-move (see Figure 18). In this move, one of the newly created folds must be an overfold while the other must be an underfold. The second fold undoes the twist introduced by the first fold since the folds have opposite signs. Figure 18 shows only a portion of an arbitrary folded ribbon diagram. Note that only a section of one edge is shown so the square-move is performed just on that section of the edge. Although the folding angles adjacent to the edge will necessarily change as a result of the move, the folds and the folding information at those folds remain unchanged. This move adds or removes two vertices and two edges simultaneously.

Next we define moves on folded ribbons similar to the three Reidemeister moves (R1, R2, and R3). These moves change the relation between the crossings in the knot diagram.

**Definition 23.** The $R1$ move allows a twist to be added or removed from the knot.
Figure 18. The folded ribbon □-move allows sides to be added to and removed from the knot diagram by simultaneously creating an underfold and an overfold.

Definition 24. Ribbon $R1$ adds an additional self-crossing and two folds to the ribbon diagram, but preserves the linking number of the diagram.

As shown in Figure [19], we allow an additional self-crossing to be added to the knot diagram. Because the ribbon has width, when the self-crossing is added to the knot diagram, two additional folds form. To preserve the linking number of the folded ribbon in this move, the new folds must both be either overfolds or underfolds.

Figure 19. Ribbon $R1$ allows self-crossings to be added to the knot diagram in such a way that the linking number of the knot is unchanged.

Note that in Figure [19] in each of the diagrams with a self-crossing, the diagonal edges of the ribbon diagram can be moved so that the folding angle at each of the vertices approaches $\pi/2$. When both of the folding angles are exactly $\pi/2$, each of the previously diagonal segments is vertical and the knot diagram for the ribbon is no longer a regular knot diagram. This position corresponds to the cusp point of the traditional $R1$ move that occurs as the twist is formed. Although this position does not add a twist to the knot diagram, and therefore is not a ribbon equivalence $R1$ move, it is a useful move when considering folded ribbons. We can call this move an accordion fold.

Again, using Figure [19] as a starting point, we define another move. By shifting one of the diagonal segments in Figure [19] past the vertical position while maintaining the location of the other diagonal segment, we obtain the change of direction fold shown in Figure [20]. Like the accordion fold, although this move does not create an additional self-crossing in
the ribbon diagram, we see that it is related to the ribbon equivalence R1 move. This move is also useful when considering folded ribbons.

![Figure 20. The change of direction fold is equivalent to ribbon R1.](image)

**Definition 25.** The $R2$ move allows two crossings to be added or removed from a knot diagram by passing a vertex of the knot over or under an edge of the knot.

This same definition can be used to define *ribbon R2*. Figure 21 shows the ribbon R2 in which a fold of the knot passes under an edge of the knot forming two new crossings. Note that the introduction of these two crossings has no effect on the linking number of the folded ribbon.

![Figure 21. Ribbon R2 creates or removes two crossings from the knot diagram by passing a vertex over or under an edge of the knot.](image)

**Definition 26.** The $R3$ move allows an edge of a knot to move from one side of a crossing to the other side of the crossing.

This move does not affect the number of crossings of the knot but it changes the location of the crossings with respect to one another. Figure 22 shows the *ribbon R3* move. Again note that this move has no effect on the linking number of the folded ribbon.

**Conjecture 1.** Two knots are ribbon equivalent if and only if their diagrams are related by a finite series of ribbon equivalence moves.

Although we have not yet attempted to prove this conjecture, we suspect that the proof may be similar to Reidemeister’s proof for equivalent knots.
4.2.2. Topological Equivalence. Recall that we consider two folded ribbons to be topologically equivalent if they have equivalent knot diagrams and if when considered as ribbons in $\mathbb{R}^3$, the ribbons are topologically equivalent. Thus, given any set of equivalent polygonal knot diagrams, we can form two groups with each group containing topologically equivalent folded ribbons. To demonstrate the topological equivalence of two ribbon diagrams, the ribbon equivalence moves can be used in combination with the topological equivalence move, $R_1^*$, shown in Figure 23.

**Definition 27.** $R_1^*$ changes a fold in the ribbon diagram from an overfold to an underfold or from an underfold to an overfold.

We call this move $R_1^*$ because like $R_1$, $R_1^*$ changes the linking number of the knot. Note that when we consider the ribbon that the diagram represents in $\mathbb{R}^3$, this move physically cuts the ribbon and introduces or removes a full twist before reconnecting it.

**Conjecture 2.** Two knots are topologically equivalent if and only if their diagrams are related by a finite series of ribbon equivalence and $R_1^*$ moves.

4.2.3. Knot Diagram Equivalence. The least restrictive type of equivalence, knot diagram equivalence, defines all folded ribbons with equivalent knot diagrams to be equivalent. Viewing our folded ribbons in $\mathbb{R}^3$, this type of equivalence preserves the structure which defines the knot type but allows twists to be introduced or eliminated from the ribbon so that non-orientable ribbons are equivalent to orientable ribbons. Returning to $\mathbb{R}^2$, ...
to demonstrate this type of equivalence on folded ribbons, we define the knot diagram equivalence move, which we call the ribbon \( \triangle \)-move (illustrated in Figure 24). This move is used together with \( R1^* \) and the ribbon equivalence moves to demonstrate knot diagram equivalence.

**Definition 28.** The *ribbon \( \triangle \)-move* introduces or eliminates a fold from the ribbon diagram with unrestricted folding direction.

We call this move the ribbon \( \triangle \)-move because it adds or removes a vertex and an edge to the ribbon diagram like the triangle move. Considering the ribbon in \( \mathbb{R}^3 \), physically, this move cuts the ribbon and introduces or removes a half twist before reconnecting the ribbon. This move also changes the linking number of the ribbon diagram.

![Figure 24. The \( \triangle \)-move adds or removes a vertex and an edge from the ribbon diagram.](image)

**Conjecture 3.** Two knots are knot diagram equivalent if and only if their diagrams are related by a finite series of ribbon equivalence, \( R1^* \) and \( \triangle \)-moves.

**5. Minimizing Ribbonlength**

**Definition 29.** The *ribbonlength*, \( R(K,F) \), of a flat folded ribbon knot \( K \) with knot diagram \( K \) and folding information \( F \) is

\[
R(K,F) = \frac{\text{length}(K,F)}{\text{width}(K,F)}.
\]

Ribbonlength may be minimized by

1. accounting for folding information and fixing length \( (K,F) \), while maximizing the width \( (K,F) \) or
2. accounting for folding information and fixing width \( (K,F) \), while minimizing the length \( (K,F) \).

We fix width and minimize length while accounting for folding information. While previous studies (Kauffman, Brennan *et al.* minimized ribbonlength in a similar manner, they did not account for folding information. We show that different folding information results in different minimized ribbonlengths of knots with the same knot diagram.
6. MINIMIZING RIBBONLENGTH FOR UNKNOTS

We first attempted to minimize ribbonlengths for \( n \)-stick unknots in order to understand local structure and develop techniques of minimization that would be useful in non-trivial knots.

6.1. MINIMIZING unknots with convex polygonal knot diagrams.

**Lemma 10.** The minimum ribbonlength of an \( n \)-stick unknot whose knot diagram is a symmetric, regular, convex \( n \)-gon is less than or equal to
\[
\frac{n \tan \left( \frac{\pi}{2} - \alpha \right)}{2},
\]
where \( \alpha \) is the folding angle and \( n \geq 4 \in \mathbb{N} \).

**Proof.** The procedure shown in Figure 25 illustrates the minimization process for a 5-stick unknot whose knot diagram is a regular, convex pentagon.

More generally, for any folded ribbon whose knot diagram is a regular, convex \( n \)-gon, where \( n \geq 4 \in \mathbb{N} \), we may simultaneously shrink the length of the knot diagram between each of the \( n \) folds until every fold meets two other folds in a corner on the boundary of the ribbon. During this shrinking process, the number of sides, \( n \), and folding angle, \( \alpha \), remain constant. Since the knot diagram is a convex polygon, the sum of the exterior angles is \( 2\pi \). By Lemma 1, we conclude that the sum of the folding angles is \( \frac{2\pi}{\frac{\pi}{2}} = \pi \). Hence, for a given convex, regular \( n \)-gon, \( \alpha = \frac{\pi}{n} \). As a concrete example, we illustrate the case where \( n = 5 \) in Figure 26. By Corollary 2, we know that the interior angle of the convex polygon, \( \angle DCE \), equals \( \angle AFB \), which is the angle opposite each fold. By Lemma 4, we know that the length of each fold is determined by the width of the ribbon \( w \) and folding angle \( \alpha \). Since the folding angle \( \alpha = \frac{\pi}{n} \) is the same for each fold, we know that each fold has the same length.

A perpendicular, \( I \), dropped down from any corner where two folds meet, bisects the knot diagram and creates two, congruent right triangles, \( \triangle AIG \) and \( \triangle AIC \). By construction of
the folded ribbon, the length of this perpendicular $I$, from the corner to the knot diagram, is $\frac{w}{2}$ where $w$ is the width of the ribbon. The piece of the knot diagram, $|CI|$, determining a side of one of the congruent right triangles, has length $\frac{w}{2} \tan(\frac{\pi}{2} - \alpha)$ as proved in Corollary 6. Since there are two congruent triangles per perpendicular and $n$ corners, the whole length of the knot diagram is $2(\frac{w}{2} \tan(\frac{\pi}{2} - \alpha))n = wn \tan(\frac{\pi}{2} - \alpha)$. And, since ribbonlength is the length of the knot diagram divided by the width of the ribbon, the minimum ribbonlength of a folded ribbon whose knot diagram is a regular, convex $n$-gon is less than or equal to $n \tan(\frac{\pi}{2} - \alpha)$.

A natural question that arises from this procedure is: what happens if the folding angles were allowed to change? We are currently working to determine minimum ribbonlength estimates in the case of non-regular $n$-gon knot diagrams. We observed that in the case where the polygon is non-regular but the interior angles are congruent, we may still apply the shrinking and cutting techniques described above. For example, when the knot diagram is a rectangle, we can minimize two equal sides simultaneously and then minimize the other two equal sides. This process, as depicted in Figure 27, results in a minimized ribbon whose knot diagram is a square and whose minimum ribbonlength is less than or equal to 4. We discovered, however, that this does not work when the $n$ folds have different folding angles. We hypothesize that by maneuvering a flat folded ribbon knot, $K$, with a non-regular $n$-gon knot diagram until it has a rotationally symmetric $n$-gon knot diagram and minimizing ribbonlength in this symmetric position, we will obtain a critical point for the general case. Open questions include whether or not this critical point is a local or global minimum and if there are critical values of ribbonlength unrelated to rotational symmetry. Our conjecture is that the symmetric minimum is in fact a global minimum.
Figure 27. Minimization process for a 4-stick unknot whose knot diagram is a rectangle [2]. The ribbonlength of this 4-stick unknot can be reduced by first shortening $AH$ and $ED$ until fold corner $A$ meets $H$, and corner $D$ meets $E$. The ribbonlength may then be further minimized by shortening $BC$ and $GF$ until fold corner $B$ meets $C$, and corner $G$ meets $F$. This minimization process results in a folded ribbon whose knot diagram is a square.

We note that, in this case, folding information doesn’t play a role in minimizing the ribbonlength because, while we are shrinking the length of the knot diagram, we are maintaining its original shape. This is in contrast to the minimization process described in the next section.

6.2. Minimizing the unknot with self-intersecting knot diagrams. After minimizing the ribbonlength of unknots with convex polygonal knot diagrams, we considered the minimization of the X-unknot. Note that although the two folded ribbons shown in Figure 28 have the same knot diagram, these represent different folded ribbons because the folding information of the upper right fold of each knot is different. In Section 4.2.2, we saw that the change in the direction of this fold results in a change in the linking number of the folded ribbon. While the linking number of the folded ribbon on the left is zero, the linking number of the folded ribbon on the right is two. Using a combination of ribbon equivalence moves, including an accordion fold, we see that the unknot on the left can be minimized to ribbonlength zero.

There are many possible methods which could be used to find an upper bound on the minimum ribbon length of the unknot on the right in Figure 28. Applying the cutting and shrinking method, we can shorten the top and bottom segments until the corner folds touch. As shown in Figure 29, we shorten $AB$ and $CD$ simultaneously until $|AB| = |CD| = 0$. Since folding along a fold is not permitted, it is not possible to further reduce the length of the top and bottom knot diagram segments. Note that as we apply this method, the folding angles must change to allow the ribbon to be reconnected. Next, the diagonal segments of the knot diagram can be shortened until $F$ and $H$, and $E$ and $G$ meet. To find the ribbonlength of the folded ribbon in the final diagram, note that $\angle EAF = \frac{\pi}{3}$ and $\angle EAP = \frac{\pi}{6}$. Using the right triangle, $\triangle EAP$, with $|AP| = w$, we find $|EP| = \frac{w}{\sqrt{3}}$ and
\[ |EF| = \frac{2w}{\sqrt{3}}. \] Since the upper and lower knot diagram segments are midsegments of \( \triangle AFE \) and \( \triangle EFC \) respectively, each of these segments has length \( \frac{w}{\sqrt{3}} \). To find the length of each diagonal segment note that quadrilateral \( AFCE \) is a rhombus and that the diagonal segments connect the midpoints of the opposite sides of the rhombus. Thus the length of each red diagonal segment is equal to \( |AE| = \frac{2w}{\sqrt{3}} \). Adding these lengths and dividing by \( w \), we find that the minimum ribbonlength of the flat folded ribbon knot with one knot diagram crossing is less than or equal to \( 2\sqrt{3} \).

Figure 29. The ribbonlength of this X-unknot can be reduced by first shortening \( AB \) and \( CD \) and then shortening the diagonal segments until \( E \) and \( G \), and \( F \) and \( H \) meet.

We now find a smaller ribbonlength using ribbon equivalence moves, as shown in Figure 30. Returning to the original knot shown on the right side in Figure 28, rather than shortening the top and bottom segments, move the upper right fold until the fold coincides
with the knot diagram crossing as shown in the center diagram of Figure 30. Next, move
the lower right fold until it also coincides with the knot diagram crossing (shown in the
diagram on the right). Shortening the ribbon segments extending beyond the central tri-
angle until the upper and lower left folds meet the sides of the triangle, the minimization
results in four congruent isosceles right triangles with congruent sides of length $w$. Using
these triangles, it is simple to compute the ribbon length of this ribbon diagram. The knot
diagram is the mid-segment of each triangle passing from one of the congruent sides to the
hypotenuse. Thus, using this minimization process, we have that the minimum ribbon-
length of this flat folded ribbon knot is less than or equal to $4\left(\frac{w^2}{2}\right)/w = 2$. Note that the
ribbon equivalence moves could be preformed on the reduced shape rather than returning
to the original folded ribbon diagram. However, the moves can be illustrated more clearly
using the original diagram.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure30.png}
\caption{The ribbon length of the one-twist unknot with linking number 2 can be further reduced using ribbon equivalence moves.}
\end{figure}

We notice that the ribbon shown in the right diagram of Figure 30 does not have a
regular knot diagram. Other mathematicians have disallowed this type of diagram in their
minimization. However, we note that a small perturbation would result in a regular knot
diagram with ribbon length just over 2.

7. Two-Polygon Links and Knots

In Section 6, a variety of folded ribbons are minimized using sequences of ribbon Rei-
demeister moves. In this section we return to Kauffman’s construction of the trefoil on
a regular polygon. Brennan et al. found minimal ribbon length for other folded torus
knots realized on regular polygons. We demonstrate that the family of $(p, 2)$ folded torus
knots can be further minimized by introducing writhe and applying a sequence of ribbon
Reidemeister moves.

**Definition 30.** *Torus knots*, $T_{(p,q)}$, are knots which are embedded on a torus. Here, $p$ is the
number of times around the meridian and $q$ is the number of times around the longitude.

For many torus knots, folded ribbons can be constructed with the same $p, q$ specifications
that characterize a torus knot in three-space, as demonstrated by Kauffman with the
example of the trefoil. The folded ribbon counterpart to the original torus knot is denoted with the same notation.

**Definition 31.** A *folded torus knot* $T(p,q)$ is a folded ribbon with knot diagram equivalent to a $T(p,q)$ torus knot. The values of $p$ and $q$ are defined as in Definition 30.

After Kauffman introduced the question of ribbonlength for folded ribbons in general, and the $T(3,2)$ as one example, Brennan et al. expanded on this work by considering families of folded torus knots. They used general properties of folded torus knots to minimize ribbonlength. Brennan et al. grouped folded torus knots into families using a linear relationship between $p$ and $q$ that also corresponds to geometric properties of the folded torus knots. For example the trefoil knot is in the $T(q+1,q)$ family of knots. They found that all folded ribbons in the $T(q+1,q)$ family have ribbonlength of $R < 2q + \cot(\frac{\pi}{2q+1})$.

Brennan et al. also used the work of DeMaranville who demonstrated that folded torus knots are realized on regular polygons. Brennan et al. expanded upon this work by determining formulas for the number of sides the regular polygons will have. For example, a $T(q+1,q)$ is realized on a regular polygon with $2q + 1$ sides. They went on to prove a theorem related to the regular polygons realized with folded ribbon constructions.

**Theorem 11.** All regular polygons of at least seven sides can be realized with torus folded ribbons.

![Diagram](https://via.placeholder.com/150)

**Figure 31.** A $T(9,2)$ folded ribbon realized as a regular nonagon and the ribbon required to construct the folded ribbon.

Ribbonlength, for a family of torus knots, is determined by using properties of the regular polygon associated with each family. As shown in Figure 31, the ribbon forms regular trapezoids when the folded torus knot is constructed on the regular polygon. The folding length determines three of sides of the trapezoids while a chord of the polygon determines the length of the base. Formulas for ribbonlength can be understood with respect to the dimensions of these trapezoids, which in turn are determined by the interior angles of the polygon. The interior angles of the trapezoid are the folding angle and the interior angle of the regular polygon on which the folded ribbon is realized. Another family which Brennan et al. studied was the $T(p,2)$ family of folded torus knots. These folded ribbons can be...
constructed such that the boundary of the folded ribbon coincides with polygons with $p$ sides. When the folded torus knots are realized as such, they have ribbonlength $p\cot(\pi/p)$.

7.1. **Two-Polygon Knots and Links.** $T_{(p,2)}$ folded torus knots can be transformed by introducing writhe and applying ribbon Reidemeister moves that allow us to realize these folded torus knots on two connected polygons. Given two connected polygons we wish to realize with folded ribbons, the construction of a folded torus link is more straightforward than the construction of a folded torus knot. Hence, we consider the link and this construction will be used in demonstrating how a folded torus knot on this figure is made.

**Definition 32.** A *Two-Polygon link*, $T\!L_{2(p,2)}$ is a folded ribbon link with a boundary that coincides with the edges of two symmetric $p$-gons joined at one edge.

The Two-Polygon link is realized with two long transversal segments that join regions of two $T_{(p,2)}$ knots which are relatively unchanged. Two other portions of the link consist of short segments that spiral around the transversal segment. We note that a $T_{(p,2)}$ torus knot and the corresponding folded ribbon can only be constructed when $p$ is odd. However, here we can construct $T\!L_{2(p,2)}$ for $p$ even or odd.

**Definition 33.** A *spiral fold* is a series of folds that result in a knot diagram equivalent to a $T_{(p,2)}$ flat folded ribbon knot, but in which one of the polygonal segments remains straight while other polygonal segments wrap around the transversal piece. This is depicted in Figure 32.

The spiral fold is also necessary for the construction of a Two-Polygon knot. However, here, this configuration is present on only one region of the folded ribbon while the fold that corresponds to Definition 24 realized as a change of direction fold is necessary for the other transversal area.

**Definition 34.** A *Two-Polygon knot*, $T_2(p,2)$, is a folded ribbon knot with a boundary that coincides with the edges of two symmetric polygons joined at one edge.
Using the formulas which Brennan et al. derived, it is possible to find the minimum ribbonlength of Two-Polygon knots and links. To determine the ribbonlength, we first observe that the segments that have been added can be divided up into regular trapezoids and parallelograms with measurements that correspond to trapezoids in the original $T_{(p,2)}$ folded ribbon and the folding length. There are two cases to consider when finding the ribbonlength of a Two-Polygon link. A Two-Polygon link where $p$ is even, has two different components. For $p$ odd, the components are symmetric.

First, we begin with the symmetric components of Two-Polygon links with $p$ odd. There are five regular trapezoids that make up the component of the link, and this number depends on $p$. By comparing this number with the same measure in other sizes of Two-Polygon links, we find this to be equal to $p - 3$. Next, the link includes a transversal piece which consists of two regular trapezoids. Finally, there are three additional ribbon segments in the spiraling fold that are parallelograms with side length $l$.

Brennan et al. found that the length of $l = \frac{w}{\sin\left(\frac{2\pi}{p}\right)}$. The base of the trapezoid has length $l(1 + \cos(\frac{2\pi}{p}))$. The ribbonlength of the trapezoid, $T_R$ equals half the sum of the folding length and base length divided by the width. The ribbonlength of the parallelograms, $P_R$ equals the folding length divided by the width. So, ribbonlength for the symmetric components of Two-Polygon links with $p$ odd is:

\[ T_R = \frac{1}{2} \left( \frac{l + l(1 + \cos(\frac{2\pi}{p}))}{w} \right) \]

\[ P_R = \frac{l}{w} \]
Symmetric Components:

\[
(p - 3)T_R + 2T_R + 3P_R
\]

\[
\frac{(p-3)}{2w} (b + l) + \frac{2}{w} (b + l) + \frac{3}{w} l
\]

\[
\frac{(p-1)}{2w} b + \frac{(p-1)}{(2w)} l + \frac{3}{w} l
\]

\[
\frac{(p-1)}{2w} b + \frac{(p+5)}{(2w)} l
\]

\[
\frac{(p-1)}{2w} \frac{w}{\sin\left(\frac{2\pi}{p}\right)} \left(1 + \cos\left(\frac{2\pi}{p}\right)\right) + \frac{(p+5)}{2w} \frac{w}{\sin(2\pi/p)}
\]

\[
\frac{(p-1)}{2} \frac{1+\cos\left(\frac{2\pi}{p}\right)}{\sin\left(\frac{2\pi}{p}\right)} + \frac{(p+5)}{2\sin(2\pi/p)}
\]

Actual values that are determined by these formulas appear in Table 1. Next we find the general formulas for the components of a Two-Polygon link where \( p \) is odd. The Spiral component contains \( p - 3 \) regular trapezoids plus the six regular parallelograms. The Transversals component contains \( p - 3 \) regular trapezoids plus the two transversal segments.

Figure 34. This figure depicts the equality between the transversal segment and two of the original parallelograms. The other new segments are similarly identified with existing measurements related to the original parallelogram.

Spiral Component Ribbonlength:

\[
(p - 3)T_R + 6P_R
\]

\[
\frac{(p-3)}{2w} (b + l) + \frac{6}{w} l
\]

\[
\frac{(p-2)}{2w} (b + l) + \frac{6}{w} l
\]

\[
\frac{(p-2)}{2w} b + \frac{(p+10)}{2w} l
\]

\[
\frac{p-2}{2w} \left(1 + 2\cos\left(\frac{2\pi}{p}\right)\right) \frac{w}{\sin(2\pi/p)} + \frac{w(p+10)}{2w\sin(2\pi/p)}
\]

\[
\frac{p-2}{2} \frac{1+2\cos(2\pi/p)}{\sin(2\pi/p)} + \frac{p+10}{2\sin(2\pi/p)}
\]
Transversals Component Ribbonlength:

\[
(p - 3)T_R + 4T_R \\
\frac{(p-3)}{2w}(b + l) + \frac{4}{2w}(b + l) \\
\frac{(p+1)}{2w}(b) + \frac{(p+1)}{2w}(l) \\
\frac{(p+1)}{2w} \left(1 + 2\cos\left(\frac{2\pi}{p}\right)\right) \frac{w}{\sin(2\pi/p)} + \frac{(p+1)}{2w} \left(\frac{w}{\sin(2\pi/p)}\right) \\
\frac{2}{2} \frac{(1+2\cos(2\pi/p))}{\sin(2\pi/p)} + \frac{(p+1)}{2}\frac{l}{\sin(2\pi/p)}
\]

Now to find the general formula for Two-Polygon knots, we follow a similar procedure as above. There will be \(2(p - 3)\) regular trapezoid segments needed to make the knot as well as two regular trapezoids and three parallelograms for the side of the knot resembling the link. The two transversal segments which have change of direction folds correspond to two regular trapezoid and two parallelograms.

\[
\text{Two-Polygon Knot Ribbonlength:}\ \ \\
2(p - 3)T_R + 4T_R + 5P_R \\
\frac{2(p-3)}{2w}(b + l) + \frac{4}{2w}(b + l) + \frac{5}{w}l \\
\frac{(2p-1)}{2w}(l) + \frac{(2p+1)}{2w}(l) + 5l \\
\frac{(2p-1)}{2w}(l) + \frac{(2p+6)}{2w}(l) \\
\frac{2}{2} \frac{(1+2\cos(2\pi/p))}{\sin(2\pi/p)} + \frac{(2p+6)}{2\sin(2\pi/p)}
\]

7.2. **Minimizing using Two-Polygon Knots.** The knot diagram of a Two-Polygon knot is equivalent to the knot diagram of a \(T_{(2,p)}\) folded ribbon. The Two-Polygon knot

---

**Figure 35.** The leftmost figure is a Two-Polygon knot constructed on two hexagons. Next is the change of direction fold. This is contrasted with the original spiral fold. Finally, the segment containing the change of direction is depicted when unfolded.
represents a version of the same folded ribbon in which ribbonlength has been further minimized. Figure 36 depicts the original $T_{(2,p)}$ and the folded torus knot in a minimized Two-Polygon knot configuration. The introduction of writhe reduces the length of the regular trapezoids which form the folded torus knot.

![Figure 36](image_url)

**Figure 36.** Above, the introduction of writhe allows minimization of ribbonlength. Below, the significant reduction in the length of the trapezoids is depicted.

Brennan *et. al.* demonstrated that all regular polygons with at least seven sides and $p$ odd, can be realized with folded ribbon constructions of $T_{(2,p)}$. We conjecture that all pairs of symmetric regular $p$-gons joined at one side with $p$ greater than or equal to six, can be realized with folded ribbon constructions of $T_{2(2,p)}$.

**Conjecture 4.** All pairs of symmetric regular polygons with six or more sides can be realized with folded ribbon constructions of folded torus knots, realized as Two-Polygon links and knots.

8. **Further Minimizing (p, 2) Torus knots**

To further minimize $(p, 2)$ torus knots, rather than continuing to perform Reidemeister moves on the Two-Polygon configurations, we return to the $(p, 2)$ torus knot diagram. All $(p, 2)$ torus knots can be represented with a diagram similar to the diagrams for the $(2, 3)$ and $(2, 5)$ torus knots shown in Figure 37. Our construction of the $(p, 2)$ torus folded
ribbons is best understood by considering the knot diagrams shown. The folded ribbon we construct has a non-regular knot diagram. By mentally rotating the knots represented by the knot diagrams in Figure 37 so that the crossings coincide, we can begin to envision the folded ribbon that results from our construction.

To construct a folded ribbon (2,3) torus knot (see Figure 38):

1. Begin by placing a ribbon strand in each hand, with the strand in the left hand passing over the strand in the right hand. Call the positions of the ends of the strands northeast, southeast, southwest, and northwest.
2. To begin constructing the knot, take the southeast end and bring it to the northwest corner, creating a fold.
3. Then bring the southwest end to the northeast corner.
4. Next take the upper strand from the northwest corner and fold it to the southeast corner.
5. Now fold the upper northeast strand to the southwest corner.
6. To complete the knot, connect the two eastern ends and the two western ends.

To generalize the construction to form other (2,p) torus knots, simply repeat steps 2 through 5. To compare this folded ribbon trefoil to Kauffman’s construction, note that this configuration of the folded ribbon trefoil is only knot diagram equivalent to Kauffman’s construction. This construction of the trefoil is topologically equivalent to an annulus while Kauffman’s construction is topologically equivalent to a Mobius strip. Also, note that our construction contains three additional folds.

Returning to the folded ribbon trefoil we have constructed, we shorten the connected strands forming four right isosceles triangles with congruent sides of length $w$ on top of 4 squares also with side length $w$. The knot diagram of the folded ribbon knot passes through each triangle and each square once. In each triangle, the knot diagram is a mid-segment of the triangle connecting the midpoint of the hypotenuse to the midpoint of another side.
Since this mid-segment is parallel to one of the congruent sides, the length of the mid-segment is $\frac{w}{2}$. In each square, the knot diagram crosses from the midpoint of one side of the square to the midpoint of the opposite side. Since the square has side length $w$, this section of the knot diagram has length $w$. Combining these pieces, we see that the ribbonlength of this $(2,3)$ torus knot is $\frac{4(\frac{w}{2}) + 4(w)}{w} = 6$.

**Lemma 12.** The minimum ribbonlength of a $(2, p)$ torus knot is less than or equal to $2p$.

Each repetition of steps 2-5 adds four squares to the ribbon diagram, so in general, the ribbonlength of a $(2, p)$ torus knot constructed in this way is $2 + 2(p - 1) = 2p$.

9. **Conclusion**

To summarize our work, we defined Reidemeister moves for flat folded ribbon knots to create a hierarchy of knot equivalence. In addition, we explored how these moves may affect bounds on minimal ribbonlength, and investigated the impact global versus local twisting has on ribbonlength minimization. We determined upper bounds on minimum ribbonlength for a variety of knot types. In the future, we hope to show that the ribbon Reidemeister moves we defined are necessary and sufficient to determine folded ribbon equivalence. We would also like to confirm our conjecture that the minimum ribbonlength for an unknot whose knot diagram is a non-regular polygon is determined by the minimum...
ribbonlength for a symmetric polygon. Additionally, we would like to prove the conjecture that $T_{2(p,2)}$ can be realized for all $p$ greater than six.

10. Acknowledgments

We would like to thank our advisor, Dr. Elizabeth Denne, for her enthusiasm and insight and acknowledge the work done by Shivani Aryal and Shorena Kalandarishvili also under her advisement. In addition, we would like to thank the Center for Women in Mathematics at Smith College. This research was funded by an NSF grant.

Appendix A. Minimizing a folded ribbon 5-stick unknot

Without ribbon Reidemeister moves:

\[ R_{nm} \leq 5 \tan 54 \approx 6.88 \]

Using ribbon equivalence moves:

\[ R_{re} \leq 3\sqrt{3} \approx 5.20 \]

Using topological equivalence moves:

\[ R_{te} \leq \sqrt{3} \approx 1.73 \]

Using knot diagram equivalence moves:
Appendix B. Ribbonlength for Two-Polygon Links and Knots

Table 1. Ribbonlength Values

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References