Knot Theory and the Alexander Polynomial

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## Contents

1 Introduction 1  

2 Defining Knots and Links 3  
  2.1 Knot Diagrams and Knot Equivalence 3  
  2.2 Links, Orientation, and Connected Sum 8  

3 Seifert Surfaces and Knot Genus 12  
  3.1 Seifert Surfaces 12  
  3.2 Surgery 14  
  3.3 Knot Genus and Factorization 16  
  3.4 Linking number 17  
  3.5 Homology 19  
  3.6 The Seifert Matrix 21  
  3.7 The Alexander Polynomial 27  

4 Resolving Trees 31  
  4.1 Resolving Trees and the Conway Polynomial 31  
  4.2 The Alexander Polynomial 34  

5 Algebraic and Topological Tools 36  
  5.1 Free Groups and Quotients 36  
  5.2 The Fundamental Group 40
# List of Figures

2.1 Two knot diagrams ............................................................. 4
2.2 Nonregular projections ......................................................... 5
2.3 The Reidemeister moves ....................................................... 7
2.4 A wild knot ................................................................. 7
2.5 Knots vs. links ............................................................ 8
2.6 An oriented trefoil knot ....................................................... 9
2.7 The connected sum of two figure eight knots ......................... 10
2.8 The square knot vs. the granny knot .................................... 10

3.1 Seifert circles for the figure eight knot .............................. 13
3.2 Seifert circles to Seifert surface ...................................... 14
3.3 Tubing and compressing .................................................. 15
3.4 Signed crossings used to compute linking numbers ............... 18
3.5 Two links ................................................................. 19
3.6 Seifert graph for the figure eight knot ............................... 20
3.7 Cycles in the Seifert graph corresponding to basis loops for the first homology group of the Seifert surface .................................................. 21
3.8 Links used to compute the Seifert matrix ............................ 25
3.9 Seifert circles and Seifert surface for the trefoil knot ............ 28
3.10 Cycles in the Seifert graph and basis loops for the trefoil ...... 29
3.11 Basis loops for the first homology group of the trefoil .......... 29
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Signed crossings used in resolving trees</td>
<td>32</td>
</tr>
<tr>
<td>4.2</td>
<td>A resolving tree</td>
<td>33</td>
</tr>
<tr>
<td>5.1</td>
<td>Loops in a topological space</td>
<td>43</td>
</tr>
<tr>
<td>6.1</td>
<td>Labeled crossing for the Dehn presentation</td>
<td>50</td>
</tr>
<tr>
<td>6.2</td>
<td>Knot diagram labeled for the Dehn Presentation</td>
<td>51</td>
</tr>
<tr>
<td>6.3</td>
<td>Crossings used to obtain relations for the Wirtinger presentation</td>
<td>52</td>
</tr>
<tr>
<td>6.4</td>
<td>Knot diagram with crossings and strands labeled for the Wirtinger presentation</td>
<td>53</td>
</tr>
<tr>
<td>6.5</td>
<td>The trefoil knot on the z = 1 plane</td>
<td>55</td>
</tr>
<tr>
<td>6.6</td>
<td>The space (X_1) used to compute the fundamental group of the knot complement</td>
<td>56</td>
</tr>
<tr>
<td>6.7</td>
<td>Loops in (X_0) give the relations of the Wirtinger presentation</td>
<td>57</td>
</tr>
<tr>
<td>7.1</td>
<td>Relations from the Wirtinger presentation</td>
<td>72</td>
</tr>
<tr>
<td>7.2</td>
<td>An unusual unknot diagram</td>
<td>74</td>
</tr>
<tr>
<td>7.3</td>
<td>Knot diagram for the trefoil knot</td>
<td>75</td>
</tr>
<tr>
<td>7.4</td>
<td>Knot diagram for the figure eight knot</td>
<td>77</td>
</tr>
<tr>
<td>7.5</td>
<td>Knot diagram for the turk’s head knot</td>
<td>79</td>
</tr>
<tr>
<td>A.1</td>
<td>Subdivision of a loop</td>
<td>86</td>
</tr>
<tr>
<td>A.2</td>
<td>Subdivision of a homotopy</td>
<td>88</td>
</tr>
<tr>
<td>A.3</td>
<td>The (n)-leafed rose</td>
<td>90</td>
</tr>
<tr>
<td>A.4</td>
<td>Subspaces (X_1) and (X_0) for the (n)-leafed rose.</td>
<td>90</td>
</tr>
<tr>
<td>A.5</td>
<td>The (T_{2,3})-torus knot</td>
<td>92</td>
</tr>
<tr>
<td>A.6</td>
<td>Another view of the (T_{2,3})-torus knot</td>
<td>93</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

There is not much about knots in their everyday context that strikes us as being particularly mathematical. Even to those who have studied some topology it is not immediate what sort of questions arise in a mathematical study of knots. Still, there is a wealth of mathematics devoted to knot theory. One of the main goals of knot theory is to differentiate between distinct knots. To even state this goal clearly requires some developed mathematics. When are two knots the same and when are they different? For example a knot in string may be loose or tight, does this effect the type of knot? If one strand of the knot is moved slightly to one side is it still the ‘same’ knot? Intuitively and mathematically, these knots are the same. Knot theorists seek to formalize the idea that it is the method of tying a knot that distinguishes it from other knots. Knots that are the same (in a sense that we define rigorously later on) belong to the same knot type, and each knot type can have many different representative knots, and may look quite different.

Of course this study is further complicated by our need to represent knots two dimensionally. One particular representative knot will appear different from each angle it’s viewed, this extends the process not only to sorting which knots belong to the same knot type, but also which knot projections belong to the same knot.

To study this we employ topological methods. By exploring both the knot, and the space containing the knot, we can formalize relationships between similar knots. These
relationships come in the form of topological invariants. Invariants can be numbers, polynomials or algebraic structures such as groups to name some. Invariants are properties such that two knots of the same type give the same value. Thus two knots of different knot types will produce distinct numbers, polynomials, or groups through a string of topological or algebraic manipulations. Thus two knots are conclusively distinct if they produce different invariants. Note that the converse is not necessarily true, as distinct knots may produce the same knot invariant.

This paper is a study of a particular knot invariant — the Alexander polynomial. The Alexander polynomial is a knot invariant that can be obtained in many ways. Here we examine three methods of obtaining the Alexander polynomial, first from geometric techniques, second by examining crossings in knot diagrams, and third, through the fundamental group of the knot complement. These three methods span many areas of knot theory and allow us to survey the subject. In addition, the tools these methods employ include group theory, ring theory, genus, surgery, topology, fundamental group and linear algebra, and thus a large portion of this paper is devoted to mastering these tools.

The first necessary step, and certainly not a trivial one, is to define a knot.
Chapter 2

Defining Knots and Links

There are several ways of defining knots, with varying degrees of formality. With a concept as familiar as a knot, it is tempting to define them in a more colloquial way. For example, we might define a knot in mathematics based on a knot in everyday life. In this case a knot would be much like a knot in string, with the exception that in mathematics an knot has its ends joined together (this way it cannot be ‘untied’) and the ‘string’ has no thickness. However, a more mathematically applicable definition will require some more formal terminology.

**Definition 1.** Let $X$ and $Y$ be topological spaces and let $f : X \to Y$ be an injective function. Then $f$ is an embedding of $X$ in $Y$ if $X$ is homeomorphic to $f(X)$.

**Definition 2.** A knot is an embedding of $S^1$ in $\mathbb{R}^3$.

Given a definition of a knot we may now concentrate on the representation of knots in knot diagrams, and define knot equivalence.

2.1 Knot Diagrams and Knot Equivalence

We should not forget that a knot is a three dimensional object, however, it is frequently to our advantage to represent them in two dimensions. Given a knot in space, we may pick a vector in $\mathbb{R}^3$ and project the knot onto a plane perpendicular to that vector. This gives
us a ‘flat’ representation of the knot. However, in this process we lose some information. Strands which lie on top of each other are indistinguishable in our projection. To fix this problem, we represent the strand of the knot running below the crossing with a line broken at the crossing. This gives us a knot diagram. A knot diagram for the trefoil knot is given in Figure 2.1.

Figure 2.1: Left is the a knot diagram for the unknot. Right is a knot diagram for the trefoil knot. Note that the broken lines at the crossings indicate that those strands of the knot lie below the rest of the knot.

For the case when more than two strands of the knot meet at a point in the projection,... well, we avoid dealing with this situation. Any projection of a knot where three parts of the knot project to a point is arbitrarily close to one without this problem in the knot projection. There are some other types of projections that we would like to avoid in our discussion of knot diagrams. We would not like two strands of the knot to overlap in our projection. We would also like to eliminate a single strand projecting to a cusp, i.e. a strand going to a point and then turning around along the same path it came from.

**Definition 3.** A knot projection whose only self intersection occurs at double points, and which crosses itself transversely is a regular projection of the knot.

An example of a regular projection of a knot is the knot diagram in Figure 2.1.

**Theorem 1.** Any projection of a knot which is not a regular projection is arbitrarily close to a regular projection.
By the theorem above any projection of a knot is arbitrarily close to a regular projection. Thus without loss of generality we consider exclusively knot diagrams which represent regular projections.

Figure 2.2: Here are three situations not found in regular projections. Left, a strand projects to a cusp. Middle, two strands overlap but do not cross. Right, three strands cross at a point.

To discuss knot diagrams, we need some terminology. In a knot diagram, a section of the knot beginning at the break in the strand at a crossing and continuing until the next break occurs is called an arc. In the neighborhood of a crossing, the strand of the knot at the top of the crossing (represented by a solid line) is called the overcrossing while the strand at the bottom of the crossing (represented by a broken line) is called the undercrossing.

We have yet to define knot equivalences. Given a knot, you might take a strand and add one twist to it, changing the embedding of the knot in space, and even changing the corresponding knot diagram. Still the knot type remains the same. We give the definition of knot equivalence using some high powered topology, and then in terms of knot diagrams.

**Definition 4.** An ambient isotopy is a homotopy (defined in Section 5.2.1) \( H : \mathbb{R}^3 \times [0, 1] \rightarrow \mathbb{R}^3 \) such that \( H(x, 0) \) is the identity and \( H(x, t) \) is a homeomorphism for all \( t \) in \([0,1]\). We denote \( H(x,1) \) by \( h(x) \)

**Definition 5.** Let \( K_1 \) and \( K_2 \) be two knots in \( \mathbb{R}^3 \), then \( K_1 \) is equivalent to \( K_2 \) if \( K_1 \) is ambient isotopic to \( K_2 \). That is, there exists an ambient isotopy \( H \) such that \( h(K_1) = K_2 \).

This concept of knot equivalence allows us to partition knots into classes called knot types. Throughout this paper, when we discuss knots being different, we mean that the
knots belong to different knot types. However, formally a ‘knot’ refers to a particular embedding of $S^1$ in $\mathbb{R}^3$ and two knots in the same knot type are different if they are distinct embeddings. Similarly the notation $[K]$ can refer to a knot type, however we typically abuse notation by denoting both a particular knot and a knot type by $K$. In addition, we label knots using the notation $n_m$ where $n$ is the crossing number of $K$, and $m$ specifies which $n$-crossing knot we are referencing.

For example, the knot type of the simple circle in $\mathbb{R}^3$ is referred to as the unknot or the trivial knot and is denoted $0_1$ as it is the first (and only) zero-crossing knot.

Our second approach to knot equivalence will be in terms of three operations on knot diagrams called Reidemeister moves. As a knot moves in space or changes under ambient isotopy its projection changes by Reidemeister moves. These moves are illustrated in Figure 2.3.

**Theorem 2.** [11] Given knot diagrams for two knots $K_1$ and $K_2$, $K_1$ is equivalent to $K_2$ if and only if the knot diagram for $K_1$ can be deformed to the knot diagram for $K_2$ by a finite sequence of Reidemeister moves.

Currently our definition of knots allows some knots that we would not like to consider in our study. For example, consider the knot pictured in Figure 2.4.

This knot is a valid embedding of the circle in space, however the knot above has infinite knotting. Types of knots that exhibit this type of behavior are called wild knots. In order to restrict our study to more well behaved knots we will restrict our study to tame knots, which we will define shortly. First we introduce polygonal knots.

**Definition 6.** A polygonal knot is a simple closed polygonal curve in $\mathbb{R}^3$.

**Definition 7.** A tame knot is a knot which is ambient isotopic to a polygonal knot.

**Note:** Henceforth, we restrict our discussion to tame knots, and we will take the term ‘knot’ to mean a tame knot.

---

1 The crossing number of a knot is the minimum number of crossings in all regular projections of knots in its knot type.
Figure 2.3: The Reidemeister moves are operations done on knot diagrams to obtain a knot diagram for an equivalent knot. Top is an illustration of the Reidemeister I move. Middle illustrates the Reidemeister II move. Bottom is an illustration of the Reidemeister III move. Note that by performing these moves the knot necessarily goes through a point where it is not in regular position. The Reidemeister I, II, and III move making a strand project to a cusp, two strands overlap, and three strands cross respectively.

Figure 2.4: An example of a wild knot which displays infinite knotting at the point $p$. 
Polygonal knots give us a ‘nice’ way of representing all tame knots and thus allows us to consider polygonal knots exclusively when forming proofs.

2.2 Links, Orientation, and Connected Sum

While our discussion is primarily concerned with knots, it is helpful for us to be somewhat familiar to a similar object, links. These will be important to our discussion of surfaces and resolving trees. Many of the results presented throughout this paper extend to links. For a text with further details on links, we refer the reader to [4].

Definition 8. A link or an n-component link is an embedding of n copies of $S^1$ in $\mathbb{R}^3$. For example, a knot is a 1-component link.

Another useful term to know is a split link. As the name suggests, a split link is a link with multiple components that are not interwoven.

Definition 9. Suppose $L$ is a link of two components, $L_1$ and $L_2$. $L$ is a split link if $L_1$ may be enclosed in a topological sphere that does not intersect $L_2$.

Two knots sitting next to each other in space, but not intertwined is an example of a split link as illustrated in Figure 2.5.

![Knot Link Split Link](image)

Figure 2.5: This figure illustrates different types of links. On the left is a one component link, or a knot. Center is a two component link, with one component a figure eight knot and one component an unknot. Right is an example of a two component split link.

Often when we refer to links, we typically use the term in the most general way. A link can refer to a knot, link or a split link.
It is sometimes to our advantage to consider knots with an orientation. With oriented knots, we consider many of the same problems. Mainly, when are two oriented knots the same and when are they different? In addition, there is a new version of this problem. When is an oriented knot equivalent to the same knot with the opposite orientation? A copy of the same knot with opposite orientation is called its reverse and knots which satisfy this property are called reversible.

Figure 2.6: Here is an oriented trefoil knot, an example of an oriented knot.

Frequently throughout this paper, we will apply orientation to a knot arbitrarily to obtain some information about the oriented knot which could not be obtained from the nonoriented version. However, we will typically discard the orientation after that point, as we are not investigating oriented knots. As we will show in Section 7.5 the Alexander polynomial cannot distinguish oriented knots.

It is to our advantage to separate a knot into two smaller knots which we know more about. Or similarly, to build a larger knot out of two small knots. The process of connecting two knots, \( K_1 \) and \( K_2 \) to form a larger knot, \( K \), illustrated in Figure 2.7, is referred to as taking the connected sum of \( K_1 \) and \( K_2 \), and is denoted \( K = K_1 \# K_2 \). This involves taking two knots, breaking a strand of each knot, and rejoining these strands to the broken strands in the other knot. The reverse process, by which you take a knot and break it into two smaller knots is called factorizing. To factor a knot, one part of the knot must be contained in a topological sphere which intersects the knot in only two points. These are the points of the knot that are broken and rejoined to create two knots.

While it may seem that taking the connected sum of two knots might produce different
knots depending on where you chose to connect them, this is not the case. To see this, imagine shrinking one of the knots, say \( K_1 \) in the connected sum, to an extremely small relative size. This knot could then slide through to any spot on \( K_2 \) without changing the knot type of the connected sum. And so taking the connected sum cannot depend on where the two knots are joined. However, if you are considering two oriented knots, this operation may produce distinct links. For example, the connected sum of two trefoils produces the granny knot, while the connected sum of the trefoil with its reverse gives the square knot. These knots are given in Figure 2.8.

This operation of taking the connected sum of knots, as well as factorizing knots, allows us to define \textit{prime} and \textit{composite} knots.

\textbf{Definition 10.} A knot is \textit{composite} if it is the connected sum of two or more nontrivial
knots. Otherwise it is \textit{prime}.

In the following section we present our first example of a knot invariant.
Chapter 3

Seifert Surfaces and Knot Genus

In this chapter we discuss the relationship between knots and surfaces. Given any knot, there is a collection of orientable surfaces whose boundary is the knot. By analyzing the topology of these surfaces, we can obtain knot invariants such as the Alexander polynomial. Here we discuss a simple way of obtaining orientable surfaces from knot diagrams, and use them to construct the knot invariants: signature, determinant, and Alexander polynomial.

3.1 Seifert Surfaces

To begin with, we will need a more rigorous description of the surfaces we are interested in.

**Definition 11.** A surface $F$ is a *spanning surface* for a knot $K$ if the boundary of $F$, $\partial F$ is ambient isotopic to $K$.

In order to obtain our knot invariant, we must first construct Seifert surfaces obtained by the following construction.

**Construction of Seifert Surfaces**

Given an oriented knot diagram, break each crossing and rejoin each strand to an adjacent strand in a fashion that maintains orientation. The resulting diagram is a collection of
circles called \textit{Seifert circles}. Note that the Seifert circles produced through this method do not depend on the initial choice of orientation. Some circles may be nested and some may be disjoint. Figure ?? shows a diagram of the figure eight knot and its corresponding Seifert circles.

![Figure 3.1: Left: the figure eight knot. Right: Seifert circles obtained from the knot diagram on the left.](image)

To construct a surface from these circles, we must think of each Seifert circle as the boundary of a disc. We then think of nested circles as discs lying at several different levels, with discs corresponding to inner circles lying at higher levels. We then add back in the crossings to connect the discs and complete our orientable surface. We do this by adding a twisted band in the place of the crossing such that the direction of the twist matches the type of crossing in the original knot diagram. The result is an orientable surface. Note that as each disc is an orientable (2-sided) surface and the discs are connected by 2 sided bands, the result necessarily has 2 distinct sides, and is thus orientable. Our construction of the Seifert surface of the figure eight knot is continued in Figure 3.2.

Note that in performing this construction for an arbitrary knot we have proved the following theorem.

\textbf{Theorem 3.} \textit{Every knot bounds an orientable surface.}

We now relate these orientable surfaces via surgery.
Figure 3.2: This illustrates the process of constructing the Seifert surface from the Seifert circles. Each Seifert circle becomes a disc in the surface, connected by twisted bands. In addition, the boundary of the Seifert surface is the figure eight knot. The image on the right is a side view of the constructed Seifert surface, courtesy of Seifertview [15].

3.2 Surgery

As we have defined spanning surfaces in terms of the knot on their boundary, it is natural for two spanning surfaces to be equivalent if their boundaries are the same knot type. Thus we next discuss transforming surfaces into other surfaces to allow us to more thoroughly explore the question of which surfaces are equivalent.

To transform a spanning surface into another while leaving the boundary knot unchanged we use two operations, described below:

Tubing

Let $F$ be a surface and let $\lambda_0$ and $\lambda_1$ be loops on $F$ which do not intersect $\partial F$. Then a new surface can be obtained by deleting the discs with boundary $\lambda_0$ and $\lambda_1$ and identifying the loops $\lambda_0$ and $\lambda_1$. This new surface is said to be equivalent to $F$ by tubing.

Note that in this construction we have created two new boundary components, $\lambda_0$ and $\lambda_1$ and then removed these boundary components by identifying the loops. Thus the tubing operation ultimately leaves the boundary unchanged. This construction can be more easily visualized by removing the discs with boundary $\lambda_0$ and $\lambda_1$ and adding in a ‘tube’ between these loops.
Figure 3.3: This figure illustrates the tubing and compressing operations. Going from left to right a surface is transformed by tubing. Going from right to left a surface is transformed by compressing. The loop $\lambda$ in the rightmost picture is an essential loop.

**Compressing**

Compressing is the opposite operation, by which we remove a ‘tube’ in our surface. While this idea may be intuitive, it is more difficult to define than tubing. For this we use the following definition.

**Definition 12.** A loop $\lambda$ in a surface $F$ is *essential* if it does not bound a disc in $F$.

Let $\lambda$ be an essential loop in the surface $F$. This is illustrated in the rightmost picture of Figure 3.3. Let $\lambda_0$ and $\lambda_1$ be boundary loops formed by cutting out an annulus around $\lambda$. Then joining a two discs to $F$ with boundary $\lambda_0$ and $\lambda_1$ respectively, gives a new surface equivalent to $F$ by compressing.

Again, we created two new boundary components by deleting the annulus, and then we attached discs, removing these new boundary loops. Thus ultimately the boundary of the surface remains unchanged through compression. These operations are illustrated in Figure 3.3. It is also important to note that an orientable surface transformed by a tubing or compressing operation is still orientable.

These operations facilitate the computation of knot invariants as they allow us to move between surfaces with the same boundary knot. Any two orientable surfaces with the same boundary can be related by a finite sequence of tubing, and compressing operations and
ambient isotopy. This idea will be used extensively to prove the invariance of signature, determinant and the Alexander polynomial in the following sections.

3.3 Knot Genus and Factorization

Applying the study of surfaces to knots gives us many new tools to apply. A familiar invariant of surfaces is the genus, which measures the number of holes in a surface. Make this a knot invariant as follows.

**Definition 13.** The *genus* of a knot $K$ is the minimum genus of all orientable spanning surfaces with boundary $K$.

While this definition is easy to comprehend, in practice it is difficult to find the genus of a knot. As one cannot actually examine all spanning surfaces of a knot, the genus of a knot is determined by finding a lower bound (i.e. proving that no spanning surface of the knot can have a genus lower that some $n$), and then displaying a spanning surface which has genus $n$.

While the statement of the following theorem may seem simple, the result has some important applications. For example, this result shows that the connected sum of two nontrivial knots cannot be the unknot.

**Theorem 4.** Let $K_1$ and $K_2$ be knots and $g(K_1)$, $g(K_2)$ be their respective genus. Then $g(K_1 \# K_2) = g(K_1) + g(K_2)$.

**Proof.** This proof is done in two steps. The first step shows that $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$ and the second shows that $g(K_1 \# K_2) \geq g(K_1) + g(K_2)$.

To show that $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$ let $F_1$ and $F_2$ be spanning surfaces for $K_1$ and $K_2$ respectively such that $F_1$ achieves the knot genus for $K_1$ and likewise for $F_2$. We then form a spanning surface $F$, for $K_1 \# K_2$ by joining $F_1$ and $F_2$ such that there is a single arc of intersection. The resulting surface $F$ now has genus $g(K_1) + g(K_2)$. Thus we have constructed a spanning surface for $K_1 \# K_2$ with genus $g(K_1) + g(K_2)$. As the genus of a knot is a minimum of the genus over all surfaces it follows that $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$.
To show that \( g(K_1 \# K_2) \geq g(K_1) + g(K_2) \) is more involved. Let \( F \) be a spanning surface for \( K_1 \# K_2 \). Then, place a topological sphere around one knot in the connected sum, say \( K_1 \). This sphere may intersect \( K_2 \) in many different ways. For example, it is possible to have a tube of surface intersecting the sphere with a closed end inside the sphere. Cutting this tube along the sphere and pasting a disc to the open end of the tube on either side of the sphere now removes that intersection. Doing this surgery leaves us with a topologically equivalent surface for \( K_2 \) which has one less intersection with the sphere. Inside the sphere, the surface for \( K_1 \) is unchanged other than an extra disjoint topological sphere due to the surgery. This extra sphere does not change the genus of the surface inside the sphere as a sphere has genus 0.

We seek to generalize this type of surgery to deal with any type of intersections. Any set of intersections is a set of loops, with some loops nested inside other loops. Beginning with the innermost loops, proceed with the surgery described above, cutting the surface along the sphere and adding a topological disc to either side. Inductively we may remove all of the intersections without changing the genus. Thus we have constructed spanning surfaces for \( K_1 \) and \( K_2 \) whose sum is the genus of \( F \). Hence \( g(K_1 \# K_2) \geq g(K_1) + g(K_2) \) and it follows that \( g(K_1 \# K_2) = g(K_1) + g(K_2) \)

Note that as the unknot is a topological disc, it has genus 0. Thus as any two nontrivial knots necessarily have positive genus, by the above theorem it is impossible to have the connected sum of two nontrivial knots be the trivial knot.

### 3.4 Linking number

Beginning here we diverge from our exclusive study of surfaces to start the construction of the Alexander polynomial from orientable surfaces. First we explore linking number. Linking number is applied in a very specific way to obtain the Alexander polynomial. However, linking number is a link invariant in its own right, as two 2-component links with different linking numbers must be distinct.
Definition 14. Consider a diagram of an oriented 2-component link, $L$ with link components $K_1$ and $K_2$. Of the crossings where $K_1$ crosses $K_2$ or $K_2$ crosses $K_1$, there are two types as displayed in Figure 3.4.

For each crossing, $c$ between $K_1$ and $K_2$, let $\epsilon(c)$ be the number assigned to the crossing of its type as given in Figure 3.4. Then the linking number of our link $L$ is

$$\text{lk}(K_1, K_2) = \frac{1}{2} \sum_c \epsilon(c).$$

![Figure 3.4: Left: A left handed crossing contributes $-1$ to the sum used to compute linking numbers. Right: A right handed crossing contributes $+1$ to the sum used to compute linking numbers.](image)

To show that this is an invariant it suffices to show that the linking number is unchanged by Reidemeister moves. This is easy to check, for example, the Reidemeister I move has no effect on the linking number as it creates only an additional crossing in one component of the link. The Reidemeister II move creates both an additional positive and negative crossing and thus does not contribute to the linking number. The Reidemeister III move neither adds crossings or changes the sign of crossings.

We will begin with an example of the simplest nontrivial link.

Example 1 (The Hopf Link). Examining the crossings which involve both link components in the leftmost link in Figure 3.5, we see that both crossings are positive crossings, making our sum over the crossings 2. Hence the linking number for the Hopf link is 1.

This example may provide some insight as to why it is necessary to divide by two to obtain the linking number.
Figure 3.5: Link diagrams used to compute linking number. Left is a link diagram for the Hopf link. Note that the center crossing in the rightmost link does not contribute to its linking number as it only involves one link component.

**Example 2.** Next we will go through a slightly more complicated example. Starting at the upper left crossing in the rightmost link in Figure 3.5 and visiting each crossing going counterclockwise we achieve the sum $\text{lk}(L) = \frac{1-1-1-1+1}{2} = -1$. Thus the linking number for this link is $-1$.

### 3.5 Homology

In finding the homology groups of a surface, we translate many topological properties into algebraic terms. As with genus, computing the homology of a surface gives us information about the embedding of the knot in $\mathbb{R}^3$.

For our purposes it suffices to be able to find the first homology group of a spanning surface for a knot, as the basis for the first homology group ultimately leads to a knot invariant. We spend this section outlining a process of finding a basis for the first homology group which bypasses much of the computation. For further references on homology see [4], [12], and [7].
3.5.1 The Seifert Graph

In Section 3.1 we gave the process of obtaining Seifert surfaces from a knot diagram. Given a knot diagram, and its decomposition into Seifert circles, we now produce a graph. In this graph, each circle in our set of Seifert circles is represented by a vertex in our graph. Two vertices are connected by an edge if and only if their corresponding Seifert circles were originally connected by a crossing in the original diagram. For example, the Seifert graph for our favorite example, the figure eight knot, is given in Figure 3.6 below.

![Seifert graph example](image)

Figure 3.6: This is an example of a Seifert graph obtained from the Seifert circles and knot diagram. Each Seifert circle is represented by a vertex and each crossing is represented by an edge. Note that the top vertex corresponds to the inner nested Seifert circle.

From the Seifert graph, we then remove edges to create a connected graph with no cycles, called a spanning tree. Adding back in any edge that was removed then creates a cycle in the Seifert graph. This cycle in the graph corresponds to a loop on the Seifert surface for the knot.

**Theorem 5.** [4] The set of loops on the Seifert surface obtained from cycles in the Seifert graph by the method described above form a basis for the first homology group for the Seifert surface of the knot.

Continuing with the example of the figure eight knot, we find when we remove edges from the Seifert graph our spanning tree is just three vertices in a line. Adding in the removed edges one at a time gives us two cycles, one between the top two vertices, and
one between the bottom two vertices. This is illustrated in Figure 3.7. These correspond to a loop $a$ between the top and main levels of the Seifert surface, and a loop $b$ between the base levels of the Seifert surface respectively. The loops are raised slightly above the surface to better illustrate their path through the crossings.

Figure 3.7: This illustrates the process of obtaining the basis loops from the Seifert graph. From left to right we first see the Seifert graph of the figure eight knot. Second is this graph with some edges removed to create a spanning tree. These edges are added in individually to create the next two graphs pictured. Finally the loops obtained from adding in edges correspond to a basis for the first homology group on the Seifert surface.

### 3.6 The Seifert Matrix

As Seifert surfaces for knots are orientable by construction, we may think of the Seifert surface as having a positive and negative side. We may make these sides more distinct by thickening the surface by some $\epsilon$ where epsilon is positive and sufficiently small to not disturb the topology of the surface. Then if our original surface is $F$, $F \times \{0\}$ now represents the negative side of $F$ and $F \times \{\epsilon\}$ represents the positive side. For any loop $l$ on our surface $F$, let $l^+$ be the loop $l$ which lies on the positive side of $F$. i.e. Let $l^+$ be the loop $l$ lying on $F \times \{\epsilon\}$. Then we define the Seifert matrix as follows.

**Definition 15.** Let $F$ be a Seifert surface for a knot with $n$ loops $l_1, l_2, \ldots, l_n$ which form
a basis for the first homology group for $F$. Then the Seifert matrix is an $n \times n$ matrix, $M$ where the entries are given by $a_{ij} = \text{lk}(l_i, l_j^+)$.}

The Seifert matrix for the figure eight knot is calculated in Example 3.

We will define knot invariants from this matrix. As we saw in Section 3.2, surfaces with boundaries of the same knot type must be related by a finite sequence of tubing and compressing operations. To define a knot invariant from the Seifert matrix, we must then define matrix equivalence such that surfaces related by tubing or compressing produce equivalent matrices. We first define matrix equivalence and then show that this definition acts as we would want under tubing and compressing operations.

**Definition 16.** Let $M$ be a Seifert matrix. Then the matrix $M$ is \textit{S-equivalent} to each of the following matrices

\[
\begin{pmatrix}
* & 0 \\
\vdots & \vdots \\
* & 0 \\
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 0 \\
\vdots & \vdots \\
0 & 0 \\
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 0
\end{pmatrix}
\]

Note that while we have only defined a way to enlarge matrices, this automatically defines a way to reduce matrices, as each of these larger matrices are S-equivalent to the reduced matrix $M$.

**Theorem 6.** \textit{Surgery equivalent surfaces give S-equivalent matrices}

\textit{Proof.} To show that surgery equivalent surfaces give S-equivalent matrices it suffices to show that matrices formed from surfaces differing by a tubing operation are S-equivalent. Let $M$ be the Seifert matrix for the surface $F$ and let $\hat{F}$ be the surface under a tubing operation. Then there is a basis for the first homology group for $\hat{F}$ which contains the basis for $H_1(F)$ as a subset. Let $m$ be the meridian of the tube and $l$ be a loop in $\hat{F}$ which runs along the longitude of the tube. Then the basis for $F$ along with the loops $l$ and $m$ form a basis for the first homology group of $\hat{F}$. Then the new Seifert matrix will remain
unchanged aside from two additional rows and columns due to the linking numbers of loops with the new basis elements.

To compute these linking numbers let us first look at how our new loops $m$ and $l$ link with themselves. Let the outside of the tube be the positive side of the surface, then it is obvious that $\text{lk}(m, m^+) = 0$ and $\text{lk}(m, l^+) = 0$. As the link $l, m^+$ has the longitudinal loop running through the center of $m^+$ we can see that $\text{lk}(l, m^+) = 1$. The linking number of $l, l^+$ depends on our choice of $l$, as a loop that spirals many times around the tube will have a different linking number than one that doesn’t. This is also tricky as we do not know how $l$ behaves outside the tube. However, we can choose a loop $l$ such that $\text{lk}(l, l^+) = 0$ by the following: choose an initial loop $l$ and suppose $\text{lk}(l, l^+) = \lambda$. Then the loop $l - \lambda m$ also runs through the longitude of the tube and additionally,

\[
\begin{align*}
\text{lk}(l - \lambda m, (l - \lambda m)^+) &= \text{lk}(l, (l - \lambda m)^+) - \text{lk}(\lambda m, (l - \lambda m)^+) \\
&= \text{lk}(l, (l - \lambda m)^+) - \lambda \text{lk}(m, (l - \lambda m)^+) \\
&= \text{lk}(l, l) - \lambda \text{lk}(l, m^+) - \lambda \text{lk}(m, l) - \lambda^2 \text{lk}(m, m^+) \\
&= 0
\end{align*}
\]

Then we may replace our original loop $l$ with the new loop $l - \lambda m$ to get our desired linking number.

Let $a_1, \ldots, a_n$ be the basis for $F$. Then as the loop $m$ is not in $F$, $\text{lk}(a_i, m^+) = 0$ and similarly $\text{lk}(m, a_i^+) = 0$ for all $i = 1, \ldots, n$. However, as we do not know how the basis elements $a_i$ sit in $F$ we cannot know $\text{lk}(a_i, l^+)$ or $\text{lk}(l, a_i^+)$. Let $\text{lk}(l, a_i^+) = \lambda_i$, then we have the following Seifert matrix for $\hat{F}$:

\[
\begin{bmatrix}
* & 0 \\
M & : & : \\
* & 0 \\
\lambda_1 & \cdots & \lambda_n & 0 & 1 \\
0 & \cdots & 0 & 0 & 0
\end{bmatrix}
\]

However, we may do a change of basis to replace all $\lambda_i$ by zero. For our change of basis we replace each generator $a_i$ by $b_i = a_i - \lambda_i m$. This does not change the linking number of
the generators $\text{lk}(b_i, b_j^\dagger)$ is zero, but does give that $\text{lk}(l, b_i^\dagger) = 0$ (which may be found by a similar computation as above). This gives us the following new Seifert matrix

$$
\begin{bmatrix}
\ast & 0 \\
M & \vdots \\
\ast & 0 \\
0 & \cdots & 0 & 0 & 1 \\
0 & \cdots & 0 & 0 & 0
\end{bmatrix}
$$

which is one of our enlarged $S$-equivalent matrices. It follows that matrices achieved through compressing a surface are also $S$-equivalent. 

The above theorem gives that equivalent knots have equivalent matrices, so it seems natural to apply matrix invariants to the Seifert matrix. This allows us to extract essential information from the matrix to compare without attempting to compare their matrices directly. We now discuss some familiar matrix invariants: signature and determinants.

### 3.6.1 Signature

**Definition 17.** Let $K$ be a knot, then the **signature** of $K$, denoted $\sigma(K)$ is $\sigma(M + M^T)$ where $M$ is any Seifert matrix for $K$.

It follows from basic linear algebra that this defines a knot invariant, as equivalent Seifert matrices will produce the same signature.

**Example 3.** We continue with our example of the figure eight knot to compute its signature. To do this, we must first finish the process of finding its Seifert matrix. We have found a basis for it homology group, and these loops are displayed on the Seifert surface for the figure eight knot in Figure 3.7. Thus to compute the Seifert matrix for the figure eight we need only to find how these two basis loops link with themselves and each other.

As we have two basis loops, our Seifert matrix is a $2 \times 2$ matrix. From Figure 3.7 we find that the links in question are as follows in Figure 3.8.
As these are simple links, we will not go through the computation of their linking numbers. The final result is the Seifert matrix below

\[
\begin{bmatrix}
-1 & 1 \\
0 & 1
\end{bmatrix}.
\]

The signature of the figure eight knot is the signature of $M + M^T$ where $M$ is the above
Seifert matrix. So,

\[ \sigma(K) = \sigma \left( \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \right) \]
\[ = \sigma \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \]
\[ = 0 \]

### 3.6.2 Determinant

**Definition 18.** Let $K$ be a knot, then the **determinant** of $K$, denoted $\det(K)$, is $|\det(M + M^T)|$ where $M$ is any Seifert matrix for $K$.

That the above definition defines a knot invariant is easy to check using basic linear algebra. It suffices to show that equivalent Seifert matrices yield the same determinant.

**Example 4.** We will use the computation of the Seifert matrix above to calculate the determinant of the figure eight knot. As we determined in the signature example, the Seifert matrix, $M$ for the figure eight knot is as follows

\[
\begin{bmatrix}
-1 & 1 \\
0 & 1
\end{bmatrix}.
\]

Then the determinant of our knot is the absolute value of the determinant of matrix

\[ |\det(M + M^T)| = \left| \det \left( \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \right) \right| = \left| \det \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \right| \]

and so $\det(K) = 5$.

**Theorem 7.** Let $K_1$ and $K_2$ be knots, then

1. $\det(K_1 \# K_2) = \det(K_1) + \det(K_2)$

2. $\det(K_1 \# K_2) = \det(K_1) + \det(K_2)$
Proof. We prove both parts of this theorem by showing a simple fact about the Seifert matrix for a connected sum of knots.

Let $F_1$ and $F_2$ be orientable spanning surfaces for $K_1$ and $K_2$ respectively, and let $M_1$ and $M_2$ be their corresponding Seifert matrices. Let $F$ be an orientable spanning surface for the connected sum $K_1 \# K_2$ obtained by adding an edge of a rectangular disc to each $F_1$ and $F_2$ such that the disc intersects these surfaces only at their boundary. Then a basis for the first homology group of $F$ is a union of the bases for $F_1$ and $F_2$. From this union we obtain the following Seifert matrix, $M$ for $K_1 \# K_2$

\[
\begin{bmatrix}
M_1 & 0 \\
0 & M_2
\end{bmatrix}.
\]

Taking the signature and determinant $M$ gives the desired result. \qed

### 3.7 The Alexander Polynomial

This section provides our first introduction to the Alexander polynomial. We will use the tools provided in the previous sections to give our first definition of the Alexander polynomial.

**Definition 19.** Given a knot $K$, the Alexander polynomial, denoted $\Delta(K)$ is given by

$\Delta(K) = \det(t^{1/2}M - t^{-1/2}M^T)$ where $M$ is a Seifert matrix for $K$.

To show that this defines an invariant it is sufficient to check that the polynomial is invariant under Seifert matrix equivalence. We leave this to the reader.

Here we will do $1\frac{1}{2}$ examples. We will finish the computation for the Alexander polynomial for the figure eight knot, and we will do one additional example from the beginning with a different knot.

**Example 5** (Figure eight knot). As we found above, the Seifert matrix, $M$ for the figure eight knot is as follows:

\[
\begin{bmatrix}
-1 & 1 \\
0 & 1
\end{bmatrix}.
\]
Then we find the determinant to be

\[
\det(t^{1/2}M - t^{-1/2}M^T) = \det \left( t^{1/2} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} - t^{-1/2} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \right) = \det \left( \begin{bmatrix} -t^{1/2} + t^{-1/2} & t^{1/2} \\ -t^{-1/2} & t^{1/2} - t^{-1/2} \end{bmatrix} \right).
\]

Thus our Alexander polynomial is

\[
\Delta(4_1) = (-t^{1/2} + t^{-1/2})(t^{1/2} - t^{-1/2}) + (t^{1/2}t^{-1/2}) = -t + 1 + 1 - t^{-1} + 1 = -t + 3 - t^{-1}.
\]

**Example 6** (The trefoil knot). Beginning with a knot diagram, we construct the Seifert circles and then the Seifert surface. This process is illustrated in Figure 3.9.

![Image of Seifert diagram](image.png)

Figure 3.9: From the oriented knot diagram of the trefoil knot (left) we form the Seifert circles (middle) and finally the Seifert surface (right).

Next we produce the Seifert graph for the trefoil knot and find the cycles in the graph corresponding to basis elements for the first homology group. This is shown in Figure 3.10.

The linking number of these basis loops are then used to compute the Seifert matrix. The basis loops are displayed in pairs in Figure 3.11 to compute the linking numbers for the Seifert matrix.

Using the linking numbers from the loops in Figure 3.11 we obtain the following Seifert matrix for the trefoil knot

\[
\begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}.
\]
Figure 3.10: This illustrates the process of obtaining the basis loops from the Seifert graph of the trefoil knot. From left to right we first see the Seifert graph of the trefoil knot. Second is this graph with some edges removed to create a spanning tree. These edges are added in individually to create the next two graphs pictured. Finally the cycles obtained from adding in edges correspond to basis loops for the first homology group of the Seifert surface.

Figure 3.11: These are the basis loops for the first homology group of a Seifert surface for the trefoil knot, used to compute linking numbers for the Seifert matrix.

We then find the Alexander matrix by computing $\det(t^{1/2}M - t^{-1/2}M^T)$ for our Seifert matrix $M$.

\[
\det(t^{1/2}M - t^{-1/2}M^T) = \det \left( t^{1/2} \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} - t^{-1/2} \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \right)
= \det \left( -t^{1/2} + t^{-1/2} -t^{1/2} \\ t^{-1/2} \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \right)
= \det \left( -t^{1/2} + t^{-1/2} -t^{1/2} \\ t^{-1/2} \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \right).
\]
Then the Alexander polynomial for the trefoil knot is

\[ \Delta(3_1) = (-t^{1/2} + t^{-1/2})^2 - (-t^{1/2})(t^{-1/2}) = t + t^{-1} - 2 + 1 = t - 1 + t^{-1}. \]

Note that the two Alexander polynomials obtained from these examples are distinct. This shows that the two knots are in fact different.
Chapter 4

Resolving Trees

A natural way to compare knots comes from changing the crossings. This section seeks to formalize this relationship between knot diagrams differing by a single crossing. Relations between knots differing by a crossing are called **skein relations**. Here we discuss resolving trees, a method by which one changes the crossings in a knot one by one until a collection of trivial knots is obtained. This method can be used with various skein relations to define several knot invariants, one of which is our old favorite, the Alexander polynomial.

4.1 Resolving Trees and the Conway Polynomial

Given a crossing in an oriented knot diagram, there are two ways that you can change the crossing while maintaining the orientation in the rest of the knot. You can switch the overcrossing and undercrossing, or you can split the strands of the knot and rejoin them to an adjacent strand with the same orientation. When doing this it is important to keep track of the signs of the crossings given previously in Definition 14.

We may use this method of deconstructing knots to define knot invariants, specifically knot polynomials. One example is the Conway polynomial, denoted $\nabla(K)$.

**Definition 20.** The Conway polynomial is constructed by three rules as follows.

1. $\nabla(0_1) = 1$
2. If $K$ is our knot and $K_+, K_-$, and $K_0$ denote $K$ with a positive, negative, or zero crossing respectively, as shown in Figure 4.1 then $\nabla(K_+) - \nabla(K_-) = z\nabla(K_0)$

3. The Conway polynomials of two ambient isotopic knots are equal.

Our goal is to reduce the knot’s complexity by changing crossings in this way until we ‘unknot’ the knot, to obtain trivial knots or links. By keeping track of the signs of the crossings as we go and using the relationships given in Definition 20 this yields the Conway polynomial. This process gives a binary tree of knot diagrams. We call these resolving trees. The polynomial found is independent of the resolving tree used [10]. The resolving tree for one knot is given in Figure 4.2 below.

Before doing an example of computing the Conway polynomial of a knot, it is useful to have the following theorem.

**Theorem 8.** If $L$ is a split link, including trivial links then $\nabla(L) = 0$

**Proof.** Let $D_0$ be the diagram of a split link with link components $L_1$ and $L_2$ and let $D_+$, $D_-$ be the link with components connected by a positive or negative crossing respectively. $D_+$ can be obtained from $D_-$ by rotating one link component, say $L_1$, by 180 degrees. Thus $D_+$ and $D_-$ are equivalent links and hence $\nabla(D_+) = \nabla(D_-)$. Applying our skein relation we have that $\nabla(D_+) - \nabla(D_-) = z\nabla(D_0)$ and so $\nabla(D_0) = 0$. 

\qed
Figure 4.2: Here is a resolving tree for a more complicated knot. The dotted circle indicates which crossing is resolved at each stage. Ambient isotopic knots are denoted with \( \sim \), while \( +, -, 0 \) denote positive, negative and zero crossings respectively.
Example 7. Consider the knot with its resolving tree as shown in Figure 4.2. We use this resolving tree to derive the Conway polynomial for this knot. The rule $\nabla(K_+) - \nabla(K_-) = z\nabla(K_0)$ actually gives us two formulas to use, depending on whether we are starting with a positive, or negative crossing. Solving for each of these we have:

$$\nabla(K_+) = \nabla(K_-) + z\nabla(K_0),$$
$$\nabla(K_-) = \nabla(K_+) - z\nabla(K_0).$$

We see that $K$ has a negative crossing in the region shown, and splits in the resolving tree to $K_0$ on the right and $K_+$ on the left. Thus, using our relationships between the signed crossings, we have that $\nabla(K) = \nabla(K_+) - z\nabla(K_0)$. As $K_+$ is equivalent to the unknot, moving to the second row we get $\nabla(K) = 1 - z(\nabla(K_0) - z\nabla(K_{0+}))$. Continuing in this manner shows that the Conway polynomial for this knot is given by,

$$\nabla(K) = 1 - z(1(1-z) - z) = 1 + 3z^2.$$  

Resolving trees can also be used to obtain our favorite knot invariant, the Alexander polynomial.

4.2 The Alexander Polynomial

To obtain the Alexander polynomial of a knot from its resolving tree, we need only to change $z$ to $t^{-1/2} - t^{1/2}$. This relationship is given by the following theorem.

Theorem 9. If $K_+, K_-$ and $K_0$ are defined as above then

$$\Delta(K_+) - \Delta(K_-) = (t^{-1/2} - t^{1/2})\Delta(K_0).$$

This result can be proved using our knowledge of Seifert surfaces and Alexander Matrices from Chapter 3. However, we omit the proof that these methods produce the same polynomial invariant.
This gives the following two formulas to employ depending on whether we are changing a positive or negative crossing:

\[
\Delta(K_+) = \Delta(K_-) + (t^{-1/2} - t^{1/2})\Delta(K_0)
\]
\[
\Delta(K_-) = \Delta(K_+) - (t^{-1/2} - t^{1/2})\Delta(K_0).
\]

To see this in action, we outline the process of obtaining the Alexander polynomial from a resolving tree for the same knot discussed above.

**Example 8.** For this we will use the resolving tree in Figure 4.2. For the Alexander polynomial, as with the Conway polynomial, we can use the fact that \(\Delta(0_1) = 1\). Additionally, we need to know the Alexander Polynomial of a split link.

**Corollary 10.** The Alexander polynomial of a split link is zero. \(\square\)

Given this we need only to change the way we express the relationship between the original knot and the knot differing by a single crossing.

Returning to our resolving tree, we find at the top of the tree that \(K\) has a negative crossing in the region shown. This knot branches to \(K_0\) (right) and \(K_+\) (left). Thus, using our relationships between the signed crossings, we have that \(\Delta(K) = \Delta(K_+) - (t^{-1/2} - t^{1/2})\Delta(K_0)\). As \(K_+\) is equivalent to the unknot, moving to the second row of our resolving tree we get \(\Delta(K) = 1 - (t^{-1/2} - t^{1/2})(\Delta(K_0) - (t^{-1/2} - t^{1/2})\Delta(K_{00})\). Continuing in this manner we finally obtain that the Alexander polynomial for this knot is given by,

\[
\nabla(K) = 1 - (t^{1/2} - t^{-1/2})(1(1(- (t^{1/2} - t^{-1/2}))) - (t^{1/2} - t^{-1/2})) - (t^{1/2} - t^{-1/2}))
= 3t - 5 + 3t^{-1}.
\]

Note that the Alexander polynomial for this knot could also have been obtained directly from the Conway polynomial by substituting \(z = t^{1/2} - t^{-1/2}\).

Tables of Alexander and Conway polynomial for knots up to nine crossings are included in most knot books, for example [4].
Chapter 5

Algebraic and Topological Tools

This chapter outlines some algebra and topology necessary to the remainder of our study of the Alexander polynomial. All results presented here are standard in modern algebra and basic topology texts and will be presented without citation. References include [8, 14, 5].

5.1 Free Groups and Quotients

5.1.1 Free Groups

This may be the appropriate time to revisit some group theory. Recall that a group is a set, $G$ with a binary operation $\cdot : G \times G \rightarrow G$ defined by $(g_1, g_2) \rightarrow g_1 \cdot g_2$ that satisfies the following properties:

- $G$ is closed under the binary operation. i.e. for $g_1, g_2 \in G$, $g_1 \cdot g_2 \in G$.
- The operation $\cdot$ is associative. i.e. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$.
- There is an identity element $e \in G$ such that for all $a \in G$, $e \cdot a = a \cdot e = a$.
- Each element has an inverse: for all $a \in G$ there exists $b \in G$ such that $ab = ba = e$.

Remark 1. Henceforth we adopt the convention that abelian groups are denoted with operation $+$ and identity $0$, while nonabelian groups are denoted with operation $\cdot$ and identity element $1$. 

36
Given only this definition of a group and a generating set of elements (for our purposes we will consider only finite generating sets), \( X = \{x_1, x_2, \ldots, x_n\} \) we can construct a free group, \( F(X) \) or equivalently \( F(n) \). This is a group consisting of words in the generators, including their powers and inverses. The only reduction that can be made in these words are those that follow directly from the group properties, for example \( x_i^k x_i^{-k} = 1 \) and \( 1w = w1 = w \). A more formal definition can be found in most algebra or group theory texts (see for instance [14]).

The following theorem allows us to address all problems in group theory with a common tool.

**Theorem 11.** Every group \( G \) is a homomorphic image of a free group, \( F \).

However, even this theorem can be made more specific. In practice, what we mean is that every group can be represented as a quotient group of a free group. But this is getting ahead of ourselves. First it is necessary to review normal subgroups and quotient groups. This will be key to further discussion of free groups, and will be a useful tool throughout the remainder of this paper. There are many equivalent definitions of a normal subgroup, which may be found in any introductory algebra text (for example [8]).

**Definition 21.** A subgroup \( N \) of a group \( G \) is normal if for all \( g \in G \), \( gN = Ng \).

From this definition it is easy to see that whether or not a subgroup is normal is only a concern for nonabelian groups, as all subgroups of abelian groups are normal. It is also interesting to note that any subgroup which contains exactly half of the group elements is normal, as there is only one coset not equal to the identity coset, and the identity commutes with all elements of the group. However, while considering these cases of normal subgroups may be useful to understanding the concept, we will be more concerned with normal subgroups of free groups.

Normal subgroups, like many algebraic structures, may be defined using a generating set. Given a set of elements \( A = \{g_1, g_2, \ldots, g_n\} \), the normal subgroup generated by \( A \), denoted \( \langle\langle A \rangle\rangle \), is the smallest normal subgroup containing \( A \) and consists of products of elements of \( A \) and their conjugates with elements of \( G \). We leave it to the reader to assure
themselves that this is in fact a normal subgroup. We are mainly interested in normal subgroups for constructing quotient groups. We define this as follows.

**Definition 22.** Let $G$ be a group. Then $H \subset G$ is a *quotient group* of $G$ if there exists a homomorphism $\varphi : G \to H$ and a normal subgroup $N$ such that $\ker \varphi = N$. This is typically denoted $H \cong G/N$.

Given this definition of quotient groups, it is clear that a subgroup being normal is necessary and sufficient to make the corresponding quotient map well defined.

**Theorem 12.** Every group is isomorphic to a quotient group of a free group.

The above theorem gives a convenient notation for groups. A quotient group of a free group can be defined entirely by the generating set of the free group, and the generating set of a normal subgroup of the free group. Thus any group can be represented in terms of the two generating sets that define an isomorphic quotient group of the free group. We call the elements of the generating set of the free group the generators of $G$ and we call the elements of the generating set of the normal subgroup the relations. Thus any group has a group presentation $G \cong \langle X : R \rangle$ where $X$ is the set of generators and $R$ is the set of relations. In terms of our previous definition, a group $G$ can be defined by the map $\varphi : F(X) \to G$ where $\ker \varphi = \langle \langle R \rangle \rangle$.

Here is another approach to looking at group presentations. Given a group and a finite set of generators, any possible element of the group may be written as a word in the generators. However, different words may define the same element of the group, for example, in any abelian group $xy = yx$. The relations tell us when words are equivalent by giving us a set of words that are equivalent to the identity. So in our abelian group where $xy = yx$ we might find the relation $xyx^{-1}y^{-1}$ in our group presentation (see the discussion of abelianizations, Section 5.1.3).

**Example 9.** Clearly any free group is a quotient by the trivial group, thus a free group on a set of generators $X$ is simply $F(x) \cong \langle X : \rangle$. 
Example 10. Another simple example is $\mathbb{Z}_n$. This is a group with a single generator, that has the property that when a group element is multiplied by itself $n$ times, it gives the identity. This is given the group presentation $\mathbb{Z}_n \cong \langle x : x^n = 1 \rangle$.

Example 11. For a more complicated example, consider the group $S_n$. This group can be generated by the two elements (in cycle notation) $(1234\ldots n)$ and the transposition $(12)$. It is clear that $(12)^2$ is equal to the identity and that $(1234\ldots n)^n$ is also the identity. Thus we get the group presentation $S_n \cong \langle x, y : x^n, y^2 \rangle$.

Henceforth we will refer to a group and its presentation interchangeably.

5.1.2 Presentation Equivalence

When dealing with presentations of groups it is important to recognize when two presentations describe isomorphic groups. For example the presentation $\langle x : x^3 \rangle$ clearly defines the same group as the presentation $\langle y : y^3 \rangle$. But what about the presentation $\langle x, y : x^{2n}, y^2, xyxy \rangle$ and $\langle x, y : x^{2n}, y^2, xyx^{-1}y^n \rangle$? In fact these too define isomorphic groups. The following Tietze transformations of group presentations allow us to formalize when two presentations are equivalent.

The first Tietze transformation describes a presentation equivalence of adding an unnecessary relation.

Definition 23. The Tietze $I$ move is defined by the following map:

$$\langle X : R \rangle \xrightarrow{I} \langle X : R \cup s \rangle.$$  

This is a presentation equivalence if $s \in \langle \langle R \rangle \rangle$.

The second Tietze transformation addresses adding an extra generator. In this case, in order for the presentation to define the same group it must be that the new generator can be written in terms of the other generators.

Definition 24. Let $y \notin X$ and let $y$ be equivalent to $w$, a word in the previous generators, then $yw^{-1}$ must be equal to the identity, and thus be in the normal subgroup generated by the relations. We define the Tietze $II$ move to be the following map:
\[ \langle X : R \rangle \xrightarrow{H} \langle X \cup y : R \cup yw^{-1} \rangle. \]

**Theorem 13.** For any two group presentations \( \langle X : R \rangle \) and \( \langle Y : S \rangle \) defining isomorphic groups there exists a finite sequence of Tietze transformations which sends \( \langle X : R \rangle \) to \( \langle Y : S \rangle \).

While this theorem allows us to define which presentation transformations preserve the group structure, in practice, given two equivalent presentations, it is very difficult to find an explicit sequence of Tietze moves that show this. However, while this definition of presentation equivalence may not be the most practical, we will later derive knot polynomials based on group relations. This concise definition of equivalence will enable us to easily show the invariance of these polynomials.

### 5.1.3 Abelianizations

An abelian group is one in which all elements commute: for \( w_i, w_j \in G \), \( w_iw_j = w_jw_i \). It is sufficient that this property holds for the generators of \( G \). Namely \( g_i g_j = g_j g_i \) for generators \( g_i, g_j \) of \( G \). This equality naturally leads to the relation \( g_i g_j g_i^{-1} g_j^{-1} \) and thus any group \( G \) can be sent to its abelianization, \( G_{ab} \) by the quotient map \( G \to \frac{G}{\langle \langle g_i g_j g_i^{-1} g_j^{-1} \rangle \rangle} = G_{ab} \). The normal subgroup \( \langle \langle g_i g_j g_i^{-1} g_j^{-1} \rangle \rangle \) is called the commutator subgroup.

### 5.2 The Fundamental Group

#### 5.2.1 The Fundamental Group

The fundamental group is a topological invariant of a space. Thus while distinct topological spaces may not have distinct fundamental groups, topological spaces with different fundamental groups are necessarily distinct spaces. The fundamental group of a space is closely connected to its homology groups, which we have discussed in Chapter 3. In fact the first homology group is the abelianized fundamental group. However, we will define
the fundamental group independently. In constructing the fundamental group from the bottom up, we get an idea of the machinery behind this algebraic tool.

The first tool necessary to the discussion of fundamental groups is a path. As the word implies, a path can be thought of as the trail a particle takes through a space. We define this more formally as follows.

**Definition 25.** Let $X$ be a topological space. A path, $a$ is a continuous function $a : [t_1, t_2] \to X$.

The starting point of the path is $a(t_1)$ and the terminal point of the path is $a(t_2)$.

**Definition 26.** A loop $a$, is a path which has the same initial and terminal point. i.e. $a(t_1) = a(t_2)$.

Henceforth, we will consider paths and loops as continuous functions from $[0, 1] \to X$. As you can always rescale the path so that the particle completes the path in this interval, this is not a limiting assumption.

**Definition 27.** Given two paths $a$ and $b$ where the terminal point of the path $a$ is the starting point of path $b$, $a(1) = b(0)$. We define the path $a \cdot b$ to be the path which starts at $a(0)$, travels along $a$ and then travels along $b$, terminating at $b(1)$ as follows

$$
(a \cdot b)(t) = \begin{cases} 
a(2t) & \text{for } 0 \leq t \leq \frac{1}{2} 
\end{cases}
$$

Note that loops are closed under path composition as the composition of paths which begin and end at point $p$ produces a path that begins and ends at point $p$. The *inverse* of a path $a$ will be the path that begins at the terminal point of $a$, and travels backwards along $a$ until finally reaching the initial point of $a$. We define the inverse of $a$, $a^{-1} : [0, 1] \to X$ by

$$
a^{-1}(t) = a(-t + 1).
$$

We would also like to be able to describe when two paths are equivalent.
Definition 28. Paths $a$ and $b$ are homotopic, denoted $a \sim b$, if there exists a continuous function $H : [0, 1] \times [0, 1] \to X$ such that $H(0, t) = a(t)$ and $H(1, t) = b(t)$.

It is easy to check that homotopy defines an equivalence relation on paths (see [7] for example). We denote the equivalence class containing the path $a$ by $[a]$ and $b \sim a$ if and only if $b \in [a]$.

At this point it’s clear where we are heading. We would like to form a group from the loops in a topological space.

Definition 29. The fundamental group of a topological space $X$ at point $p$, $\pi_1(X, p)$ is the group of equivalence classes of loops from point $p$ under the operation $\cdot$ defined by $[a] \cdot [b] := [a \cdot b]$.

To show that this is a group we must check that the operation $\cdot$ is well defined, associative, that the equivalences classes are closed under this operation, that there is an identity class, and that each element has an inverse.

- As loops are closed under path composition, it follows that the the operation $\cdot$ is well defined. In addition, this yields that equivalence classes of loops are closed under composition.

- By Definition 27 it is clear that the paths $a \cdot (b \cdot c)$ and $(a \cdot b) \cdot c$ are the same up to rescaling, so clearly $a \cdot (b \cdot c) \sim (a \cdot b) \cdot c$. Therefore the operation $\cdot$ is associative.

- The identity will be the constant path, $e(t) = p$. Note that $e \cdot a \neq a$, is path which remains at the point $p$ until $t = \frac{1}{2}$, then traverses $a$ from $t = \frac{1}{2}$ to $t = 1$. Of course the path $a$ may leave $p$ in $[0, \frac{1}{2}]$. However, the equivalence class of the identity behaves as it should, and $[e] \cdot [a] = [a] \cdot [e] = [a]$ for all loops $a$.

- We have already defined inverse paths. However we have yet to show that the equivalence class of a path composed with its inverse yields the class of the constant path. By definition, $a \cdot a^{-1} = \begin{cases} a(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ a(2 - 2t) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$.
so the continuous family of functions

\[ H(s, t) = \begin{cases} 
  a(2t(1 - s)) & \text{for } 0 \leq t \leq \frac{1}{2} \\
  a((2 - 2t)(1 - s)) & \text{for } \frac{1}{2} \leq t \leq 1 
\end{cases} \]

decoms \( a \cdot a^{-1} \) to the constant path. Hence \([a] \cdot [a^{-1}] = [e]\) for all paths \(a\) in \(X\).

Thus the equivalence classes of loops under path composition form a group.

This group is one tool used to describe the structure of a topological space. For example, consider the annulus shown in Figure 5.1 with the subspace topology inherited from \(\mathbb{R}^2\).

It is clear that the lower loop, \(\gamma_2\) in this space is of the same class as the identity loop.

However, the loop which travels around the ‘hole’ in the space, \(\gamma_1\) is a distinct loop as it cannot be deformed to the constant path. Composing the second loop with itself yields another distinct loop. In fact the nontrivial loop pictured here generates the fundamental group of this space, an infinite cyclic group. The single generator of our fundamental group indicates that there is a single hole in our topological space.

Of course our discussion of the fundamental group of this space has been very informal, we now provide some tools which facilitate computation of the fundamental group.

**5.2.2 Computational Tools**

First, let us highlight some properties of the fundamental group.
**Theorem 14.** If $X$ is a pathwise connected space and $p$ and $p'$ are points in $X$, then $\pi_1(X, p) \cong \pi_1(X, p')$.

*Proof.* Because $X$ is pathwise connected, there exists a path $a$ such that $a(0) = p'$ and $a(1) = p$. Let $\alpha = [a]$ and let $\beta$ be a class of paths in $\pi_1(X, p)$. Then we define $f : \pi_1(X, p) \to \pi_1(X, p')$ by $f(\beta) = \alpha \beta \alpha^{-1}$. This map is clearly an injection. In addition, for any $\gamma \in \pi_1(X, p')$, $f(\alpha^{-1} \gamma \alpha) = \gamma$. It remains to show that it is a homomorphism. Let $\beta_1$ and $\beta_2$ be classes of paths in $\pi_1(X, p)$, then by definition of $f$ we have that

$$
 f(\beta_1) f(\beta_2) = (\alpha \beta_1 \alpha^{-1})(\alpha \beta_2 \alpha^{-1})
 = \alpha \beta_1 (\alpha^{-1} \alpha) \beta_2 \alpha^{-1}
 = \alpha \beta_1 \beta_2 \alpha^{-1}
 = f(\beta_1 \beta_2).
$$

In this paper we restrict our discussion of topological spaces to those which are pathwise connected, and thus the notation $\pi_1(X, p)$ may be replaced by $\pi_1(X)$ without any loss of information.

**Definition 30.** A topological space $X$ is *simply connected* if $\pi_1(X, p)$ is the trivial group.

**Theorem 15.** Every convex set is simply connected.

*Proof.* If $X$ is convex, every two points can be joined by a straight line. Let $a$ be a $p$ based loop in $X$. Then the map $f : [0, 1] \times [0, 1] \to X$ defined by $f(s, t) = (1 - s)a(t)$ continuously deforms $a$ to the constant path.

Observing that $\mathbb{R}^n$ is convex yields the following:

**Corollary 16.** The space $\mathbb{R}^n$ is simply connected.

Additionally, it is necessary to discuss the fundamental group of one more topological space, the fundamental group of $S^1$.

**Theorem 17.** The fundamental group of the circle, $S^1$, is isomorphic to $\mathbb{Z}$. 
As with our previous example of the annulus, this result is not surprising. It is immediate that any nontrivial path must make at least one full rotation around the circle. Some extra work convinces us that each multiple of rotations around the circle in either direction corresponds to a distinct class of paths. With the group structure previously defined, this gives the group of integers.

However, while this may seem intuitive, the formal proof of this is quite extensive and involves path lifting, homotopy lifting, winding numbers, and other machinery not discussed here. For a formal proof of this result see [12].

While the definitions and theorems above define the fundamental group, it is very difficult to calculate fundamental groups of spaces with only these tools. For this reason, we now discuss two further results which simplify the calculation of fundamental groups.

5.2.3 Deformation Retracts and the Seifert Van Kampen Theorem

In this section we explore the relationship between the fundamental group of spaces related by functions. For this it is necessary to introduce the following proposition, giving a solid foundation for this study.

**Proposition 18.** Let $X$ and $Y$ be topological spaces and let $f : X \to Y$ be a continuous function. Then $f$ induces a homomorphism $f_* : \pi_1(X, p) \to \pi_1(Y, f(p))$.

**Proof.** Given a class of loops $[a]$ in $X$, we define $f_*([a]) = [f(a)]$. To show that this is a homomorphism we must show that this map is well defined, and that $f ([a] \cdot [b]) = f[a] \cdot f[b]$.

Consider loops $a$ and $b$ such that $a \sim b$. Let $\{h_s\}$ be a homotopy that deforms $a$ to $b$. Then $\{fh_s\}$ is a homotopy that deforms $f(a)$ to $f(b)$ and so $f_*$ is well defined.
To show that it is a homomorphism,
\[
f(a \cdot b)(t) = f(a \cdot b)(t)
= \begin{cases} 
  f(a(t)) & \text{for } 0 \leq t \leq \frac{1}{2} \\
  f(b(t + \frac{1}{2})) & \text{for } \frac{1}{2} \leq t \leq 1
\end{cases}
= \begin{cases} 
  fa(t) & \text{for } 0 \leq t \leq \frac{1}{2} \\
  fb(t + \frac{1}{2}) & \text{for } \frac{1}{2} \leq t \leq 1
\end{cases}
= (fa \cdot fb)(t).
\]
\[
\]

We now introduce some maps particularly useful for the induced homomorphism of the fundamental group.

**Definition 31.** A *retraction* of a topological space $X$ onto a subspace $Y$ is a continuous function $\rho : X \to Y$ such that $\rho$ is the identity map on $Y$. A subspace $Y$ of a space $X$ is a *retract* of $X$ if there exists a retraction $\rho : X \to Y$.

**Example 12.** Consider the square $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$. A retraction of this space onto the interval $[0, 1]$ is given by the following map:
\[
\rho(x, y) = (x, 0) \quad \text{for } 0 \leq x, y \leq 1.
\]

**Theorem 19.** If $\rho : X \to Y$ is a retraction and $X$ is pathwise connected, then for a given basepoint $p \in X$, the induced homomorphism $\rho_* : \pi_1(X, p) \to \pi_1(Y, \rho(p))$ is onto.

**Proof.** By Theorem 14 the fundamental group $\pi_1(X, p)$ is not affected by a change of basepoint. It then suffices to prove the above theorem for $p \in Y \subset X$. Consider the following sequence of homomorphisms:
\[
\pi_1(Y, p) \xrightarrow{i_*} \pi_1(X, p) \xrightarrow{\rho_*} \pi_1(Y, p)
\]
where $i_*$ is the inclusion map. Because $\rho$ is a retraction map, $\rho_* i_*$ is the identity on $\pi_1(Y, p)$ and thus $\rho_*$ is onto. \qed
Definition 32. A deformation of a topological space $X$ is a family of functions $h_s : X \rightarrow X$ for $s \in [0, 1]$ such that $h_0$ is the identity and for $p \in X$, $h_s(p)$ is continuous in both variables.

Example 13. Consider the disc in $\mathbb{R}^2$ with radius 1 centered at the origin. This disc can be deformed to the point at the origin by the following family of functions:

$$h_s(r, \theta) = (r(1-s), \theta), \quad \text{for} \quad \begin{cases} 0 \leq r, s \leq 1, \\ 0 \leq \theta < 2\pi. \end{cases}$$

Combining these two previous definitions, it is clear what we mean by the term deformation retract. In simple terms (simple if you’ve mastered the previous definitions!) a deformation retract of $X$ onto $Y$ is a deformation of $X$ such that the final map, $h_1$ is a retraction of $X$ onto $Y$. However, because of its importance in the following sections, we formally define it here.

Definition 33. Given a topological space $X$ and a subspace $Y$, a deformation retract of $X$ onto $Y$ is a family of functions $h_s : X \rightarrow X$ where $s \in [0, 1]$ such that $h_0$ is the identity and for $p \in X$, $h_s(p)$ is continuous in both variables. In addition the map $h_1$ is a retraction of $X$ onto the subspace $Y$.

Note that the deformation of the disc given in Example 13 is a deformation retract of the disc onto the origin. Finally, the importance of these definitions becomes clear in the following theorem.

Theorem 20 ([5]). If a subspace $Y$ is a deformation retract of a pathwise connected topological space $X$, then $\pi_1(X)$ is isomorphic to $\pi_1(Y)$.

The following theorem allows us to examine a topological space in terms of simpler components. Given topological spaces $X$ and $Y$ satisfying some criteria the Seifert Van Kampen theorem allows us to find the fundamental group of $X \cup Y$ given the fundamental groups $\pi_1(X)$ and $\pi_1(Y)$. This theorem is used extensively in the proof of the knot group in the following chapter. The proof of the Siefert Van Kampen theorem is found in Appendix A.
Theorem 21. The Seifert Van Kampen Theorem

Let $X$ be a topological space and let $X = X_1 \cup X_2$ where $X_1$ and $X_2$ are open subsets such that $X_1, X_2$ and $X_0 = X_1 \cap X_2$ are nonempty and pathwise connected. Then $\pi_1(X)$ is generated by $\pi_1(X_1), \pi_1(X_2)$ and $\pi_1(X_0)$ by the following commutative diagram:

\[
\begin{array}{ccc}
\pi_1(X_1, p) & \xrightarrow{j_1*} & \pi_1(X_0, p) \\
\xrightarrow{i_1*} & & \xrightarrow{j_0*} \\
\pi_1(X_2, p)
\end{array}
\]

In addition, if $H$ is a group and $\psi_i : \pi_i(X_i) \to H$ are homomorphisms such that $\psi_0 = \psi_1 i_1* = \psi_2 i_2*$, then there is a unique $\lambda : \pi_1(X) \to H$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\pi_1(X_1, p) & \xrightarrow{j_1*} & \pi_1(X_0, p) \\
\xrightarrow{i_1*} & & \xrightarrow{j_0*} \\
\pi_1(X_2, p)
\end{array}
\]
Chapter 6

Knot Groups

The knot groups are an important knot invariant. In Section 6.2 we prove that the knot group describes the topology of the knot complement. But first we provide a simpler and perhaps more intuitive definition of knot groups in terms of knot diagrams. Later, we will show that these groups are in fact the fundamental group of $\mathbb{R}^3 \setminus K$. For the reader who is interested in the knot groups as presented here, see [9].

6.1 Two Presentations

In this section we outline the methods of producing two knot presentations. We start from a knot diagram and produce a presentation of a group isomorphic to the knot group.

6.1.1 Dehn Presentations

To find the Dehn presentation of the knot group beginning with an oriented knot diagram, we separate the plane into regions separated by strands of the knot called faces. We next label the faces of the knot, including the unbounded region of the knot as a face. These faces will become the generators of the knot presentation. To find the relations of the presentation, look at the crossings of the knot. Any crossing is of the form shown in Figure 6.1 below. Here $x_i, x_j, x_k, x_l$ are the labeled faces of the knot. Starting with the
Figure 6.1: A labeled crossing for the Dehn Presentation. Here $x_i, x_j, x_k$ and $x_l$ are labeled faces of the knot diagram. A crossing of this type yields the relation $x_i x_j^{-1} x_k x_l^{-1}$ in the Dehn presentation.

face counterclockwise from the undercrossing strand leaving the crossing, $x_i$, and continuing counterclockwise around the crossing we obtain the relation $x_i x_j^{-1} x_k x_l^{-1}$. The knot group has generators $x_i$ obtained from labeling the faces of the knot, and relations, $r_i$ found from the crossings. However one more step is necessary to complete the group presentation. We must add the relation $x_i = 1$ for some $i$. This last step is equivalent to removing the generator $x_m$ from the presentation all together. We then obtain the following presentation for the knot group of knot $K$

$$G = \langle x_1, \ldots, x_n : r_1, \ldots, r_m, x_i \rangle.$$  

**Example 14.** Consider the oriented knot diagram with faces labeled $a, \ldots, g$ and crossings labeled $1, \ldots, 5$ as shown in Figure 6.2. The faces labeled $x_1, \ldots, x_7$ are the generators of our group, with relations given by the five crossings. As we have five crossings, we will obtain five relations from the crossings found below.

$$\begin{align*}
r_1 &= x_3 x_7^{-1} x_1 x_2^{-1} \\
r_2 &= x_3 x_2^{-1} x_4 x_1^{-1} \\
r_3 &= x_5 x_4^{-1} x_1 x_7^{-1} \\
r_4 &= x_3 x_4^{-1} x_6 x_7^{-1} \\
r_5 &= x_5 x_7^{-1} x_6 x_4^{-1}\
\end{align*}$$
Figure 6.2: Knot diagram with crossings and faces labeled to compute the Dehn presentation.

Observe that the presentation is greatly simplified by setting $x_7 = 1$. This gives us the following Dehn presentation for the knot group

$$G = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 : x_3x_7^{-1}x_1x_2^{-1}, x_3x_2^{-1}x_1x_4^{-1}, x_5x_4^{-1}x_1x_7^{-1}, x_3x_4^{-1}x_6x_7^{-1}, x_5x_7^{-1}x_6x_4^{-1}, x_7 \rangle.$$

Or equivalently, using a Tietze I move we obtain,

$$G = \langle x_1, x_2, x_3, x_4, x_5, x_6 : x_3x_1x_2^{-1}, x_3x_2^{-1}x_1x_4^{-1}, x_5x_4^{-1}x_1, x_3x_4^{-1}x_6, x_5x_6x_4^{-1} \rangle.$$

From the first relation we obtain the equality $x_2 = x_3x_1$. We remove generator $x_2$ by performing a Tietze II operation to get the following:

$$G = \langle x_1, x_3, x_4, x_5, x_6 : x_3x_1x_2^{-1}, x_3x_2^{-1}x_1x_4^{-1}, x_5x_4^{-1}x_1, x_3x_4^{-1}x_6, x_5x_6x_4^{-1} \rangle.$$

Using the equalities $x_1 = x_4x_5^{-1}$ and $x_6 = x_4x_3^{-1}$, we reduce further to obtain:

$$G = \langle x_3, x_4, x_5 : x_3x_5x_4^{-1}x_3^{-1}x_4x_5^{-1}x_4^{-1}, x_5x_4^{-1}x_3^{-1}x_4^{-1} \rangle.$$

We use one more Tietze move to simplify, this time, using the equality $x_5^{-1} = x_4x_3^{-1}x_4^{-1}$:

$$G = \langle x_3, x_4 : x_3x_4x_3x_4^{-2}x_3^{-1}x_4^{-1}x_3^{-1}x_4^{-2} \rangle.$$

This gives us a fairly simple presentation of the knot in Figure 6.2, with only two generators and one relation.
6.1.2 Wirtinger Presentations

Take an oriented knot diagram. Any arbitrary crossing may be represented as one of the two diagrams below, where the labels $x_j, x_k, x_l$ represent arcs of the knot.

![Diagram of Wirtinger crossings]

Figure 6.3: Here is an arbitrary labeled positive (right) and negative (left) crossing in a knot diagram, where $x_j, x_k, x_l, x_m, x_n$ and $x_p$ are labeled arcs of the knot. The relations for the Wirtinger presentation are obtained by reading around the crossing.

For each crossing in our knot diagram we define a path $a_i$ which travels counterclockwise around a small neighborhood of the crossing. Let $r_i$ be the signed product of strands that $a_i$ crosses, i.e. if the strand $x_i$ is pointing toward the crossing then $x_i$ is incorporated into the product and if the strand $x_i$ is pointing away from the crossing then $x_i^{-1}$ is incorporated into the product. Let us adopt the convention to begin our path at the strand of the undercrossing leaving the crossing. For example, from the negative crossing on the left in Figure 6.3 we obtain the relation $x_j^{-1}x_l^{-1}x_kx_l$ and from the positive crossing on the right we obtain the relation $x_p^{-1}x_mx_nx_m^{-1}$. We form a group as follows:

$$G = \langle x_1, \ldots, x_n : r_1, \ldots, r_n \rangle,$$

where $x_i$ is a strand of the knot and the relation $r_i$ is obtained by the above construction. However, one of these relations is necessarily in the normal subgroup generated by the other $n - 1$, we simplify our presentation by omitting one of the relations, yielding the Wirtinger presentation of the knot group,

$$G = \langle x_1, \ldots, x_n : r_1, \ldots, \hat{r}_i, \ldots r_n \rangle.$$
Example 15. Here we construct the Wirtinger presentation for the same example we examined with the Dehn presentation. Consider the oriented knot diagram with strands labeled $x_1, \ldots, x_5$ and crossings labeled $1, \ldots, 5$ as shown in Figure 6.4. For the first crossing of the knot diagram travel counterclockwise around crossing 1, and visit the strands $x_2$ (leaving), $x_4$ (approaching), $x_1$ (approaching), and $x_4$ (leaving). This yields the relation $r_1 = x_2^{-1}x_4x_1x_4^{-1}$. Continuing this method with the remainder of the crossings yields the following relations

$$
\begin{align*}
    r_1 &= x_2^{-1}x_4x_1x_4^{-1} \\
    r_2 &= x_5^{-1}x_2x_4x_2^{-1} \\
    r_3 &= x_3^{-1}x_1x_2x_1^{-1} \\
    r_4 &= x_4^{-1}x_5x_3x_5^{-1} \\
    r_5 &= x_1^{-1}x_3x_5x_3^{-1}.
\end{align*}
$$

This the following Wirtinger presentation:

$$
G = \langle x_1, x_2, x_3, x_4, x_5 : x_2^{-1}x_4x_1x_4^{-1}, x_5^{-1}x_2x_4x_2^{-1}, x_3^{-1}x_1x_2x_1^{-1}, x_4^{-1}x_5x_3x_5^{-1}, x_1^{-1}x_3x_5x_3^{-1} \rangle.
$$

Equivalently,

$$
G = \langle x_1, x_2, x_3, x_4, x_5 : x_2^{-1}x_4x_1x_4^{-1}, x_5^{-1}x_2x_4x_2^{-1}, x_3^{-1}x_1x_2x_1^{-1}, x_4^{-1}x_5x_3x_5^{-1} \rangle.
$$
To show that the Dehn and Wirtinger presentations of this knot are equivalent requires a bit of work. First, reduce this presentation via Tietze moves. Beginning with the equality
\[ x_5 = x_2 x_4 x_2^{-1} \]
we get the following presentation:
\[ G = \langle x_1, x_2, x_3, x_4 : x_2^{-1} x_4 x_1 x_4^{-1}, x_3^{-1} x_1 x_2 x_1^{-1}, x_4^{-1} x_2 x_4 x_2^{-1} x_3 x_2 x_4 x_2^{-1} x_2^{-1} \rangle. \]

Using the equality \( x_3 = x_1 x_2 x_1^{-1} \) we obtain
\[ G = \langle x_1, x_2, x_4 : x_2^{-1} x_4 x_1 x_4^{-1}, x_4^{-1} x_2 x_4 x_2^{-1} x_1 x_2 x_1^{-1} x_2 x_4 x_2^{-1} x_2^{-1} \rangle. \]

Finally we substitute \( x_2 = x_4 x_1 x_4^{-1} \) to simplify to a presentation with two generators and one relation.
\[ G = \langle x_1, x_4 : x_1 x_4 x_1^{-1} x_4^{-1} x_1 x_4 x_1^{-1} x_4^{-1} x_1 x_4 x_1^{-1} x_4^{-1} \rangle. \]

To show that the Dehn and Wirtinger presentations are a knot invariant it suffices to prove that the presentations are invariant under Reidemeister moves. For this see [9].

### 6.2 The Fundamental Group of the Knot Complement

Two knots are equivalent if they are ambient isotopic. Thus two equivalent knots have ambient isotopic knot complements and hence, isomorphic fundamental groups. Here we provide a proof that the Wirtinger presentation of the knot is isomorphic to the fundamental group of the knot complement, and is thus an invariant of the knot. A similar proof of this fact may be found in [5] or [9]. Note that the following proof makes use of Van Kampen’s Theorem, found in Appendix A.

**Theorem 22.** For a knot \( K \), the Wirtinger presentation described above is isomorphic to \( \pi_1(\mathbb{R}^3 \setminus K) \).

**Proof.** Given a smooth knot \( K \) let us consider the knot as lying along the \( z = 1 \) plane. Clearly the entire knot cannot lie on this plane, as it would intersect itself at a crossing. To account for this, in the neighborhood of a crossing, let the undercrossing strand of the knot jump down to the \( z = 0 \) plane, as shown in Figure 6.5 below.
Figure 6.5: This is an illustration of the type of embedding we wish to consider for our construction of the fundamental group of the knot complement. Here we see a trefoil knot which lies nearly completely on the $z = 1$ plane, with the undercrossings lying on the $z = 0$ plane.

Now, in order to consider an open space with a fundamental group isomorphic to that of the knot complement, we ‘thicken’ the knot. Let the knot be contained in a closed tube of radius $\epsilon$, as shown in Figure 6.6. By making $\epsilon$ sufficiently small, we can ensure that for any two strands of the knot, their $\epsilon$ tubes do not intersect. Thus the complement of the $\epsilon$ tube of the knot is a deformation retract of $\mathbb{R}^3 \setminus K$. In addition, because this $\epsilon$ tube of $K$ is closed, its complement is an open subspace of $\mathbb{R}^3$. Let us denote the $\epsilon$ tube of the knot by $N$ and its complement by $\mathbb{R}^3 \setminus N$. Note that $\mathbb{R}^3 \setminus N$ is a deformation retract of $\mathbb{R}^3 \setminus K$, and so their fundamental groups are isomorphic.

To find the fundamental group of the knot complement, we apply the Van Kampen theorem to this space. For this, separate $\mathbb{R}^3 \setminus N$ into two components, one component which lies strictly above the $z = 0$ plane called $X_1$, and one which lies below the $z = \frac{\epsilon}{2}$ plane called $X_2$. As usual, their intersection is denoted $X_0$.

To apply the Seifert Van Kampen theorem we must first compute the fundamental group of these spaces. We begin with $X_1$. This is a space which has $n$ handles corresponding to the overcrossing strands of the $n$ crossings in the knot. Thus this space deformation retracts to the $n$ leafed rose (see [9]), and the fundamental group of this space is isomorphic
to $F(n)$ as discussed in Appendix A. For convenience in notation (or maybe a bit of insight into the direction of the proof) we will denote the generators of this free group $x_1, \ldots, x_n$.

Figure 6.6: This figure is an illustration of the component space $X_1$ used to apply the Van Kampen theorem to the computation of the fundamental group of the knot complement. A generating loop of the fundamental group is shown. Note that the plane cuts the lower level of the knot in half.

Now considering the space $X_2$, we see that $X_2$ is mainly just the lower half of $\mathbb{R}^3$, the one difference is that there are some groves in the top due to the segments of $N$ containing undercrossings. However, these groves make no major changes to the topology of the space and $X_2$ is a deformation retract of $\mathbb{R}^3$. As $\pi_1(X_2) \cong \pi_1(\mathbb{R}^3)$, and $\mathbb{R}^3$ is a convex space, $X_2$ is simply connected.

Given that $X_0$ is the intersection of the portion of $\mathbb{R}^3 \setminus N$ lying above $z = 0$ and the portion of $\mathbb{R}^3 \setminus N$ lying below $z = \frac{1}{2}$ we can see that $X_0$ is a thickened plane with $n$ holes in it, each hole corresponding to an undercrossing. This space also deformation retracts to the $n$ leafed rose and so $\pi_1(X_0) = F(n)$.

To apply the Van Kampen theorem we must now see how the generators of $\pi_1(X_0)$ include in our two spaces $X_1$ and $X_2$. Clearly as $X_2$ is simply connected, the generator of $\pi_1(X_0)$ is equivalent to the trivial loop. Thus by our commutative diagram, the generator
of $\pi_1(X_0)$ maps to the trivial loop in $X$ through $j_1, i_1$ as well. In this sense, the inclusion of the generator of $\pi_1(X_0)$ in $X_1$ will give us the relations for $\pi_1(X)$.

Now, each generator of $\pi_1(X_0)$ is a loop around a hole in this thickened plane, and each of these holes is due to an undercrossing of the knot. This undercrossing is then connected to two overcrossings, as this knot naturally returns to the $z = 1$ plane on either side of the undercrossing. In addition, as this hole corresponds to an undercrossing, there is necessarily an overcrossing of the knot passing above it. Thus to find what is homotopic to this loop in $X_1$, we observe that the loop gets caught under the handles in four places, twice under the overcrossing segment passing over our undercrossing, and once from each overcrossing segment connected to our undercrossing segment. If we pay attention to the order that we come to each of these handles as we travel around our loop we get $x_ix_jx_kx_j$. However, keeping in mind that these loops have orientation, and our loop has direction as well, we obtain relations like $x_i^{-1}x_j^{-1}x_kx_j$, the same as our relations found from the Wirtinger presentation.

Figure 6.7: Here we see that a basis loop in $X_0$, when lifted into the space $X_1$, catches on the knot in four places. By gathering the loop to the basepoint between each time it passes under the knot we can see that the loop shown is homotopic to the loop described by a relation of the Wirtinger presentation. Note that the loop displayed above shows a midpoint of the homotopy.
To complete this proof, we must show that the fundamental group of the knot complement actually is the Wirtinger presentation. As was mentioned above, by our commutative Van Kampen diagram, the generators of $\pi_1(X_0)$ are trivial in $\pi_1(X)$. As the fundamental groups of our component spaces generate the fundamental group of our larger space, $X$, we can be sure that $\pi_1(X)$ is generated by $x_1, \ldots, x_n$ corresponding to the labels of overcrossing strands. In addition, we have that the generators of $\pi_1(X_0)$ are trivial in $\pi_1(X)$, thus giving the generators and relations of both $\pi_1(X)$ and the Wirtinger presentation.

Given that the fundamental group of the knot complement is a topological invariant of the knot, we may now use it without hesitation to derive knot polynomials. This somewhat lengthy process is described in the following chapter.
Chapter 7

The Fox Calculus and Alexander Ideals

This chapter uses ring theory extensively. For background in rings, ideals, and greatest common divisors, see Appendix B.

7.1 The Free Calculus

Definition 34. Given a group $G$ we define the group ring, $\mathbb{Z}G$ to be the ring consisting of elements of the form $\sum n_i g_i$, where $n_i = 0$ for all but finitely many $i$, and with addition and multiplication defined as follows:

\[
\text{addition : } \sum n_i g_i + \sum m_i g_i = \sum (n_i + m_i) g_i \\
\text{multiplication : } \sum n_i g_i \sum m_j h_j = \sum (n_i m_j) g_i h_j
\]

where $g, h \in G$ and $n, m \in \mathbb{Z}$

Lemma 23. The set $\mathbb{Z}G$ with addition and multiplication defined as above is a ring.

Proof. Here we give the key points proving that $\mathbb{Z}G$ is indeed a ring.

The group ring $\mathbb{Z}G$ is an abelain group under addition. Note that $0 = \sum 0 g_i$ acts as the additive identity as $\sum 0 g_i + \sum m_i g_i = \sum (0 + m_i) g_i = \sum m_i g_i = \sum m_i g_i + \sum 0 g_i$. The
inverse of $\sum n_i g_i$ is $\sum -n_i g_i$. Addition in the group ring is also associative as addition in $\mathbb{Z}$ is associative. Similarly, because addition in $\mathbb{Z}$ is commutative, the group ring forms an abelian group under addition.

The group ring is closed under multiplication. As we have defined multiplication, we have that $\sum n_i g_i \sum m_j h_j = \sum n_i m_j g_i h_j$. Because the integers are closed under multiplication $n_i m_j \in \mathbb{Z}$, and because $G$ is a group (and is therefore closed), $g_i h_j \in G$. In addition, because finitely many terms in each sum $\sum n_i g_i$ and $\sum m_j h_j$ are nonzero, there are finitely many nonzero terms in the product.

Note that if $e$ is the identity in the group $G$, then $1e$ acts as a multiplicative identity in $\mathbb{Z}G$. In addition, based on the definition of multiplication, we see that $\mathbb{Z}G$ is a commutative ring if and only if $G$ is an abelian group.

As defined, $\mathbb{Z}G$ also satisfies the required distribution properties. For example:

$$\sum k_j h_j (\sum n_i g_i + \sum m_i g_i) = \sum k_j h_j (\sum (n_i + m_i) g_i)$$
$$= \sum k_j (n_i + m_i) h_j g_i$$
$$= \sum (k_j n_i + k_j m_i) h_j g_i$$
$$= \sum k_j n_i h_j g_i + \sum k_j m_i h_j g_i$$
$$= \sum k_j h_j \sum n_i g_i + \sum k_j h_j \sum m_i g_i$$

So $\mathbb{Z}G$ is a ring. 

**Example 16.** First, a simple example: the group ring of the trivial group is $\mathbb{Z}$.

**Example 17.** Probably the most natural example of a group ring is the group ring of an infinite cyclic group. Let us denote the infinite cyclic group $\langle t \rangle$. Then elements of the group ring $\mathbb{Z}\langle t \rangle$ are elements of the form $\sum n_i t^i$ where finitely many of the $n_i \neq 0$. These ring elements are polynomials with integer coefficients and positive and negative powers of $t$. An alternative notation for this ring is $\mathbb{Z}[t, t^{-1}]$.

**Example 18.** For another less familiar example let’s consider the group ring $\mathbb{Z}\mathbb{S}_3$. Elements of this ring are of the form $\sum n_i g_i$ where $n_i \in \mathbb{Z}$ and $g_i \in \mathbb{S}_3$ describes a permutation on three letters which we write in cycle notation. Then $3(123) - 2(123) = 1(123)$ and
\[3(123) \cdot -2(12) = -6(13).\] Notice that even though \(S_3\) is a group of order 6, the group ring \(\mathbb{Z}S_3\) is infinite.

Next we will discuss a calculus on the group ring. This is typically referred to as the free calculus, or Fox calculus.

**Definition 35.** We define a derivative to be a map \(D : \mathbb{Z}G \rightarrow \mathbb{Z}G\) which satisfies the following properties:

(i) \(D(w_1 + w_2) = D(w_1) + D(w_2)\)

(ii) \(D(w_1w_2) = D(w_1)\epsilon(w_2) + w_1D(w_2)\)

where \(w_i \in \mathbb{Z}G\) and \(\epsilon\) is the augmentation map \(\epsilon : \mathbb{Z}G \rightarrow \mathbb{Z}\) is defined by \(\epsilon(\sum n_ig_i) = \sum n_i\).

Note that for elements of the group, the second property simplifies to:
\(D(g_1g_2) = D(g_1) + g_1D(g_2)\).

The second property of the derivative allows us to define the derivative of powers of a group element, and also products of group elements. We are thus able to define a derivative on the group ring \(\mathbb{Z}G\) by first defining how the derivative acts on generators of the group.

For our purposes we shall deal exclusively with derivatives of elements of the the group ring of the free group \(F(x_1, \ldots, x_n)\). We define the “partial derivative” as follows.

\[
\frac{\partial(x_i)}{\partial x_j} = \delta_{ij} \quad \text{where} \quad \delta_{ij} = \begin{cases} 0 & \text{for} \quad i \neq j \\ 1 & \text{for} \quad i = j \end{cases}
\]

At this point we outline some useful properties of the derivative.

- \(\frac{\partial(x^{-1})}{\partial x} = -x^{-1}\).

Observe that \(\frac{\partial(xx^{-1})}{\partial x} = \frac{\partial(1)}{\partial x} = 0\). In addition, by property (ii) of derivatives,

\[
\frac{\partial(xx^{-1})}{\partial x} = \frac{\partial(x)}{\partial x} + x\frac{\partial(x^{-1})}{\partial x} = 1 + x\frac{\partial(x^{-1})}{\partial x}.
\]

Hence \(1 + x\frac{\partial(x^{-1})}{\partial x} = 0\), so \(x\frac{\partial(x^{-1})}{\partial x} = -1\) and \(\frac{\partial(x^{-1})}{\partial x} = -x^{-1}\).
• \( \frac{\partial x^n}{\partial x} = \frac{x^{n-1}}{x-1} \)

This can be shown inductively in much the same way as the last property by using property (ii) of the derivative. For \( k \in \mathbb{N} \) let \( \frac{\partial(x^k)}{\partial x} = \frac{x^{k-1}}{x-1} \), then

\[
\frac{\partial(x^{k+1})}{\partial x} = \frac{\partial(x-x^k)}{\partial x} = \frac{\partial(x)}{\partial x} + x \frac{\partial(x^k)}{\partial x} = 1 + x \frac{x^{k-1}}{x-1} = \frac{x^{k+1}}{x-1}.
\]

An analogous argument holds for negative \( n \).

**Example 19.** Now that we have the basic tools for taking partial derivatives, here is a more complicated example.

\[
\frac{\partial(x^3yx^{-1}y^2)}{\partial x} = \frac{\partial(x^3)}{\partial x} + x^3 \frac{\partial(yx^{-1}y^2)}{\partial x} = \frac{x^3}{x-1} + x^3 \left( \frac{\partial(y)}{\partial x} + y \frac{\partial(x^{-1}y^2)}{\partial x} \right) = \frac{x^3}{x-1} + x^3 y \left( \frac{\partial(x^{-1})}{\partial x} + x^{-1} \frac{\partial(y^2)}{\partial x} \right) = \frac{x^3}{x-1} + x^3 y(-x^{-1}) + x^3 yx^{-1} \frac{\partial(y^2)}{\partial x} = \frac{x^3}{x-1} - x^3 yx^{-1}.
\]

### 7.2 Elementary Ideals

While the group ring and free calculus may seem removed from our discussion of knot invariants, it turns out to be a key tool in deriving knot polynomials.

**Definition 36.** Let \( G \) be a group such that \( G \cong \frac{F(n)}{\langle R \rangle} \) and let \( F = F(n) \). Consider the following chain of maps

\[
ZF \xrightarrow{\partial} ZF \xrightarrow{\gamma} ZG \xrightarrow{\alpha} ZG_{ab}
\]

where \( \gamma \) is the quotient map by the relations of \( G \) and \( \alpha \) is the abelianization map. The *Alexander matrix* of a group \( G \) with presentation \( \langle x_1, \ldots, x_n : r_1, \ldots, r_m \rangle \) is the matrix \( |a_{ij}| \) where \( a_{ij} = \alpha \gamma \left( \frac{\partial(r_i)}{\partial x_j} \right) \).
Example 20 (The dihedral group of order 3). Consider the group $D_3$ with presentation \( \langle x_1, x_2 : x_1^2, x_2^3, x_1x_2x_1x_2 \rangle \). Taking the derivatives we have

\[
\begin{align*}
\frac{\partial(x_1^2)}{\partial x_1} &= 1 + x_1 \\
\frac{\partial(x_1^3)}{\partial x_1} &= 0 \\
\frac{\partial(x_1x_2x_1x_2)}{\partial x_1} &= 1 + x_1x_2
\end{align*}
\]

\[
\begin{align*}
\frac{\partial(x_1^2)}{\partial x_2} &= 0 \\
\frac{\partial(x_1^3)}{\partial x_2} &= 1 + x_2 + x_2^2 \\
\frac{\partial(x_1x_2x_1x_2)}{\partial x_2} &= x_1 + x_1x_2x_1.
\end{align*}
\]

From our relations we find that $x_1x_2x_1 = x_2^{-1} = x_2^2$ so after applying $\gamma$ to our derivatives we simplify our equations to

\[
\begin{align*}
\gamma \left( \frac{\partial(x_1^2)}{\partial x_1} \right) &= 1 + x_1 \\
\gamma \left( \frac{\partial(x_1^3)}{\partial x_1} \right) &= 0 \\
\gamma \left( \frac{\partial(x_1x_2x_1x_2)}{\partial x_1} \right) &= 1 + x_1x_2
\end{align*}
\]

\[
\begin{align*}
\gamma \left( \frac{\partial(x_1^2)}{\partial x_2} \right) &= 1 + x_2 + x_2^2 \\
\gamma \left( \frac{\partial(x_1^3)}{\partial x_2} \right) &= x_1 + x_2^2.
\end{align*}
\]

Note that applying $\alpha$ to these expressions does not simplify anything. Thus we end with the Alexander matrix below:

\[
\begin{bmatrix}
1 + x_1 & 0 \\
0 & 1 + x_2 + x_2^2 \\
1 + x_1x_2 & x_1 + x_2^2
\end{bmatrix}
\]

Example 21. For our next example, we will calculate the Alexander matrix for the group with the presentation

\[
\langle x_1, x_2, x_3, x_4 : x_i = [x_{i+3}, x_{i+2}]x_{i+1}[x_{i+1}^{-1}, x_{i+3}] \rangle
\]

where $i = 1, 2, 3, 4$ and addition is mod 4. We take the derivative of the relation in its general form and for convenience, let $r_i = x_i - [x_{i+3}, x_{i+2}]x_{i+1}[x_{i+1}^{-1}, x_{i+3}]$ for $i = 1, 2, 3, 4$.

\[
\begin{align*}
\frac{\partial(r_i)}{\partial x_i} &= 1 \\
\frac{\partial(r_i)}{\partial x_{i+1}} &= -[x_{i+3}, x_{i+2}]^{-1} \\
\frac{\partial(r_i)}{\partial x_{i+2}} &= x_{i+3}x_{i+2}^{-1} - x_{i+3}[x_{i+2}]^{-1}x_{i+3}^{-1} + [x_{i+3}, x_{i+2}]^{-1}x_{i+1}x_{i+2}^{-1} - [x_{i+3}, x_{i+2}]^{-1}x_{i+1}x_{i+2}^{-2}x_{i+3} \\
\frac{\partial(r_i)}{\partial x_{i+3}} &= -1 + x_{i+3}x_{i+2}^{-1}x_{i+3}^{-1} - [x_{i+3}, x_{i+2}]^{-1}x_{i+1}x_{i+2}^{-1} + [x_{i+3}, x_{i+2}]^{-1}x_{i+1}x_{i+2}^{-1}x_{i+3}^{-1}.
\end{align*}
\]
Considering these derivatives under the \( \gamma \) map we have

\[
\begin{align*}
\gamma \left( \frac{\partial (r_i)}{\partial x_i} \right) &= 1 \\
\gamma \left( \frac{\partial (r_i)}{\partial x_{i+1}} \right) &= -1 \\
\gamma \left( \frac{\partial (r_i)}{\partial x_{i+2}} \right) &= x_{i+3} x_{i+1}^{-1} - x_{i+3} x_{i+2} x_{i+3}^{-1} + [x_{i+3}, x_{i+2}^{-1}] x_{i+2}^{-1} x_{i+3}^{-1} x_{i+2} - x_{i+3} x_{i+2}^{-1} x_{i+3}^{-1} x_{i+2} \\
\gamma \left( \frac{\partial (r_i)}{\partial x_{i+3}} \right) &= -1 + x_{i+3} x_{i+2}^{-1} x_{i+3}^{-1} - [x_{i+3}, x_{i+2}^{-1}] x_{i+1}^{-1} x_{i+2}^{-1} + x_i.
\end{align*}
\]

To simplify the next computation, we write this in the following equivalent way

\[
\begin{align*}
\gamma \left( \frac{\partial (r_i)}{\partial x_i} \right) &= 1 \\
\gamma \left( \frac{\partial (r_i)}{\partial x_{i+1}} \right) &= -[x_{i+3}, x_{i+2}^{-1}] \\
\gamma \left( \frac{\partial (r_i)}{\partial x_{i+2}} \right) &= x_{i+3} x_{i+1}^{-1} - [x_{i+3}, x_{i+2}^{-1}] x_{i+2}^{-1} + [x_{i+3}, x_{i+2}^{-1}] x_{i+1}^{-1} x_{i+2} - x_{i+3} x_{i+2}^{-1} x_{i+3}^{-1} x_{i+2} \\
\gamma \left( \frac{\partial (r_i)}{\partial x_{i+3}} \right) &= -1 + [x_{i+3} x_{i+2}^{-1} x_{i+3}^{-1} - [x_{i+3}, x_{i+2}^{-1}] x_{i+1}^{-1} x_{i+2}^{-1} + x_i.
\end{align*}
\]

Applying the abelianization map immediately gives

\[
\begin{align*}
\alpha \gamma \left( \frac{\partial (r_i)}{\partial x_i} \right) &= 1 \\
\alpha \gamma \left( \frac{\partial (r_i)}{\partial x_{i+1}} \right) &= -1 \\
\alpha \gamma \left( \frac{\partial (r_i)}{\partial x_{i+2}} \right) &= x_{i+3} x_{i+1}^{-1} - x_{i+2}^{-1} + x_{i+1} x_{i+2}^{-1} - x_i x_{i+3} x_{i+2} \\
\alpha \gamma \left( \frac{\partial (r_i)}{\partial x_{i+3}} \right) &= -1 + x_{i+2}^{-1} - x_{i+1} x_{i+2}^{-1} + x_i.
\end{align*}
\]

However, further inspection of this group shows that the abelianized group is cyclic. This can be seen by adding the commutators into the group presentation, which simplifies the relations to \( x_i = x_{i+1} \) for \( i = 1, 2, 3, 4 \). Thus we may send all four generators \( x_i \) to a single generator \( t \). This gives

\[
\begin{align*}
\alpha \gamma \left( \frac{\partial (r_i)}{\partial x_i} \right) &= 1 \\
\alpha \gamma \left( \frac{\partial (r_i)}{\partial x_{i+1}} \right) &= -1 \\
\alpha \gamma \left( \frac{\partial (r_i)}{\partial x_{i+2}} \right) &= -t + 2 - t^{-1} \\
\alpha \gamma \left( \frac{\partial (r_i)}{\partial x_{i+3}} \right) &= t - 2 + t^{-1}.
\end{align*}
\]

Taking into consideration that this is only for the general relation \( r_i \) we may substitute \( i = 1, \ldots, 4 \) to produce the necessary maps of the relations and obtain the Alexander matrix.
below
\[
\begin{bmatrix}
1 & -1 & -t + 2 - t^{-1} & t - 2 + t^{-1} \\
\ t - 2 + t^{-1} & 1 & -1 & -t + 2 - t^{-1} \\
-t + 2 - t^{-1} & t - 2 + t^{-1} & 1 & -1 \\
-1 & -t + 2 - t^{-1} & -t + 2 - t^{-1} & 1
\end{bmatrix}.
\]

**Definition 37.** Given an \(m \times n\) matrix \(A\) with coefficients in a ring \(R\), the \(k^{th}\) elementary ideal of \(A\) is defined by:

\[
E_k(A) = \begin{cases}
0 & \text{if } (n-k) > m \\
R & \text{if } (n-k) \leq 0
\end{cases}
\]

The ideal generated by the determinants of all \((n-k) \times (n-k)\) minors of \(A\)

In addition the elementary ideals form an ascending chain

\[
0 = E_1 \subseteq E_2 \subseteq \cdots \subseteq E_{n-1} \subseteq E_n = E_{n+1} = \cdots = R.
\]

As applied to knots, we will find the elementary ideals of the Alexander matrix. In this case the ring we work with is \(\mathbb{Z} G_{ab}\) where \(G_{ab}\) is the abelianized knot group.

We will compute some examples of these, but first, given that we are interested in Alexander matrices to extract this sequence of ideals, it is useful to define matrix equivalence such that \(A \sim A' \iff E_k(A) = E_k(A')\).

**Lemma 24.** The sequence of elementary ideals is invariant under the following matrix operations.

1. Permuting rows,
2. Permuting columns,
3. Adding a row of zeros,
4. Adding a linear combination of rows to an existing row,
5. Adding a linear combination of columns to an existing column,

6. Adding a new row and column of zeros with a 1 in the intersection. Note that by adding linear combinations of columns 6. is equivalent to adding a new row and column with a 1 in the intersection, and zeros in the remaining column entries:

\[
\begin{bmatrix}
A \\
a \\
1
\end{bmatrix} \rightarrow
\begin{bmatrix}
A & 0 \\
a & 1
\end{bmatrix}
\text{ or }
\begin{bmatrix}
A \\
0 \\
1
\end{bmatrix} \rightarrow
\begin{bmatrix}
A & 0 \\
0 & 1
\end{bmatrix}.
\]

Proof. It is necessary to show that these matrix operations leave the elementary ideals unchanged. Much of this relies on basic properties of determinants, and for the invariance of determinants based on matrix operations we will refer you to a linear algebra textbook, for instance [3]. Here we show the invariance of the elementary ideals under operation 3 and 6. These are less trivial as it changes the number of rows and columns from \(n\) to \(n + 1\), and thus instead of considering the determinants of \((n - k) \times (n - k)\) minors, we are now considering the determinants of \((n - k + 1) \times (n - k + 1)\) minors.

Let \(A\) be the original Alexander matrix and let \(A'\) be the new Alexander matrix obtained by operation 6. Since they are equivalent by column operations, we consider the simpler version of operation 6: \(A \rightarrow \begin{bmatrix} A & 0 \\ a & 1 \end{bmatrix}\). Note that if \(B\) is an \((n - k) \times (n - k)\) minor of \(A\), then \(\det(B) \in E_k(A')\) as \(\begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix}\) is an \((n - k + 1) \times (n - k + 1)\) minor of \(A'\). Thus we have the inclusion \(E_k(A) \subseteq E_k(A')\). By inspection, we find that there are two other types of minors in our new matrix \(A'\). There are \((n - k + 1) \times (n - k + 1)\) minors which contain a row or a column of zeros, and there are \((n - k + 1) \times (n - k + 1)\) minors of our original matrix \(A\).

We need not consider the minors of the first type, as their determinant is zero and they therefore make no contribution to generating the ideal. Note that the determinants of the \((n - k + 1) \times (n - k + 1)\) minors of our matrix \(A\) are contained in \(E_{k-1}(A)\). As the elementary ideals form an ascending chain we have that \(E_{k-1}(A) \subseteq E_k(A)\). Hence the determinants of these larger minors are contained in \(E_k(A)\) and thus \(E_k(A') \subseteq E_k(A)\). As we have inclusions in both directions, it follows that \(E_k(A') = E_k(A)\). \(\square\)

These matrix operations are very useful in calculating the elementary ideals, as taking
determinants is easiest when matrices are as reduced as possible. Let’s continue with our Alexander matrices found in the examples above and determine their chain of elementary ideals. Starting with Example 20, where \( G = D_3 \), we have the following \( 3 \times 2 \) Alexander matrix:

\[
\begin{bmatrix}
1 + x_1 & 0 \\
0 & 1 + x_2 + x_2^2 \\
1 + x_1x_2 & x_1 + x_2^2
\end{bmatrix}.
\]

By definition, \( E_0 = 0 \), thus our first nontrivial ideal is \( E_1 \) which is generated by the determinants of all \((n - 1) \times (n - 1)\) minors. Here \( n = 2 \), so \( E_1 \) is generated by the determinants of all \( 1 \times 1 \) minors. Therefore \( E_1 = \langle 1 + x_1, 1 + x_2 + x_2^2, 1 + x_1x_2, x_1 + x_2^2 \rangle \).

Since \( n - 2 \leq 0 \) then \( E_2 = R = \mathbb{Z}(D_3)_{ab} \). This gives us our chain of elementary ideals for the group \( D_3 \)

\[
E_0 = 0 \subseteq E_1 = \langle 1 + x_1, 1 + x_2 + x_2^2, 1 + x_1x_2, x_1 + x_2^2 \rangle \subseteq E_2 = \mathbb{Z}(D_3)_{ab}
\]

Returning to example 21 we have the following Alexander matrix

\[
\begin{bmatrix}
1 & -1 & -t + 2 - t^{-1} & t - 2 + t^{-1} \\
t - 2 + t^{-1} & 1 & -1 & -t + 2 - t^{-1} \\
-t + 2 - t^{-1} & t - 2 + t^{-1} & 1 & -1 \\
-1 & -t + 2 - t^{-1} & -t + 2 - t^{-1} & 1
\end{bmatrix}.
\]

In this case it is to our advantage to reduce this matrix by the operations found in
Lemma 24.

The elementary ideal $E$ generated by the determinants of all one minor with a nonzero determinant. Therefore the Alexander matrix. As we have a column of zeros in our reduced matrix, there is only one minor with a nonzero determinant. Therefore $E_0 = 0$. For $k = 1$, $E_1$ is the ideal generated by the determinants of all $3 \times 3$ minors of the Alexander matrix. As we have a column of zeros in our reduced matrix, there is only one minor with a nonzero determinant. Therefore $E_1 = \langle (-t + 3 - t^{-1})(-t + 1 - t^{-1})^2 \rangle$. The elementary ideal $E_2$ has $k = 2$ and is thus generated by the determinants of $2 \times 2$ minors. Therefore $E_2 = \langle (-t + 1 - t^{-1})^2, -(t + 3 - t^{-1})(-t + 1 - t^{-1}) \rangle$. The ideal $E_3$ is the ideal generated by the determinants of all $1 \times 1$ minors of our Alexander matrix and so $E_3 = \langle -t + 1 - t^{-1}, -t + 3 - t^{-1} \rangle$. For $k \geq 4$, $n - k \leq 0$ and so $E_k = \mathbb{Z}(t)$ where $G = \langle x_1, x_2, x_3, x_4 : x_i = [x_{i+3}, x_{i+2}^{-1}, x_{i+1}^{-1}, x_{i+3}] \rangle$. In summary, we have the following:

\[
\begin{bmatrix}
1 & -1 & -t + 2 - t^{-1} & t - 2 + t^{-1} \\
-t + 1 - t^{-1} & 0 & -t + 1 - t^{-1} & 0 \\
-t + 1 - t^{-1} & 0 & t - 1 + t^{-1} & 0 \\
-1 & -t + 2 - t^{-1} & t - 2 + t^{-1} & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -1 & -t + 2 - t^{-1} & t - 2 + t^{-1} \\
0 & 0 & 0 & 0 \\
1 & -1 & -t + 2 - t^{-1} & t - 2 + t^{-1} \\
-t + 1 - t^{-1} & 0 & t - 1 + t^{-1} & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -1 & -t + 3 - t^{-1} & 0 \\
-t + 1 - t^{-1} & 0 & 0 & 0 \\
0 & -t + 1 - t^{-1} & 0 & 0
\end{bmatrix}
\]
elementary ideals for our group presentation

\[ E_k = \begin{cases} 
0 & \text{for } k = 0 \\
\langle -t + 3 - t^{-1} \rangle & \text{for } k = 1 \\
\langle -t + 1 - t^{-1} \rangle, \langle -t + 3 - t^{-1} \rangle & \text{for } k = 2 \\
\langle -t + 1 - t^{-1}, -t + 3 - t^{-1} \rangle & \text{for } k = 3 \\
\mathbb{Z}(t) & \text{for } k \geq 4.
\end{cases} \]

In addition to the ideals being invariant under matrix operations, we would like to know that isomorphic groups produce isomorphic chains of elementary ideals. We do this with the following proposition.

**Proposition 25.** The elementary ideals are invariant under group presentation equivalence.

**Proof.** To prove this we will show that the ideals are invariant under Tietze transformations.

**Tietze I:**

We would like to show that ideals remain unchanged under the following presentation map

\[ \langle X : R \rangle \overset{I}{\to} \langle X : R \cup s \rangle \]

where \( s \in \langle \langle R \rangle \rangle \). Given that \( a_{ij} = \alpha \gamma \left( \frac{\partial r_i}{\partial x_j} \right) \). Our Alexander matrix remains unchanged other than a new row due to \( \alpha \gamma \left( \frac{\partial s}{\partial x_j} \right) \).

As the new relation \( s \) is in the normal closure of \( R = \{r_1, r_2, \ldots r_n\} \), \( s \) is a product of conjugates of these relations. Therefore, we may write it as follows

\[ s = \prod_{k=1}^{p} g_k r_{i_k}^m g_k^{-1} \]

where \( g_k \in F(X) \). Note that in this product a given relation \( r_i \) may occur more than once.

By the definition of the derivative we have

\[ \frac{\partial (s)}{\partial x_j} = \frac{\partial (g_{1} r_{i_1}^{m_{1}} g_{1}^{-1})}{\partial x_j} + g_{k} r_{i_k}^{m_{k}} g_{k}^{-1} \cdot \frac{\partial (g_{2} r_{i_2}^{m_{2}} g_{2}^{-1})}{\partial x_j} + \cdots + \prod_{k=1}^{p} g_{k} r_{i_k}^{m_{k}} g_{k}^{-1} \cdot \frac{\partial (g_{p} r_{i_p}^{m_{p}} g_{p}^{-1})}{\partial x_j}. \]
However, because \( \gamma(r_i) = 1 \), this reduces as follows

\[
\gamma \left( \frac{\partial(s)}{\partial x_j} \right) = \gamma \sum_{k=1}^{p} \left( \frac{\partial(g_k r_{ik}^m g_k^{-1})}{\partial x_j} \right).
\]

By the definition of the free derivative, we may simplify this even further, separating over the products in the derivative

\[
\gamma \left( \frac{\partial(s)}{\partial x_j} \right) = \gamma \sum_{k=1}^{p} \left( \frac{\partial(g_k)}{\partial x_j} + \frac{\partial(r_{ik}^m)}{\partial x_j} + \frac{\partial(g_k^{-1})}{\partial x_j} \right).
\]

Because \( \gamma \left( \frac{r_{ik}^m}{r_{ik}^{-1}} \right) = 1 \), we have \( \gamma \left( \frac{r_{ik}^m}{r_{ik}^{-1}} \right) = m_k \) and thus

\[
\gamma \left( \frac{\partial(s)}{\partial x_j} \right) = \sum_{k=1}^{p} m_k \gamma(g_k) \gamma \left( \frac{\partial(r_{ik})}{\partial x_j} \right).
\]

Applying \( \alpha \) yields

\[
\alpha \gamma \left( \frac{\partial(s)}{\partial x_j} \right) = \sum_{k=1}^{p} m_k \alpha \gamma(g_k) \alpha \gamma \left( \frac{\partial(r_{ik})}{\partial x_j} \right).
\]

Thus the row in the Alexander matrix due to adding a new relation is a linear combination of the other rows. This corresponds to adding a row of zeros to the original Alexander matrix (operation 3) and adding to it a linear combination of rows (operation 4). By Lemma 24, the Alexander matrix produced from the new group presentation \( \langle X : R \cup s \rangle \) is equivalent to the matrix produced from the original presentation and therefore the ideals are unchanged.
Tietze II:
We would like to show that ideals remain unchanged under the following presentation map
\[ \langle X : R \rangle \xrightarrow{II} \langle X \cup y : R \cup yw^{-1} \rangle \]
where \( w \in F(X) \). Under this transformation, the Alexander matrix will remain unchanged
other than one new row and column due to \( \alpha \gamma \left( \frac{\partial (yw^{-1})}{\partial x} \right) \) and \( \alpha \gamma \left( \frac{\partial r}{\partial y} \right) \) respectively.
Note that for all relations \( r_i \in R \), \( r_i \in F(X) \) and thus \( \alpha \gamma \left( \frac{\partial r}{\partial y} \right) = 0 \). Also note that
\( \alpha \gamma \left( \frac{\partial (yw^{-1})}{\partial y} \right) = 1 \). Thus this presentation transformation is equivalent to matrix opera-
tion 6. in Lemma 24 and the elementary ideals are unchanged.

7.3 Knot Polynomials

**Definition 38.** Let \( K \) be a tame knot and let \( E_k \) be the \( k \)th elementary ideal obtained
from \( \pi_1(\mathbb{R}^3 \setminus K) \). Then the \( k \)th knot polynomial of \( K \), denoted \( \Delta_k(K) \) is the generator of
the smallest principal ideal containing \( E_k \).

**Corollary 26.** The knot polynomials are knot invariants.

**Proof.** We have shown in Proposition 25 that the ideals are invariant under presentation
equivalence. In addition, \( \pi_1(\mathbb{R}^3 \setminus K) \) is an invariant of the knot. Thus the knot polynomials
are an invariant of the knot. \( \square \)

The Alexander polynomial is the first knot polynomial and is typically denoted \( \Delta(K) \).
While many of the results of this chapter are relevant to all of the knot polynomials
obtained from the elementary ideals, our discussion is mainly concerned with the Alexander
polynomial. The following two theorems greatly reduce the computation required to obtain
knot polynomials.

**Theorem 27.** The abelianized knot group is infinite cyclic.

**Proof.** For this we refer to the Wirtinger presentation. Given a crossing, as shown in
Figure 7.1 the relation given by that crossing will be of the form \( x_{i+1}x_{k(i)}x_i^{-1}x_{k(i)}^{-1} \) or
similarly, \( x_{i+1}x_{k(i)}^{-1}x_i^{-1}x_{k(i)} \).
Figure 7.1: Recall that for an arbitrary crossing like the two pictured above, the relation from the Wirtinger presentation is \( x_j^{-1} x_k^{-1} x_i x_k \) if it is a negative crossing (left) and \( x_p^{-1} x_m x_n x_m^{-1} \) if it is a positive crossing (right).

Thus the knot group for any knot has a presentation of the form

\[
\pi(\mathbb{R}^3 \setminus K) = \langle x_1, \ldots, x_n : x_{i+1} x_k^{\pm 1} x_i^{-1} x_k^{-1} \rangle \quad \text{for } i = 1, \ldots, n.
\]

The abelianized knot group is isomorphic to the quotient group of our knot group with the commutator subgroup, or equivalently, adding the generators of the commutator subgroup to our relations. This gives

\[
\alpha \left( \pi(\mathbb{R}^3 \setminus K) \right) = \langle x_1, \ldots, x_n : x_{i+1} x_k^{\pm 1} x_i^{-1} x_k^{-1}, [x_i, x_j] \rangle \quad \text{for } i, j = 1, \ldots, n.
\]

The previous relations, reduce to the following equivalent presentation of the abelianized knot group

\[
\alpha \left( \pi(\mathbb{R}^3 \setminus K) \right) = \langle x_1, \ldots, x_n : x_{i+1} x_i^{-1}, [x_i, x_j] \rangle \quad \text{for } i, j = 1, \ldots, n.
\]

From this new set of relations it is clear that \( x_{i+1} x_i^{-1} = 1 \) and thus \( x_{i+1} = x_i \) in the abelianized knot group. However, this relation alone demands that the group not only be commutative, but also cyclic. Thus our relations \([x_i, x_j]\) are redundant and can be removed through Tietze transformations, which gives

\[
\alpha \left( \pi(\mathbb{R}^3 \setminus K) \right) = \langle x_1, \ldots, x_n : x_{i+1} x_i^{-1} \rangle \quad \text{for } i = 1, \ldots, n.
\]

This presentation is clearly isomorphic to the infinite cyclic group.
Theorem 28. [5] The Alexander matrix of any Wirtinger presentation of a knot group is equivalent to the matrix with any column replaced by a column of zeros.

From the Wirtinger presentation of the knot, we need only to take derivatives of the relations with respect to \( n - 1 \) generators to obtain a matrix equivalent to the Alexander matrix of the knot, as columns in the Alexander matrix correspond to generators of the Wirtinger presentation. We use this shortcut in several of the following examples.

Before we compute the Alexander polynomial of some example knots it is important to know that Alexander polynomials are only unique up to multiplication by \( \pm t^k \), units in the group ring. This is explained further in Appendix B.

### 7.4 Examples

We have just completed the construction of the following machinery.

1. Starting with a knot, we produce a knot diagram.
2. Given the knot diagram, we produce its knot group via the Wirtinger presentation.
3. We then take derivatives to find elementary ideals.
4. Finally we use the elementary ideals to obtain the Alexander polynomial.

We now outline the entire process for a few knots.

#### 7.4.1 The Unknot

Some topological intuition will tell you that the fundamental group of the unknot complement is infinite cyclic. Here we illustrate another method of obtaining the fundamental group of the unknot complement by considering an unknot diagram with 2 crossings. This allows us to calculate the knot group for the unknot by the Wirtinger presentation and gives us relations to consider when constructing the Alexander polynomial.

Consider the unknot diagram in Figure 7.2 with strands labeled \( x_1 \) and \( x_2 \) and crossings labeled 1 and 2 as shown.
Using the Wirtinger presentation of the knot group we obtain the following two relations:

\[ r_1 = x_2^{-1}x_1x_1^{-1} = x_2^{-1}x_1 \quad \text{and} \quad r_2 = x_1^{-1}x_1^{-1}x_2x_1. \]

These relations give us the following presentation for the fundamental group of the unknot complement.

\[ \pi_1(\mathbb{R}^3 \setminus 0_1) = \langle x_1, x_2 : x_2^{-1}x_1, x_1^{-1}x_1^{-1}x_2x_1 \rangle \]

As we are using the Wirtinger presentation, we may discard one of the relations, as it is the consequence of the other relation. We only need to consider relation 2. From Theorem 28 the Alexander matrix from this group presentation is equivalent to the matrix with one column replaced with a column of zeros. Taking the derivative of the second relation with respect to \( x_2 \) we obtain

\[ \frac{\partial(x_1^{-1}x_2^{-1}x_2x_1)}{\partial x_2} = 1. \]

This remains unchanged after applying \( \gamma \) and the abelianization maps, hence the Alexander matrix for the unknot is \( [1] \) and thus the chain of ideals for the unknot are as follows:

\[ E_k(A) = \begin{cases} 0 & \text{for } k = 0 \\ \langle 1 \rangle = \mathbb{Z}\langle t \rangle & \text{for } k \geq 1 \end{cases} \]
And so the Alexander polynomial for the unknot is the generator of the smallest principle ideal containing $R$, $\Delta(0_1) = 1$.

### 7.4.2 Trefoil Knot

![Knot Diagram](knot-diagram.png)

**Figure 7.3:** Knot diagram for the trefoil knot used in computing the Alexander matrix, elementary ideals, and Alexander polynomial.

Consider the trefoil knot diagram above with strands labeled $x_1, x_2, x_3$ and crossings 1, 2, 3 as shown. Using the Wirtinger presentation of the knot and going counterclockwise around the crossings we have the following relations:

$$r_1 = x_2^{-1}x_1x_3^{-1} \quad r_2 = x_1^{-1}x_3x_2^{-1} \quad r_3 = x_3^{-1}x_2x_1^{-1}.$$  

As we are using the Wirtinger presentation, we know that any one relation is a consequence of the other two, thus we discard the second relation. This gives us the following group presentation for the knot group

$$\pi_1(\mathbb{R}^3 \setminus 3_1) = \langle x_1, x_2, x_3 : x_2^{-1}x_1x_3^{-1}, x_3^{-1}x_2x_1^{-1} \rangle.$$  

We use Tietze transformations to simplify this group presentation, namely substituting $x_3 = x_2x_1x_2^{-1}$ gives us the new group presentation,

$$\pi_1(\mathbb{R}^3 \setminus 3_1) = \langle x_1, x_2 : x_2^{-1}x_1x_2x_1^{-1}x_2^{-1}x_1^{-1} \rangle.$$
Equivalently,
\[ \pi_1(\mathbb{R}^3 \setminus 3_1) = \langle x_1, x_2 : x_1x_2x_1 = x_2x_1x_2 \rangle. \]

In order to find the Alexander matrix of the trefoil knot, we need only the single relation 
\[ x_1x_2x_1 - x_2x_1x_2 = 0. \]

Taking derivatives with respect to \( x_1 \) we obtain 
\[ \frac{\partial(x_1x_2x_1 - x_2x_1x_2)}{\partial x_1} = 1 + x_1x_2 - x_2. \]

Note that the map \( \gamma \) (which mods out by the group relations) does not simplify this expression. The abelianization map then has the effect of sending both \( x_1 \) and \( x_2 \) to the generator of the infinite cyclic group, \( t \), giving us the following:
\[ \alpha \gamma \left( \frac{\partial(x_1x_2x_1 - x_2x_1x_2)}{\partial x_1} \right) = t^2 - t + 1. \]

The Alexander matrix for the trefoil knot is matrix equivalent to
\[ A = \begin{bmatrix} t^2 - t + 1 & 0 \end{bmatrix}. \]

We now take determinants of the minors to find the chain of ideals.
\[ E_k(A) = \begin{cases} 
0 & \text{for } k = 0 \\
\langle t^2 - t + 1 \rangle & \text{for } k = 1 \\
\mathbb{Z}(t) & \text{for } k \geq 1 
\end{cases} \]

where \( G \) is the knot group of the trefoil. The Alexander polynomial is then the generator of the smallest principle ideal containing \( E_1 = \langle t^2 - t + 1 \rangle \) and thus \( \Delta(3_1) = t^2 - t + 1. \)

Note that when multiplied by \( t^{-1} \), this is the same as the Alexander polynomial for the trefoil knot obtained by skein relations in Chapter 4.

### 7.4.3 Figure Eight Knot

Consider the figure eight diagram found in Figure 7.4 with strands labeled \( x_1, x_2, x_3, x_4 \) and crossings labeled 1, 2, 3, 4 as shown. Using the Wirtinger presentation of the knot and going counterclockwise around the crossings we have the following relations
\[
\begin{align*}
    r_1 &= x_1^{-1}x_2^{-1}x_4x_2 \\
    r_2 &= x_3^{-1}x_4^{-1}x_2x_4 \\
    r_3 &= x_4^{-1}x_1x_3x_1^{-1} \\
    r_4 &= x_2^{-1}x_3x_1x_3^{-1}.
\end{align*}
\]
This gives us the group presentation for the knot group of the figure eight knot

\[ \pi_1(\mathbb{R}^3 \setminus \overline{K}) = \langle x, y : yx^{-1}xyy^{-1} = x^{-1}yxy^{-1}x \rangle. \]

Looking at the knot group as a group ring, we can see it has the single relation

\[ yx^{-1}xy^{-1} - x^{-1}yxy^{-1}x = 0. \]

Taking derivatives with respect to \( x \) we obtain

\[ \frac{\partial(yx^{-1}xy^{-1} - x^{-1}yxy^{-1}x)}{\partial x} = yx^{-1} + yx^{-1}y + x^{-1} - x^{-1}y - x^{-1}yxy^{-1}. \]
Note that the map $\gamma$ which mods out by the group relations does not simplify our derivative. The abelianization map has the effect of sending both $x$ and $y$ to the generator of the infinite cyclic group, $t$, giving us:

$$\alpha \gamma \left( \frac{\partial (yx^{-1}xy^{-1} - x^{-1}xy^{-1}x)}{\partial x} \right) = t - 3 + t^{-1}.$$

Then the Alexander matrix for the figure eight knot is equivalent to:

$$A = \begin{bmatrix} t - 3 + t^{-1} & 0 \end{bmatrix}.$$

Taking determinants of the minors to find the chain of ideals yields

$$E_k(A) = \begin{cases} 0 & \text{for } k = 0 \\ \langle t - 3 + t^{-1} \rangle & \text{for } k = 1 \\ \mathbb{Z}G & \text{for } k \geq 1 \end{cases}.$$

The Alexander Polynomial is the generator of the smallest principle ideal containing $E_1 = \langle t - 3 + t^{-1} \rangle$ and thus $\Delta(4_1) = t - 3 + t^{-1}$. Note that this polynomial is the same Alexander polynomial we computed for the figure eight knot via resolving trees in Chapter 4.

### 7.4.4 Turk Head’s Knot

Consider the Turk Head’s knot diagram found in Figure 7.5 with strands labeled $a, \ldots, h$ and crossings labeled $1, \ldots, 8$. Using the Wirtinger presentation of the knot and going counterclockwise around the crossings we have the following relations:

\[
\begin{align*}
    r_1 &= f^{-1}b^{-1}ab \\
    r_2 &= g^{-1}c^{-1}bc \\
    r_3 &= h^{-1}d^{-1}cd \\
    r_4 &= e^{-1}a^{-1}da \\
    r_5 &= c^{-1}fe^{-1} \\
    r_6 &= d^{-1}gf^{-1} \\
    r_7 &= a^{-1}gh^{-1} \\
    r_8 &= b^{-1}ehe^{-1}.
\end{align*}
\]

We use Tietze transformations to simplify the group presentation for the Turk’s head knot. This gives us the following reduced presentation

$$\pi(\mathbb{R}^3 \setminus K) = \langle x_1, x_2, x_3, x_4 : x_i = [x_{i+3}, x_{i+2}^{-1}]x_{i+1}[x_{i+1}^{-1}, x_i] \rangle$$
where $i = 1, \ldots, 4$ and addition is mod 4.

Note that this is the group presentation we started with in Example 21 computing Alexander matrices and elementary ideals. Therefore, we already showed the elementary ideals for this knot to be:

\[
E_k = \begin{cases} 
0 & \text{for } k = 0 \\
\langle (t - 3 + t^{-1})(t - 1 + t^{-1})^2 \rangle & \text{for } k = 1 \\
\langle (t - 3 + t^{-1})(t - 1 + t^{-1}), (t - 1 + t^{-1}) \rangle & \text{for } k = 2 \\
\mathbb{Z}\langle t \rangle & \text{for } k \geq 3
\end{cases}
\]

The Alexander polynomial is the generator of the smallest principle ideal containing $E_1$ and thus the smallest principle ideal containing $\langle (t - 3 + t^{-1})(t - 1 + t^{-1})^2 \rangle$. Therefore $\Delta(t) = (t - 3 + t^{-1})(t - 1 + t^{-1})^2$.

Note that all of the different knots in these examples have distinct knot polynomials. As we have shown previously that the knot group is an invariant of the knot, and that the ideals are an invariant of the knot group, we come through this long chain of constructions, to the convenient fact that the knot polynomials are an invariant of the knot. Thus the knots with different knot polynomials given in the examples above must in fact be different.
knots! However, this is by no means a total invariant. For an example of a nontrivial knot with the Alexander polynomial of the unknot and a pair of knots with identical knot polynomials and distinct sets of elementary ideals see [5].

7.5 Properties of the Alexander polynomial

In this section we aim to prove two main properties of the Alexander polynomial. The first is that for any knot $K$, $\Delta_K(1) = \pm 1$. In the examples we’ve looked at thus far, the coefficient of the highest term of the polynomial is the same as the coefficient of the lowest term of the polynomial. Our second aim in this section is to show that for some $n \in \mathbb{Z}$, $\Delta(t) = t^n \Delta(t^{-1})$.

Note that $\Delta_K(1)$ is the image of $\Delta_K(t)$ under the augmentation map. Thus as $\langle \Delta_K(t) \rangle = E_1$, by the definition of the knot polynomials, it follows that $\epsilon(\langle \Delta(t) \rangle) = \epsilon(\langle \Delta(t) \rangle) = \epsilon(E_1)$. Therefore, the statement that $\Delta(1) = \pm 1$ is equivalent to saying that $\epsilon(E_1) = \mathbb{Z}$. We shall give a proof of this alternative statement, leaving $\Delta(1) = \pm 1$ as a corollary.

**Theorem 29.** For any tame knot $K$, $\epsilon(E_1(K)) = \mathbb{Z}$.

**Proof.** Consider the group presentation for the infinite cyclic group $\langle x : \rangle$. Let the Alexander matrix formed from this presentation be $A'$ and note that $A'$ is the empty matrix. Because our Alexander matrix has no columns, $n = 0$ and the elementary ideals are formed from the extreme cases, not from determinants of minors. More specifically,

$$E_k(A') = \begin{cases} 0 & \text{for } k = 0 \\ R = \mathbb{Z}\langle t \rangle & \text{for } k \geq 1. \end{cases}$$

Thus, under the augmentation map, we can see that this matrix gives us the desired outcome

$$\epsilon(E_k(A')) = \begin{cases} 0 & \text{for } k = 0 \\ \mathbb{Z} = \langle 1 \rangle & \text{for } k \geq 1. \end{cases}$$

We will show that for any Alexander matrix, $A$ of a knot group, $\epsilon(E_k(A)) = \epsilon(E_k(A'))$.

First observe that $A = |a_{ij}|$ and $\epsilon(|a_{ij}|) = |\epsilon(a_{ij})|$. This gives us that $\epsilon(E_k(A)) = \ldots$
If $E_k (\epsilon(A))$ and similarly, $\epsilon (E_k (A')) = E_k (\epsilon(A'))$. The proof reduces to showing that $\epsilon(A) = \epsilon(A')$. For this we consider another group presentation of the infinite cyclic group, one more closely resembling the abelianized knot group. Let $\langle X : R \rangle$ be the knot group for our knot $K$. Then the abelianized knot group can be written as $\langle X : R, [x_i, x_j] \rangle$ for $i, j = i, \ldots, n$. Because the abelianized knot group is infinite cyclic, this group is isomorphic to the presentation $\langle x : \rangle$ that generates the matrix $A'$. Thus the matrix generated by this group presentation, $A''$, and $A'$ are equivalent matrices. Note that $A''$ is identical to $A$ excepting some new rows due to the relations in the commutator subgroup. We now take the derivatives of these new relations to determine their contribution to the Alexander matrix $A''$:

$$\frac{\partial (x_i x_j x_i^{-1} x_j^{-1})}{\partial x_k} = (1 - x_i x_j x_i^{-1}) \delta_{ik} + \left( x_i - x_i x_j x_i^{-1} x_j^{-1} \right) \delta_{jk}.$$ 

Considering this map under the augmentation map gives us the relevant differences between $\epsilon(A)$ and $\epsilon(A'')$. Now,

$$\epsilon \left( \frac{\partial (x_i x_j x_i^{-1} x_j^{-1})}{\partial x_k} \right) = \epsilon \left( (1 - x_i x_j x_i^{-1}) \delta_{ik} + \left( x_i - x_i x_j x_i^{-1} x_j^{-1} \right) \delta_{jk} \right) = 0,$$

thus $\epsilon(A)$ and $\epsilon(A'')$ are related as follows

$$\epsilon(A'') = \begin{bmatrix} ||\epsilon(a_{ij})|| & 0 \\ 0 & ||\epsilon(a_{ij})|| = \epsilon(A). \end{bmatrix}$$

By Lemma 24, $A''$ and $A$ have the same ideals under the augmentation map. As matrix $A'$ is matrix equivalent to $A''$, for any knot group,

$$\epsilon \left( E_k (A') \right) = \begin{cases} 0 & \text{for } k = 0 \\ \mathbb{Z} = \langle 1 \rangle & \text{for } k \geq 1 \end{cases}$$

and $\epsilon(E_1 (K)) = \mathbb{Z}$ for any knot $K$. \hfill \Box

**Corollary 30.** $\Delta_K(1) = \pm 1$. \hfill \Box

**Theorem 31.** For some $n \in \mathbb{Z}$, $\Delta(t) = t^n \Delta(t^{-1})$. 
This theorem can be proved in several ways. Using the material in Chapter 3 we can examine the changes to the Seifert matrix achieved by taking the mirror image of knots. While this change switches the role of $t$ and $t^{-1}$ in the determinant producing the Alexander polynomial, the output is the same [4]. This result can also be shown by showing that two types of knot presentations are dual presentations [5]. However, both of these methods employ tools we have not discussed, so we omit this proof.

These results shows us a shortcoming of our invariant, as it cannot distinguish mirror images of knots. To distinguish knots which vary in this fashion requires different invariants.
Chapter 8

Conclusion

We have shown three methods of obtaining the Alexander polynomial. First, using orientable surfaces whose boundary is the knot, we showed how the homology of these surfaces leads to the Alexander polynomial. We then showed that the Alexander polynomial can be obtained using skein relations. Last, we showed a method of obtaining the Alexander polynomial through the fundamental group of the knot complement. While we do not prove that these methods produce the same invariant, it should be convincing that the same knots examined by different methods here produce the same polynomial. To connect these methods requires more advanced topology and can be found in [6].

The Alexander polynomial was the first polynomial invariant to be applied to knots, and carries with it some important history of the subject. The Alexander polynomial was first obtained through the Seifert surfaces in 1927, and was at the time a simple and valuable tool for distinguishing knots. It was not 1969 that Conway discovered a way to obtain the Alexander polynomial using skein relations. This created a relatively quick and simple way of obtaining the Alexander polynomial. While the Alexander polynomial is by no means a total invariant, the Alexander polynomial continues to be an important tool for distinguishing knots.
Appendix A

Seifert Van Kampen

In this section we outline a method that greatly reduces the computation required in finding the fundamental group of topological spaces. In addition we apply this result to some important examples.

**Theorem 32. The Seifert Van Kampen Theorem**

Let $X$ be a topological space and let $X = X_1 \cup X_2$ where $X_1$ and $X_2$ are open subsets of $X$ such that $X_1, X_2$ and $X_0 = X_1 \cap X_2$ are nonempty and pathwise connected. Then $\pi_1(X)$ is generated by $\pi_1(X_1), \pi_1(X_2)$ and $\pi_1(X_0)$ by the following commutative diagram.

In addition, if $H$ is a group and $\psi_i : \pi_1(X_i) \to H$ are homomorphisms such that $\psi_0 = \psi_1i_1* = \psi_2i_2*$, then there is a unique $\lambda : \pi_1(X) \to H$ such that the following diagram commutes.

The following proof uses some basic topological tools, which we will not review in this
section. For more details see an introductory topology text, for example [7].

Proof. Let \( \alpha \) be the class of a path in \( \pi_1(X, p) \) where \( p \in X_0 \) and let \([ a ] = \alpha\). By the Lebesgue lemma, we may divide the domain of \( a \), \([0, 1]\) into a finite number of smaller intervals

\[
0 = t_0 < t_1 < t_2 < \cdots < t_n = 1
\]

such that each subinterval is contained in some \( a^{-1}(X_i) \) for \( i = 0, 1, 2 \). In other words, for an interval \([t_i, t_{i+1}]\), \( a([t_i, t_{i+1}]) \subset X_k \) for some \( k \in \{0, 1, 2\} \). To be more specific, we will let \( \mu(i) \) be a function \( \mu : i = 0, \ldots, n \rightarrow \{0, 1, 2\} \) such that \( a([t_i, t_{i+1}]) \subset X_{\mu(i)} \). For each \( t_j \), choose some path \( b_j \) with the following properties:

- \( b_j(0) = p \) and \( b_j(1) = a(t_j) \). The path \( b_j \) starts at point \( p \) and ends at the point \( a(t_j) \) along the path \( a \).
- If \( a(t_j) = p \) then \( b_j(t) = p \) for all \( t \in [0, 1] \). If the terminal point of \( b_j \) is \( p \), then \( b_j \) is the constant path.
- \( b_j(t) \in X_{\mu(i)} \cap X_{\mu(i+1)} \) for \( i = 1, \ldots, n - 1 \). Equivalently, the loops \( b_j a[t_j, t_{j+1}] b_{j+1}^{-1} \) are contained in \( X_i \) for some \( i = 0, 1, 2 \).

The paths \( b_j \) are illustrated for a few time steps in Figure A.

Let \( a_i \) denote the path \( a[t_i, t_{i+1}] \) and note that \( a = \prod_{i=1}^n a_i \). It follows that \( a \sim \prod_{i=1}^n b_{i-1} a_i b_i^{-1} \). By the properties above, we know that \( b_{i-1} a_i b_i^{-1} \) is a \( p \) based loop in
Figure A.1: Here we show an application of the Seifert Van Kampen theorem to a topological space. Displayed are topological spaces $X_1$ and $X_2$ with $X_1 \cap X_2 = X_0$ and $X_1 \cup X_2$ is the whole space. In the proof of the Seifert Van Kampen theorem, loop $a$ is subdivided into paths $a[t_j, t_{j+1}]$ and made into a composition of loops $b_j a[t_j, t_{j+1}] b_j^{-1}$.

$X_{\mu(i)}$. Thus each $b_{i-1} a_i b_i^{-1}$ belongs to a class of loops which is an element of the group $\pi_1(X_{\mu(i)}, p)$, and thus the groups $\pi_1(X_i, p)$ generate the fundamental group $\pi_1(X, p)$.

It remains to show the second part of the Van Kampen theorem. Let $\alpha$ be a class of paths in $\pi(X, p)$. The previous argument showed that a path $a \in \alpha$ can be represented as a product of loops from $\pi(X, p)$, $a = \prod_{i=1}^n j_{\mu(i)} a_i$. Thus we may represent a class of paths $\alpha$ as $\alpha = \prod_{i=1}^n j_{\mu(i)} a_i$ so let us define $\lambda : \pi_1(x, p) \to H$ to be $\lambda \alpha = \prod_{i=1}^n \psi_{\mu(i)} a_i$. Clearly, this is the only map which satisfies the commutative diagram. Thus if this map is well defined, it is the unique map from $\pi_1(X, p) \to H$.

To show that $\lambda$ is well defined we must show that for $a, b \in \alpha$, $\lambda(a) \lambda(b^{-1}) = 1$, as we have defined $\lambda$ based on the maps $\psi_i$ this amounts to showing that $j_{\mu(i)}(a_i) j_{\mu(i)}(b_i)^{-1} = 1$ implies that $\psi_{\mu(i)}(a_i) \psi_{\mu(i)}(b_i)^{-1}$. A more general way of showing that $\lambda$ is well defined shows that for any finite set of $a_i \in \pi_i(X_{\mu(i)}, p)$ that $\prod_{i=1}^r j_{\mu(i)} a_i = 1$ implies that $\prod_{i=1}^r \psi_{\mu(i)} a_i = 1$.

By our assumption, since $a = \prod_{i=1}^r j_{\mu(i)} a_i = 1$, then $a$ must be homotopic to the constant path at $p$. Or, equivalently, there exists a continuous function $h : [0, 1] \times [0, 1]$ such that $h(t, 0) = a$, $h(t, 1) = p$ and $h(0, s) = p$. Note that the domain of this map is also
a compact set, so again by the Lebesgue lemma, there exist divisions

\[ 0 = t_0 < t_1 < t_2 < \cdots < t_n = 1 \]
\[ 0 = s_0 < s_1 < s_2 < \cdots < s_m = 1 \]

such that each rectangle in this subdivision is contained in \( h^{-1}X_i \) for some \( i = 0, 1, 2 \).

Note that these subdivisions may be finer than our original subdivisions of the path \( a \) into \( a_1, \ldots, a_r \). Let us denote a rectangle in this subdivision by \( R_{ij} \) such that \( R_{ij} = [t_{i-1}, t_i] \times [s_{j-1}, s_j] \). These rectangles represent a subdivision of the homotopy of \( a[t_{i-1}, t_i] \). Let \( v(i, j) \) be a function \( v : i = 0, \ldots, n, j = 1, \ldots, m \to \{0, 1, 2\} \) such that \( h([t_{i-1}, t_i] \times [s_{j-1}, s_j]) \subset X_{v(i,j)} \). We now do a construction similar to the one done to obtain \( a \) as a product of \( a_i \).

For each corner of a rectangle we can find a path \( e_{ij} \) such that \( e_{ij} \) satisfies the following properties:

- \( e_{ij}(0) = p \) and \( e_{ij}(1) = h(t_i, s_j) \). The path begins at \( p \) and ends at \( h(t_i, s_j) \), the image of the top right corner of \( R_{ij} \) under the homotopy.
- If \( h(t_i, s_j) = p \), then \( e_{ij} = p \). If the terminal point of \( e_{ij} \) is \( p \), then \( e_{ij} \) is the constant path.
- \( e_{ij} \subset X_{v(i,j)} \cap X_{v(i+1,j)} \cap X_{v(i,j+1)} \cap X_{v(i+1,j+1)} \). The path \( e_{ij} \) is contained in the \( X_k \) corresponding to each of the next adjacent adjacent subdivisions of the homotopy.

Using these properties, we may define paths whose domain is each edge of the rectangle.

We do this as follows,

\[ c_{ij}(t) = h(t + t_{i-1}, s_j) \quad 0 \leq t \leq t_i - t_{i-1} \]
\[ d_{ij}(s) = h(t_i, s + s_{j-1}) \quad 0 \leq s \leq s_j - s_{j-1} \]

These paths are illustrated in Figure A.2. Then we may define the following paths

\[ a_{ij} = e_{i-1,j} c_{ij} e_{ij}^{-1} \quad i = 1, \ldots, n \text{ and } j = 1, \ldots, m \]
\[ b_{ij} = e_{i,j-1} d_{ij} e_{ij}^{-1} \quad i = 1, \ldots, n \text{ and } j = 1, \ldots, m. \]

Then \( a_{ij} \) is a path which begins at \( p \), travels to the point \( h(t_{i-1}, s_j) \) then continues along the \( t \) direction according to \( h \) until it reaches \( h(t_i, s_j) \) and then returns to \( p \). Similarly \( b_{ij} \) is
Figure A.2: This displays the subdivision of the homotopy into rectangles. The paths $e_{ij}$ originate at point $p$ and travel to $h[t_i, s_j]$. The paths $c_{ij}$ and $d_{ij}$ are the horizontal and vertical paths in the subdivision respectively.

A path which begins at $p$, travels to the point $h(t_i, s_{j-1})$, then continues along the $s$ direction according to $h$ until it reaches $h(t_i, s_j)$ and then returns to $p$. By the third property of the paths $e_{ij}$ listed above, the paths $a_{ij}, b_{ij}, a_{i,j-1}$ and $b_{i-1,j}$ are contained in $X_{v(i,j)}$ and thus they are representative paths of equivalence classes $\alpha_{ij}, \beta_{ij}, \alpha_{i,j-1}$ and $\beta_{i-1,j}$ in $\pi_1(X_{v(i,j)})$. As the path $a_{i,j-1}b_{ij}a_{ij}^{-1}b_{i-1,j}$ completes a cycle around $R_{ij}$ in the domain, it is contractible in $X$ and therefore contractible in $X_{v(i,j)}$. Hence $\alpha_{i,j-1}\beta_{ij}\alpha_{ij}^{-1}\beta_{i-1,j} = 1$.

Now we must show that if $\alpha \in \pi_1(X_i)$ and $\beta \in \pi_1(X_j)$ have a common representative loop then $\psi_i(\alpha) = \psi_j(\beta)$. For this, note that if $\alpha$ and $\beta$ have a common representative loop, $a$, then the image of $a$ is contained in $X_i \cap X_j$. Also note that by the way we have defined these component spaces, $X_i \cap X_j = X_k$ for some $k = 0, 1, 2$. Then $X_k \xrightarrow{\eta_i} X_i$ and $X_k \xrightarrow{\eta_j} X_j$ are inclusion mappings. Thus by our commutative diagram we have $\psi_i \eta_i = \psi_k = \psi_j \eta_j$. 
We use this result to conclude that
\[ \psi_{v(i,j)}(\alpha_{ij}) \left( \psi_{v(i,j+1)}(\alpha_{ij}) \right)^{-1} = 1 \]
\[ \psi_{v(i,j)}(\alpha_{ij}) \left( \psi_{v(i,j+1)}(\alpha_{ij}) \right)^{-1} = 1. \]
In other words, applying the \( \psi \) maps to edges of adjacent rectangles with opposite orientation cancels to give the identity. Our previous result that \( \alpha_{i,j-1} \beta_{ij} \alpha_{ij}^{-1} \beta_{i-1,j} = 1 \) means that going counterclockwise around each rectangle \( R_{ij} \) gives the identity through the \( \psi \) maps. As we have a finite subdivision of the space \([0,1] \times [0,1]\), it follows inductively that traveling around the outside of \([0,1] \times [0,1]\) gives the identity. In addition, by the construction of our map \( h \), only the elements with \( s = 0 \) are nontrivial. Thus their product must produce the identity, giving us
\[ \prod_{i=1}^{n} \psi_{v(i,1)} \alpha_{i0} = 1. \]
It remains to show that
\[ \prod_{i=1}^{r} \psi_{v(i,1)} \alpha_{i0} = 1. \]
By our construction, we have that \( \alpha_{ij} \) possesses a common representative loop with \( \alpha_{i,j+1} \) and so inductively it is clear that \( \psi_{v(i,1)} \alpha_{i0} = 1 \) and so
\[ \prod_{i=1}^{r} \psi_{v(i,1)} \alpha_{i0} = 1. \]
This completes our proof that \( \lambda \) is well defined. We have shown the desired result. \( \square \)

This theorem greatly simplifies the calculation of fundamental groups, as it allows us to find the fundamental group of the space in terms of simpler subspaces. We will now outline some examples, calculating the fundamental group of several spaces using the Van Kampen theorem.

**Example 22** \((n\text{-leaved rose})\). In this example we will use both the Van Kampen theorem and our tool of deformation retracts to calculate the fundamental group of the following spaces.
A diagram of the \( n \)-leafed rose is found in Figure A.3. To calculate its fundamental group we need to employ the Van Kampen theorem inductively.

As we know the fundamental group of the circle, it is natural to divide the space of the \( n \)-leafed rose into its ‘leaves’, each homeomorphic to \( S^1 \). Let \( X \) denote the \( n \)-leafed rose and let \( p \) be the center point. To apply the Seifert Van Kampen Theorem, it is necessary that \( X_i \) be an open subspace of \( X \). To accommodate this requirement, we take an open subset of \( X \) which contains \( X_i \) as shown in Figure A.4. As \( X_i \) is a deformation retract of the larger space, they have isomorphic fundamental groups. For the remainder of the proof we abuse notation by referring to \( X_i \) and the open subspace containing \( X_i \) interchangeably.

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Figure A.4: Left is the open subspace containing \( X_i \) for \( i = 1, \ldots, n \). Right is the open subspace containing \( X_0 \). Note that the spokes originating from point \( p \) are partial segments of the other \( n-1 \) leaves of the rose.
Then $X = X_1 \cup X_2 \cup \cdots \cup X_n$ where $X_i$ is a 'leaf' and, $\pi_1(X_i, p) \cong \mathbb{Z}$. Let $C_k = \bigcup_{i=1}^k X_i$. We will show that $\pi_1(C_k, p) = F(k)$. Our base case is already done. As $X_1$ is homeomorphic to $S^1$, $\pi_1(X_1, p) \cong \mathbb{Z} \cong F(1)$.

Let $C_k = C_{k-1} \cup X_k$. By our inductive hypothesis we know that $\pi_1(C_{k-1}, p) = F(\{x_1, \ldots, x_{k-1}\})$. As $X_k$ is homeomorphic to $S^1$, it follows that $\pi_1(X_k, p) = \mathbb{Z} \cong F(x_k)$. Furthermore, we know that $X_0 = X_k \cap C_{k-1} = \{p\}$. As $\{p\}$ is only a single point, it is clear that $X_0$ is simply connected. This gives us the following commutative diagram from the Van Kampen theorem:

\[
\begin{array}{ccc}
F(x_k) & \xrightarrow{j_1} & \pi_1(C_k, p) \\
i_1* & & \downarrow j_1* \\
1 & \xrightarrow{j_0*} & \pi_1(C, p) \\
& i_2* & \downarrow j_2* \\
& F(\{x_1, \ldots, x_{k-1}\}) & \\
\end{array}
\]

By the first part of the Seifert Van Kampen theorem, it follows that $\pi_1(C_k, p)$ is generated by $\{x_1, \ldots, x_k\}$. It remains to show that $\pi_1(C_k, p) \cong F(k)$. Let $H$ be the free group, $F(\{a_1, \ldots, a_k\})$. The free group property (see [5]), states that if $F$ is a free group and $G$ is a group, and there exists a map from the generating set of the $F$ to the generating set of $G$, then there exists a homomorphism $F \to G$. Let the maps $\psi_i$ be homomorphisms induced by the free group property by the set maps $x_k \mapsto a_k$ for the generators $x_k$ in $F(x_k)$ and $x_i \mapsto a_i$ for the generators $x_i$ of $F(\{x_1, \ldots, x_{k-1}\})$. The second part of the Seifert Van Kampen theorem gives the following commutative diagram:

Because $\pi_1(C_k, p)$ is generated by $\{x_1, \ldots, x_k\}$ the map $a_i \mapsto x_i$ induces a homomorphism $\mu : F(\{a_1, \ldots, a_k\}) \to \pi_1(C_k, p)$ such that $\mu(x_k) = j_1*(x_k)$ and $\mu(x_i) = j_2*(x_i)$ for $i = 1, \ldots, k-1$. Then each $\lambda \mu$ and $\mu \lambda$ must compose to the identity, hence $\pi_1(C_k, p) \cong F(\{x_1, \ldots, x_k\})$. 
Example 23 (The $T_{2,3}$ Torus Knot). Before we go into this example we must define a torus knot.

Definition 39. A torus knot is a knot that can be embedded on the surface of a torus.

Torus knots are a very specific kind of knot. In fact the only information needed to distinguish torus knots is two numbers; one which indicates the number of times the knot circles around the meridional curve, and one which indicates the number of times the knot circles around the longitudinal curve. Of course, it can be shown that $T_{p,q}$ and $T_{q,p}$ are equivalent knots. In the following example we discuss the torus knot which revolves twice around one of the curves, and thrice around the other, denoted $T_{2,3}$. This knot is illustrated embedded on the surface of the torus in Figure A.5.

Figure A.5: Here is an illustration of the $T_{2,3}$ torus knot embedded on the surface of a torus. The dotted lines corresponds to strands running along the underside of the torus. This knot cycles 3 times around the longitude of the torus and twice around the meridian.

We now determine the fundamental group of $\mathbb{R}^3 \setminus T_{2,3}$ torus knot, but the argument
given applies to any torus knot. See [9] for a more general treatment of the problem.

Let us denote the solid torus to be $T$ and $K$ to be $T_{2,3}$. In addition let us denote $T^\circ$ to be the interior of $T$. As $K$ is embedded on the surface of a torus, the knot complement is also the union of $T \setminus K$ and $\mathbb{R}^3 \setminus (T^\circ \cup K)$. The intersection of these two sets is $(T \setminus K) \cap (\mathbb{R}^3 \setminus (T^\circ \cup K)) = T \setminus (T^\circ \cup K)$.

To apply the Seifert Van Kampen theorem we first compute the fundamental group of each of these spaces separately. As there is a deformation retract from $T \setminus K$ to the circle at the center of the solid torus, $\pi_1(T \setminus K) \cong \pi_1(S^1) \cong \mathbb{Z}$. In addition $\mathbb{R}^3 \setminus (T^\circ \cup K)$ is homeomorphic to $\mathbb{R}^3 \setminus S^1$ and so the fundamental group $\mathbb{R}^3 \setminus (T^\circ \cup K) \cong \mathbb{Z}$ (this can be seen as any nontrivial path must go through the hole in the center of the torus).

Finding the fundamental group of $T \setminus (T^\circ \cup K)$ requires some more thought. As the surface if the torus is homeomorphic to $S^1 \times S^1$ we can visualize it as a rectangle with the edges identified in the usual way as found in Figure A.6. Our knot, $K$ is then a series of diagonal lines stretching across the rectangle. Between each of these lines is a strip of the surface of the torus. When the sides of the rectangle are identified, the diagonal lines form $K$. Similarly after identification these strips of surface connect to form an annulus. Thus $T \setminus (T^\circ \cup K)$ is homeomorphic to an annulus which deformation retracts to a circle and so $\pi_1(T \setminus (T^\circ \cup K)) \cong \mathbb{Z}$.

![Figure A.6: Here we see an embedding of $K$ on the torus represented as a rectangle with the edges identified in the usual manner. The curve $\gamma$ is a representative generator of the fundamental group $\pi_1(T \setminus (T^\circ \cup K))$.](image)

It is easy to see that the generator of the fundamental group of the intersection maps
to a power of the generator of $T \setminus K$. Let $\gamma$ be the path described in Figure A.6 which generates the fundamental group of the annulus. This can be deformed to travel around the central circle of the torus in $T \setminus K$ twice. This can be seen in Figure A.6 as the path $\gamma$ intersects the left edge of our rectangle twice. Similarly, the path $\gamma$ wraps around the center of the torus three times.

Denote the generators of $\pi_1(T \setminus K)$ and $\pi_1(\mathbb{R}^3 \setminus (T^o \cup K))$ by $a$ and $b$ respectively. Then $a^2 = b^3$, and it follows that

$$\pi_1(\mathbb{R}^3 \setminus K) = \langle a, b : a^2 = b^3 \rangle.$$
Appendix B

Some Necessary Ring Theory

In this section we are mostly concerned with providing the tools needed to compute Alexander polynomials through the elementary ideals. For this reason most of the content in this appendix is devoted to outlining some key properties of polynomial rings. However, it is also useful to review the definitions and properties of rings, ideals, and principle ideals, and so some of those concepts are briefly discussed here. It is important to understand that this appendix only serves to complement the content of Chapter 7 and serves to refresh a reader who may not have studied ring theory recently. To the reader that is new to ring theory, a more extensive algebra book may be a valuable, for example [8].

In addition, the theorems and proofs presented here are thus very narrow in scope, and in many cases more general proofs are available. For the general proofs we refer the reader to [5].

**Definition 40.** A ring is a set and two operations \( \langle R, +, \cdot \rangle \) such that \( \langle R, + \rangle \) is an abelian group and \( R \) is closed under \( \cdot \). \( R \) must also have the following distribution properties. Let \( r_1, r_2, r_3 \in R \)

\[
\begin{align*}
    r_1(r_2 + r_3) &= r_1r_2 + r_1r_3, \\
    (r_1 + r_2)r_3 &= r_1r_3 + r_2r_3.
\end{align*}
\]

**Definition 41.** A subset \( S \) of a ring \( R \) is a subring if \( \langle S, +, \cdot \rangle \) is a ring.

**Definition 42.** Let \( I \) be a subring of \( R \) with the property that for all \( r \in R \) and all \( s \in I \)

95
If $rs \in I$, then $I$ is a \textit{left ideal} of $R$. Similarly, $I$ is a right ideal if for all $r \in R$ and all $s \in I$, $sr \in I$.

It naturally follows from the above definition that if $R$ is a commutative ring, any ideal $I$ is both a right and a left ideal. As our study of rings is restricted to commutative rings, we use the term ideal to mean both a right and left ideal.

**Definition 43.** A \textit{principal ideal} is an ideal generated by one element.

The remainder of this appendix aims to build the tools necessary to understand the greatest common divisors of elements in the infinite cyclic group ring.

**Definition 44.** For $a, b \in R$ we say that $a$ \textit{divides} $b$ (denoted $a|b$) if there exists some $c \in R$ such that $a = cb$.

**Definition 45.** Given a finite set of elements $\{r_1, r_2, \ldots, r_n\}$ of a ring $R$, $d$ is a \textit{common divisor} if $d|r_i$, for each $i = 1, \ldots, n$. If for any common divisor $g$, $g|d$ then $d$ is a \textit{greatest common divisor} or g.c.d. .

**Definition 46.** Let $R$ be a ring, then $R$ is a g.c.d. \textit{domain} if every finite set of elements $\{r_1, r_2, \ldots, r_n\}$ in $R$ has a g.c.d..

**Definition 47.** For $X = \{x_1, \ldots, x_n\}$, $\mathbb{Z}[X]$ is the ring of polynomials with coefficients in $\mathbb{Z}$ in the variables $x_1, \ldots, x_n$.

**Theorem 33.** $\mathbb{Z}[t, t^{-1}]$ is a g.c.d. domain

**Proof.** For this proof we will rely on the fact that $\mathbb{Z}[t]$ is a g.c.d. domain. A proof of this may be found in [5].

Let $f_1, \ldots, f_n$ be polynomials in $\mathbb{Z}[t, t^{-1}]$, then there exists some $m_1, \ldots, m_n \in \mathbb{N}$ such that $t^{m_i}f_i \in \mathbb{Z}[t]$ for $i = 1, \ldots, n$. The set $\{t^{m_1}f_1, \ldots, t^{m_n}f_n\}$ is a set of elements in $\mathbb{Z}[t]$, and hence has a g.c.d., $h$. Then $h|t^{m_i}f_i$ for $i = 1, \ldots, n$ and there exists polynomials $g_1, \ldots, g_n$ such that $t^{m_i}f_i = g_i h$. Thus $f_i = (g_i t^{-m_i})h$ and $h|f_i$ in $\mathbb{Z}[t, t^{-1}]$ and so $h$ is a common divisor of $\{f_1, \ldots, f_n\}$.
It remains to show that \( h \) is a greatest common divisor. To do this we must show that for another common divisor \( h' \), that \( h' | h \). Because \( h' \) is a common divisor of \( f_1, \ldots, f_n \), there exists \( \{g'_1, \ldots, g'_n\} \) such that \( f_i = g'_i h' \). Then \( t^{m_i} f_i = (t^{m_i} g'_i) h' \) and \( h' \) is a common divisor of \( \{t^{m_1} f_1, \ldots, t^{m_n} f_n\} \). Because \( h \) is the \( \text{g.c.d.} \) of \( \{t^{m_1} f_1, \ldots, t^{m_n} f_n\} \) and \( h' \) is a common divisor of \( \{t^{m_1} f_1, \ldots, t^{m_n} f_n\} \), \( h' | h \). Hence \( h \) is a \( \text{g.c.d.} \) of the elements \( f_1, \ldots, f_n \) in \( \mathbb{Z}[t, t^{-1}] \), and thus \( \mathbb{Z}[t, t^{-1}] \) is a \( \text{g.c.d.} \) domain. 

**Remark 2.** By a similar line of reasoning by above, it is clear that the \( \text{g.c.d.} \) of a set of elements in \( \mathbb{Z}[t, t^{-1}] \) is unique only up to powers of \( \pm t^k \). For this reason, the knot polynomials obtained form the sequence of elementary ideals are unique up to powers of \( \pm t^k \).

**Remark 3.** As the knot polynomials are defined to be the generator of the smallest principal ideal containing the elementary ideal \( E_k \) (see section 7.3) it is crucial to be able to construct the generator of the smallest principle ideal containing the ideal in question. As the elementary ideals are finitely generated, we find the \( \text{g.c.d.} \) of the generators of the ideal and this provides us with the generator of the desired principle ideal. What enables us to do this is that \( \mathbb{Z}[t, t^{-1}] \) is a \( \text{g.c.d.} \) domain.
Bibliography


