Squares and other polygons inscribed in curves

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A proof from “the book”

Theorem (Proof: H. Vaughan 1977)

Let $\gamma$ be a continuous closed curve in the plane without self-intersections, then there are four points on $\gamma$ that are the vertices of a rectangle.

The proof is purely topological.

Principle

- Two objects are the “same” if there is a homeomorphism between them.
- **Homeomorphism:** 1-to-1, onto, continuous map with a continuous inverse.
- Topological properties are preserved by homeomorphisms (ex. connectedness).
Setting up the proof — a topological detour

- Let $\gamma$ be a continuous simple closed curve in the plane.
- Let $M(\gamma) = \{\{u, v\} | u, v \in \gamma\}$ (order doesn’t matter).
- Since $\gamma$ is homeomorphic to the standard unit circle $S^1$, we understand $M(\gamma)$ by understanding $M(S^1)$.
- Observe that $S^1 \times S^1$ is represented by

![Diagram showing the representation of $S^1 \times S^1$.]
Visualizing a torus $S^1 \times S^1$

Gluing a Torus - YouTube

http://www.youtube.com/watch?v=0H5_h-RB0T8
Structure of $M(\gamma) = \{\{u, v\} \mid u, v \in \gamma\}$

- $S^1 \times S^1$ has both $(u, v)$ and $(v, u)$.
- Keep one half of $S^1 \times S^1$. Cut this in half then reglue.

Fact

$M(\gamma)$ is homeomorphic to a Möbius band.

Also, $M(\gamma)$’s boundary corresponds to pairs of points $\{u, u\}$. (This can be thought of as a copy of $\gamma$.)
Möbius bands are non-orientable surfaces

M.C. Escher
Real Projective Plane

Definition
Real projective plane $\mathbb{R}P^2 = \{\text{lines in } \mathbb{R}^3 \text{ through the origin}\}$

- Each line meets the unit sphere $S^2$ in two antipodal points.
- Identify a line $\ell$ with $\ell \cap S^2 = \{x, -x\}$.
- Take half of $S^2$ and glue together $y$ and $-y$ on the equator.
- $\mathbb{R}P^2$ is a surface without boundary.
Visualizing $\mathbb{RP}^2$

Learn more about surfaces: 
*Beginning Topology*, Sue Goodman.
Boy’s surface

How can you visualize topological objects?
A *Topological Picture Book* by George Francis
Real Projective Space and Möbius Bands

Fact

\[ \mathbb{R}P^2 - disc = Mb \text{ with boundary of disc as edge, or} \]

\[ \mathbb{R}P^2 = Mb \cup disc \text{ joined along their boundaries.} \]
A different view of the Möbius band...

... the Sudanese Möbius band.
Theorem (Proof: H. Vaughan 1977)

Let $\gamma$ be a continuous closed curve in the plane without self-intersections, then there are four points on $\gamma$ that are the vertices of a rectangle.

- $\gamma$ simple closed curve in the plane.
- $\gamma$ bounds a disc $D$ (Schoenflies Theorem).
- $M(\gamma) = \{\{u, v\} \mid u, v \in \gamma\}$ homeomorphic to Möbius band.
- Boundary of $M(\gamma)$ is a copy of $\gamma$.
- We know that gluing together a disc and Möbius band along their boundary gives $\mathbb{R}P^2$.
- $\mathbb{R}P^2 = M(\gamma) \cup \gamma D$
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- We know that gluing together a disc and Möbius band along their boundary gives $\mathbb{RP}^2$.
- $\mathbb{RP}^2 = M(\gamma) \cup_\gamma D$
Proof continued. . .

Define $F : \mathbb{RP}^2 \to \mathbb{R}^3$ by

$$F|_{\text{disc}} \text{ is an inclusion of } \mathbb{R}^2 \hookrightarrow \mathbb{R}^3, \text{ and } F|_{M(\gamma)} \text{ is }$$

$$F(\{u, v\}) = \left( \frac{u+v}{2}, \|u - v\| \right) \text{ (take midpoint in } \mathbb{R}^2, \text{ length } \overline{uv}).$$

- $F$ agrees on $\gamma$ and is a continuous function.
- Fact: $F$ cannot be an embedding (since $\mathbb{RP}^2$ is a closed non-orientable surface).
- $F|_{\text{disc}}$ is an embedding (by construction).
- Problem points are on $F|_{M(\gamma)}$.
- There are pairs $\{u, v\}, \{a, b\}$ with $\frac{u+v}{2} = \frac{a+b}{2}$ and $\|u - v\| = \|a - b\|$.
- Thus $\overline{uv}$ and $\overline{ab}$ are the diagonals of a rectangle.
Proof continued.

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The Square Peg problem

Question (Toeplitz, 1911)

Let $\gamma$ be a closed curve in the plane with no self-intersections. Are there four points on $\gamma$ that are the vertices of a square?

Still an open question!
Known results require $\gamma$ to be “smooth enough”.

History of the Square Peg Problem

1. Emch, 1917, proved for piecewise analytic arcs
2. Schnirelmann, 1929, proved for bounded curvature*
3. Christiansen, 1950, for convex curves
4. Ogilvy, 1950, for “nice enough” curves
5. Jerrard, 1959, for analytic curves
6. Guggenheim, 1965, fixes an error in Schnirelmann*
7. Stromquist, 1989, for a class somewhat larger than $C^1$
8. (Griffiths, 1991, contains serious errors)
9. Nielsen, Wright, 2002, centrally symmetric $C^0$ curves
Our results: Squares

Theorem (with Cantarella, McCleary)

Let $\gamma$ be a simple closed continuously differentiable curve in $\mathbb{R}^2$. Then there are an odd (or infinite) number of inscribed squares $abcd$ (in order) on $\gamma$.

What about space curves, or curves in $\mathbb{R}^k$?
Our results: Squares

Theorem (with Cantarella, McCleary)
Let $\gamma$ be a simple closed continuously differentiable curve in $\mathbb{R}^k$. Then there are an odd (or infinite) number of square-like quadrilaterals $abcd$ inscribed in order on $\gamma$ with equal sides $|ab| = |bc| = |cd| = |da|$ and equal diagonals $|ac| = |bd|$.

Corollary
For a simple closed $C^1$ plane curve, there are an odd (or infinite) number of inscribed squares.

Theorem (corners don’t matter)
For a curve of finite total curvature without cusps, there is at least one inscribed square.
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Theorem (corners don’t matter)

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How would you prove something like this?

**Idea**

If a pair of “surfaces” $\alpha, \beta$ of dimension $d_1, d_2$ intersect in a space of dimension $k$, then the intersection has dimension $d_1 + d_2 - k$.

On the left, $d_1 = d_2 = 2$ and $k = 3$, so the intersection has dimension 1.

**Principle**

As long as the boundary of $\beta$ stays in $Y$ and the boundary of $\alpha$ stays in $X$, we can deform the tube $\beta$ and the sheet $\alpha$ however we want, and the intersection will still contain a curve that goes all the way around $\beta$. 
A point in $\mathbb{R}^2$ has two coordinates $(x, y)$. Four points in $\mathbb{R}^2$ have eight coordinates $(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$. The space of quadruples of points in $\mathbb{R}^2$ is eight-dimensional.

A point on a curve $\gamma$ is described by one coordinate (how far it is from the start). The space of quadruples of points on $\gamma$ is four-dimensional.

A square in $\mathbb{R}^2$ is described by four coordinates: the coordinates of the center $(x, y)$, an angle $\theta$ by which the square is rotated, and the length $\ell$ of a side of the square. The space of squares in $\mathbb{R}^2$ is four-dimensional.

The space of squares which are inscribed in a curve is $4 + 4 - 8 = 0$ dimensional!
Dimension count for squares

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Why an odd number of squares?

An ellipse contains exactly *one* square:

Proof: Construct inscribed rectangles and use the Intermediate Value Theorem.
Odd number of squares, continued... 

- If we move the ellipse to some other curve, we deform the 4-dimensional space of quadruples of points on the curve.
- We can show that during the deformation, the boundary of this space stays away from the boundary of the space of squares, and... 
- ...the intersection will contain at least one point.
- More intersections may appear, but they appear in pairs.
- Hence # of squares is 1+ an even number, so it is odd.

Arguments like this are found in *Differential Topology* by Victor Guillemin, Alan Pollack.
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Our results: Loops of polygons

The next result doesn’t have a parallel in the literature...

Theorem (with Cantarella, McCleary)

For a generic simple closed $C^1$-smooth curve in $\mathbb{R}^k$, there is a smooth curve of inscribed equilateral $n$-gons so that one point on the $n$-gon travels all of the way around the curve.

This is best understood by watching a movie...
Equilaterial Triangles on the Jellybean Curve

The set of equilateral triangles on this curve fragments into 7 distinct loops. One goes all the way around, the other two groups of three don’t.
Why is there a loop of polygons?

A circle contains such a loop for any $n$:

and the rest of the proof (dimension count, deform the circle to your curve and watch the loop of $n$-gons deform) goes along the same lines as before.
In fact our Loop of Polygons Theorem holds for \( n \)-gons 
\((x_1, x_2, \ldots, x_n)\) whose side-lengths are in any constructible ratio 
\( R = r_1 : \cdots : r_n \). That is \(|x_2 - x_1| : \cdots : |x_1 - x_n| = r_1 : \cdots : r_n\).

Constructible: we assume each \( r_i > 0 \) and \( R \) obeys the triangle inequalities: 
\[ r_i \leq r_1 + \cdots + r_{i-1} + r_{i+1} + \cdots + r_n. \]

**Theorem (with Cantarella, McCleary)**

Given a constructible ratio \( R \) and a \( C^1 \)-smooth curve in \( \mathbb{R}^k \),
there is an arbitrarily \( C^1 \)-close smooth curve \( \gamma \) such that 
\( \text{Pol}_R \cap C_n[\gamma] \) is a collection of smooth, disjoint oriented circles \( C \).

Moreover the collection \([C]\) represents \(+1\) in \( H_1(C_n[\gamma]; \mathbb{Z}) \approx \mathbb{Z}\).
Thank you!

Special thanks to

Jason Cantarella
John McCleary

The movies and some pictures in this talk were created by Jason Cantarella
Coda 1: A bold conjecture

Among our equilateral quadrilaterals are four squares. These are the unique concyclic equilateral quadrilaterals.

Conjecture

In any simple closed plane curve, there are four inscribed concyclic quadrilaterals with any admissible edgelength ratio $r_1 : \cdots : r_4$.

(Partial results by Makeev.)
Why is this so important?

This would give a purely topological proof of the
Theorem (Four-Vertex Theorem)

*There are at least four critical points for curvature (vertices) on any simple closed $C^2$ curve in the plane.*

**Proof.**
The limit of 4 concyclic points coming together on a curve is a vertex (Mukhopadyaya). This is one of the degenerate concyclic 4-tuples $(p_1, \ldots, p_4)$ with

$$x_1 : \cdots : x_4 \rightarrow 1 : 1 : 1 : 3.$$ (1)
Coda 2: A natural question

What happens to the loops of polygons for immersed curves?