Abstract

In this paper, we present a particularly nice Binet-style formula that can be used to produce the $k$-generalized Fibonacci numbers (that is, the Tribonacci, Tetranacci, etc.). Furthermore, we show that in fact one needs only take the integer closest to the first term of this Binet-style formula in order to generate the desired sequence. Update: these results were discovered independently and simultaneously by Zhaohui Du of Singapore; see the Wikipedia page on "Generalizations of Fibonacci Numbers"; the reference is to Zhaohui’s Chinese/English math blog.

1 Introduction

Let $k \geq 2$ and define $F_n^{(k)}$, the $n$th $k$-generalized Fibonacci number, to satisfy the recurrence relation

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} \quad (k \text{ terms})$$

... and with initial conditions $0, 0, \ldots, 0, 1$ ($k$ terms) such that the first non-zero term is $F_1^{(k)} = 1$.

These numbers are also called the Fibonacci $k$-step numbers, Fibonacci $k$-sequences, or $k$-bonacci numbers. Note that for $k = 2$, we have $F_n^{(2)} = F_n$, our familiar Fibonacci numbers. For $k = 3$ we have the so-called Tribonacci (sequence number A000073 in Sloane’s Encyclopedia of Integer Sequences), followed by the Tetranacci (A000078) for $k = 4$, and so on. According to Kessler and Schiff [6], these numbers also appear in probability theory and in certain sorting algorithms. We present here a chart of these numbers for the first few values of $k$:

<table>
<thead>
<tr>
<th>$k$</th>
<th>name</th>
<th>i.c.</th>
<th>first few non-zero terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Fibonacci</td>
<td>0, 1</td>
<td>1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots</td>
</tr>
<tr>
<td>3</td>
<td>Tribonacci</td>
<td>0, 0, 1</td>
<td>1, 1, 2, 4, 7, 13, 24, 44, 81, \ldots</td>
</tr>
<tr>
<td>4</td>
<td>Tetranacci</td>
<td>0, 0, 0, 1</td>
<td>1, 1, 2, 4, 8, 15, 29, 56, 108, \ldots</td>
</tr>
<tr>
<td>5</td>
<td>Pentanacci</td>
<td>0, 0, 0, 0, 1</td>
<td>1, 1, 2, 4, 8, 16, 31, 61, 120, \ldots</td>
</tr>
</tbody>
</table>
We remind the reader of the famous Binet formula (also known as the de Moivre formula) that can be used to calculate \( F_n \), the Fibonacci numbers:

\[
F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]
\]

\[
= \frac{\alpha^n - \beta^n}{\alpha - \beta}
\]

...for \( \alpha > \beta \) the two roots of \( x^2 - x - 1 = 0 \). For our purposes, it is convenient (and not particularly difficult) to rewrite this formula as follows:

\[
F_n = \frac{\alpha - 1}{2 + 3(\alpha - 2)} \alpha^{n-1} + \frac{\beta - 1}{2 + 3(\beta - 2)} \beta^{n-1}
\]

We leave the details to the reader.

Our first (and very minor) result is the following representation of \( F_n^{(k)} \):

**Theorem 1.** For \( F_n^{(k)} \) the \( n \)th \( k \)-generalized Fibonacci number, then

\[
F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k + 1)(\alpha_i - 2)} \alpha_i^{n-1}
\]

for \( \alpha_1, \ldots, \alpha_k \) the roots of \( x^k - x^{k-1} - \cdots - 1 = 0 \).

This is a new presentation, but hardly a new result. There are many other ways of representing these \( k \)-generalized Fibonacci numbers, as seen in the articles [2], [3], [4], [5], [7], [8], [9]. Our equation (2) of Theorem 1 is perhaps slightly easier to understand, and it also allows us to do some analysis (as seen below). We point out that for \( k = 2 \), equation (2) reduces to the variant of the Binet formula (for the standard Fibonacci numbers) from equation (1).

As shown in three distinct proofs ([9], [10], and [13]), the equation \( x^k - x^{k-1} - \cdots - 1 = 0 \) from Theorem 1 has just one root \( \alpha \) such that \( |\alpha| > 1 \), and the other roots are strictly inside the unit circle. We can conclude that the contribution of the other roots in formula 2 will quickly become trivial, and thus:

\[
F_n^{(k)} \approx \frac{\alpha - 1}{2 + (k + 1)(\alpha - 2)} \alpha^{n-1} \quad \text{... for } n \text{ sufficiently large.} \quad (3)
\]

It’s well known that for the Fibonacci sequence \( F_n^{(2)} = F_n \), the “sufficiently large” \( n \) in equation (3) is \( n = 0 \), as shown here:

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_n )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n )</td>
<td>0.447</td>
<td>0.724</td>
<td>1.171</td>
<td>1.894</td>
<td>3.065</td>
<td>4.960</td>
<td>8.025</td>
</tr>
<tr>
<td>(</td>
<td>\text{error}</td>
<td>)</td>
<td>.447</td>
<td>.277</td>
<td>.171</td>
<td>.106</td>
<td>.065</td>
</tr>
</tbody>
</table>
It is perhaps surprising to discover that a similar statement holds for all the $k$-generalized Fibonacci numbers. Our main result is the following:

**Theorem 2.** For $F_n^{(k)}$ the $n^{th}$ $k$-generalized Fibonacci number, then

$$F_n^{(k)} = \text{Round} \left[ \frac{\alpha - 1}{2 + (k+1)(\alpha - 2)} \alpha^{n-1} \right]$$

for all $n \geq 2 - k$ and for $\alpha$ the unique positive root of $x^k - x^{k-1} - \cdots - 1 = 0$.

**UPDATE:** We’ve since learned that this result was found independently and simultaneously by Zhaohui Du of Singapore; see his web page [here](#).

We point out that this theorem is not as trivial as one might think. Note the error for $k = 6$, as seen in the following chart; it is not monotone decreasing.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$F_n^{(6)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
</tr>
<tr>
<td>6</td>
<td>32</td>
</tr>
<tr>
<td>7</td>
<td>64</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\frac{\alpha - 1}{2 + 7(\alpha - 2)} \alpha^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.263</td>
</tr>
<tr>
<td>1</td>
<td>0.522</td>
</tr>
<tr>
<td>2</td>
<td>1.035</td>
</tr>
<tr>
<td>3</td>
<td>2.053</td>
</tr>
<tr>
<td>4</td>
<td>4.072</td>
</tr>
<tr>
<td>5</td>
<td>8.078</td>
</tr>
<tr>
<td>6</td>
<td>16.023</td>
</tr>
<tr>
<td>7</td>
<td>31.782</td>
</tr>
</tbody>
</table>

| $n$ | $\text{|error|}$ |
|-----|-----------------|
| 0   | 0.263           |
| 1   | 0.478           |
| 2   | 0.035           |
| 3   | 0.053           |
| 4   | 0.072           |
| 5   | 0.078           |
| 6   | 0.023           |
| 7   | 0.218           |

We also point out that not every recurrence sequence admits such a nice formula as seen in Theorem 2. Consider, for example, the scaled Fibonacci sequence $10, 10, 20, 30, 50, 80, \ldots$, which has Binet formula:

$$\frac{10}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{10}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n.$$

This can be written as $\text{Round} \left[ \frac{10}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n \right]$, but only for $n \geq 5$. As another example, the sequence $1, 2, 8, 24, 80, \ldots$ (defined by $G_n = 2G_{n-1} + 4G_{n-2}$) can be written as

$$G_n = \frac{(1 + \sqrt{5})^n}{2\sqrt{5}} - \frac{(1 - \sqrt{5})^n}{2\sqrt{5}},$$

but because both $1 + \sqrt{5}$ and $1 - \sqrt{5}$ have absolute value greater than 1, then it would be impossible to express $G_n$ in terms of just one of these two numbers.

### 2 Previous Results

We point out that for $k = 3$ (the Tribonacci numbers), our Theorem 2 was found earlier by Spickerman [11]. His formula (modified slightly to match our notation) reads as follows, where $\alpha$ is the real root, and $\sigma$ and $\overline{\sigma}$ are the two complex roots, of $x^3 - x^2 - x - 1 = 0$:

$$F_n^{(3)} = \text{Round} \left[ \frac{\alpha^2}{(\alpha - \sigma)(\alpha - \overline{\sigma})} \alpha^{n-1} \right]$$

(4)
It is not hard to show that for $k = 3$, our coefficient $\frac{\alpha - 1}{2 + (k+1)(\alpha - 2)}$ from Theorem 2 is equal to Spickerman’s coefficient $\frac{\alpha^2}{(\alpha - \sigma)(\alpha - \sigma_i)}$. We leave the details to the reader.

In a subsequent article [12], Spickerman and Joyner developed a more complex version of our Theorem 1 to represent the generalized Fibonacci numbers. Using our notation, and with $\{\alpha_i\}$ the set of roots of $x^k - x^{k-1} - \cdots - 1 = 0$, their formula reads

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i^{k+1} - \alpha_i^k}{2\alpha_i^k - (k+1)} \alpha_i^{n-1}\quad (5)$$

It is surprising that even after calculating out the appropriate constants in their equation (5) for $2 \leq k \leq 10$, neither Spickerman nor Joyner noted that they could have simply taken the first term in equation (5) for all $n \geq 0$, as Spickerman did in equation (4) for $k = 3$.

The Spickerman-Joyner formula (5) was extended by Wolfram [13] to the case with arbitrary starting conditions (rather than the initial sequence $0, 0, \ldots, 0, 1$). In the next section we will show that our formula (2) in Theorem 1 is equivalent to the Spickerman-Joyner formula given above (and thus is a special case of Wolfram’s formula).

Finally, we note that the polynomials $x^k - x^{k-1} - \cdots - 1$ in Theorem 1 have been studied rather extensively. They are irreducible polynomials with just one zero outside the unit circle. That single zero is located between $2(1 - 2^{-k})$ and $2$ (as seen in Wolfram’s article [13]; Miles [9] gave earlier and less precise results). It is also known [13, Lemma 3.11] that the polynomials have Galois group $S_k$ for $k \leq 11$; in particular, their zeros can not be expressed in radicals for $5 \leq k \leq 11$. Wolfram conjectured that the Galois group is always $S_k$. Cipu and Luca [1] were able to show that the Galois group is not contained in the alternating group $A_k$, and for $k \geq 3$ it is not 2-nilpotent. They point out that this means the zeros of the polynomials $x^k - x^{k-1} - \cdots - 1$ for $k \geq 3$ can not be constructed by ruler and compass, but the question of whether they are expressible using radicals remains open.

3 Preliminary Lemmas

First, a few statements about the the number $\alpha$.

**Lemma 3.** Let $\alpha > 1$ be the real positive root of $x^k - x^{k-1} - \cdots - x - 1 = 0$. Then,

$$2 - \frac{1}{k} < \alpha < 2\quad (6)$$

In addition,

$$2 - \frac{1}{3k} < \alpha < 2\quad \text{for } k \geq 4\quad (7)$$

**Proof.** We begin by computing the following chart for $k \leq 5$:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$2 - \frac{1}{k}$</th>
<th>$2 - \frac{1}{3k}$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.5</td>
<td>1.833...</td>
<td>1.618...</td>
</tr>
<tr>
<td>3</td>
<td>1.666...</td>
<td>1.889...</td>
<td>1.839...</td>
</tr>
<tr>
<td>4</td>
<td>1.75</td>
<td>1.916...</td>
<td>1.928...</td>
</tr>
<tr>
<td>5</td>
<td>1.8</td>
<td>1.933...</td>
<td>1.966...</td>
</tr>
</tbody>
</table>
It’s clear that \(2 - \frac{1}{k} < \alpha < 2\) for \(2 \leq k \leq 5\) and that \(2 - \frac{1}{3k} < \alpha < 2\) for \(4 \leq k \leq 5\). We now focus on \(k \geq 6\). At this point, we could finish the proof by appealing to \(2(1 - 2^{-k}) < \alpha < 2\) as seen in the article [13, Lemma 3.6], but we present here a simpler proof.

Let \(f(x) = (x - 1)(x^k - x^{k-1} - \ldots - x - 1) = x^{k+1} - 2x^k + 1\). We know from our earlier discussion that \(f(x)\) has one real zero \(\alpha > 1\). Writing \(f(x)\) as \(x^k(x - 2) + 1\), we have

\[
f \left( 2 - \frac{1}{3k} \right) = \left( 2 - \frac{1}{3k} \right)^k \left( \frac{-1}{3k} \right) + 1 \tag{8}
\]

For \(k \geq 6\), it’s easy to show

\[
3k < \left( \frac{5}{3} \right)^k < \left( 2 - \frac{1}{3} \right)^k < \left( 2 - \frac{1}{3k} \right)^k
\]

Substituting this inequality into the right-hand side of (8), we can re-write (8) as:

\[
f \left( 2 - \frac{1}{3k} \right) < (3k) \cdot \left( \frac{-1}{3k} \right) + 1 = 0.
\]

Finally, we note that

\[
f(2) = 2^{k+1} - 2 \cdot 2^k + 1 = 1 > 0,
\]

so we can conclude that our root \(\alpha\) is within the desired bounds of \(2 - 1/3k\) and 2 for \(k \geq 6\).

We now have a lemma about the coefficients of \(\alpha^{n-1}\) in Theorems 1 and 2.

**Lemma 4.** Let \(k \geq 2\) be an integer, and let \(m^{(k)}(x) = \frac{x - 1}{2 + (k + 1)(x - 2)}\). Then,

1. \(m^{(k)}(2 - 1/k) = 1\).
2. \(m^{(k)}(2) = \frac{1}{2}\).
3. \(m^{(k)}(x)\) is continuous and decreasing on the interval \([2 - 1/k, \infty)\).
4. \(m^{(k)}(x) > \frac{1}{x}\) on the interval \((2 - 1/k, 2)\).

**Proof.** Parts 1 and 2 are immediate. As for 3, note that we can rewrite \(m^{(k)}(x)\) as:

\[
m^{(k)}(x) = \frac{1}{k + 1} \left[ 1 + \frac{1}{x - (2 - \frac{2}{k+1})} \right]
\]

which is simply a scaled translation of the map \(y = 1/x\). In particular, since this \(m^{(k)}(x)\) has a vertical asymptote at \(x = 2 - \frac{2}{k+1}\), then by parts 1 and 2 we can conclude that \(m^{(k)}(x)\) is indeed continuous and decreasing on the desired interval.

To show part 4, we first note that in solving \(\frac{1}{x} = m^{(k)}(x)\), we obtain a quadratic equation with the two intersection points \(x = 2\) and \(x = k\). It’s easy to show that \(\frac{1}{x} < m^{(k)}(x)\) at \(x = 2 - 1/k\), and since both functions \(\frac{1}{x}\) and \(m^{(k)}(x)\) are continuous on the interval \([2 - 1/k, \infty)\) and intersect only at \(x = 2\) and \(x = k \geq 2\), we can conclude that \(\frac{1}{x} < m^{(k)}(x)\) on the desired interval.

\[\Box\]
Lemma 5. For a fixed value of $k \geq 2$ and for $n \geq 2 - k$, define $E_n$ to be the error in our Binet approximation of Theorem 2, as follows:

$$E_n = F_n^{(k)} - \frac{\alpha - 1}{2 + (k + 1)(\alpha - 2)} \cdot \alpha^{n-1}$$

$$= F_n^{(k)} - m^{(k)}(\alpha) \cdot \alpha^{n-1},$$

... for $\alpha$ the positive real root of $x^k - x^{k-1} - \cdots - x - 1 = 0$ and $m^{(k)}$ as defined in Lemma 4. Then, $E_n$ satisfies the same recurrence relation as $F_n^{(k)}$:

$$E_n = E_{n-1} + E_{n-2} + \cdots + E_{n-k} \quad \text{(for } n \geq 2)$$

Proof. By definition, we know that $F_n^{(k)}$ satisfies the recurrence relation:

$$F_n^{(k)} = F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{(9)}$$

As for the term $m^{(k)}(\alpha) \cdot \alpha^{n-1}$, note that $\alpha$ is a root of $x^k - x^{k-1} - \cdots - 1 = 0$, which means that $\alpha^k = \alpha^{k-1} + \cdots + 1$, which implies

$$m^{(k)}(\alpha) \cdot \alpha^{n-1} = m^{(k)}(\alpha)\alpha^{n-2} + \cdots + m^{(k)}(\alpha)\alpha^{n-(k+1)} \quad \text{(10)}$$

We combine Equations (9) and (10) to obtain the desired result. □

4 Proof of Theorem 1

As mentioned above, Spickerman and Joyner [12] proved the following formula for the $k$-generalized Fibonacci numbers:

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i^{k+1} - \alpha_i^k}{2\alpha_i^k - (k + 1)} \cdot \alpha_i^{n-i} \quad \text{(11)}$$

Recall that the set $\{\alpha_i\}$ is the set of roots of $x^k - x^{k-1} - \cdots - 1 = 0$. We now show that this formula is equivalent to our equation (2) in Theorem 1:

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k + 1)(\alpha_i - 2)} \cdot \alpha_i^{n-i} \quad \text{(12)}$$

Since $\alpha_i^k - \alpha_i^{k-1} - \cdots - 1 = 0$, we can multiply by $\alpha_i - 1$ to get $\alpha_i^{k+1} - 2\alpha_i^k = -1$, which implies $(\alpha_i - 2) = -1 \cdot \alpha_i^{-k}$. We use this last equation to transform (12) as follows:

$$\frac{\alpha_i - 1}{2 + (k + 1)(\alpha_i - 2)} = \frac{\alpha_i - 1}{2 + (k + 1)(-\alpha_i^{-k})} = \frac{\alpha_i^{k+1} - \alpha_i^k}{2\alpha_i^k - (k + 1)}$$

This establishes the equivalence of the two formulas (11) and (12), as desired. □
5 Proof of Theorem 2

Let $E_n$ be as defined in Lemma 5. We wish to show that $|E_n| < \frac{1}{2}$ for all $n \geq 2 - k$. We proceed by first showing that $|E_n| < \frac{1}{2}$ for $n = 0$, then for $n = -1, -2, -3, \ldots, 2 - k$, then for $n = 1$, and finally that this implies $|E_n| < \frac{1}{2}$ for all $n \geq 2 - k$.

To begin, we note that since our initial conditions give us that $E_n^{(k)} = 0$ for $n = 0, -1, -2, \ldots, 2 - k$, then we need only show $|m^{(k)}(\alpha) \cdot \alpha^{n-1}| < 1/2$ for those values of $n$. Starting with $n = 0$, it’s easy to check by hand that $m^{(k)}(\alpha) \cdot \alpha^{-1} < 1/2$ for $k = 2$ and $3$, and as for $k \geq 4$, we have the following inequality from Lemma 3:

$$2 - \frac{1}{3} < \alpha,$$

which implies

$$\alpha^{-1} < \frac{3}{6k-1}.$$ 

Also, by Lemma 4,

$$m^{(k)}(\alpha) < m^{(k)}(2 - 1/3k) = \frac{3k - 1}{5k - 1},$$

so thus:

$$m^{(k)}(\alpha) \cdot \alpha^{-1} < \frac{3k - 1}{5k - 1} \cdot \frac{3k}{6k - 1} < \frac{(3k) \cdot 1}{(5k - 1) \cdot 2} < \frac{1}{2},$$

as desired. Thus, $0 < |m^{(k)}(\alpha) \cdot \alpha^{-1}| < 1/2$ for all $k$, as desired.

Since $\alpha^{-1} < 1$, we can conclude that for $n = -1, -2, \ldots, 2 - k$, then $|E_n| = m^{(k)}(\alpha) \cdot \alpha^n < 1/2$.

Turning our attention now to $E_1$, we note that $E_1^{(k)} = 1$ (again by definition of our initial conditions) and that

$$\frac{1}{2} = m(2) < m(\alpha) < m(2 - 1/k) = 1$$

which immediately gives us $|E_1| < 1/2$.

As for $E_n$ with $n \geq 2$, we know from Lemma 5 that

$$E_n = E_{n-1} + E_{n-2} + \cdots + E_{n-k} \quad \text{(for } n \geq 2\text{)}$$

Suppose for some $n \geq 2$ that $|E_n| \geq 1/2$. Let $n_0$ be the smallest positive such $n$. Now, subtracting the following two equations:

$$E_{n_0+1} = E_{n_0} + E_{n_0-1} + \cdots + E_{n_0-(k-1)}$$
$$E_{n_0} = E_{n_0-1} + E_{n_0-2} + \cdots + E_{n_0-k}$$

gives us:

$$E_{n_0+1} = 2E_{n_0} - E_{n_0-k}$$

Since $|E_{n_0}| \geq |E_{n_0-k}|$ (the first, by assumption, being larger than, and the second smaller than, $1/2$), we can conclude that $|E_{n_0+1}| > |E_{n_0}|$. In fact, we can apply this argument repeatedly to show that $|E_{n_0+i}| > \cdots > |E_{n_0+1}| > |E_{n_0}|$. However, this contradicts the observation from equation (3) that the error must eventually go to 0. We conclude that $|E_n| < 1/2$ for all $n \geq 2$, and thus for all $n \geq 2 - k$. \qed
6 Acknowledgement

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References


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(Concerned with sequences A000073, A000078, and A001591.)

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