Experiment # 4
Mersenne's Law and the Harmonic Series for Strings
PHYS/MUS 102—Spring 2016

Background

Standing waves are the basis for the ***harmonic series*** that a stringed instrument plays. Standing waves occur at certain ***resonant frequencies*** at which an integer number of "humps" (1/2 wavelengths) can fit on the string which is fixed at both ends. The shape of the standing wave waveform (how many nodes and anti-nodes) and the frequency at which that shape is achieved are intimiately linked; together they are known as a ***mode***. Therefore, the harmonic series defines what notes (pitches) and instrument can play, thus it determines, in part, an instruments sound quality, or ***timbre***. The resonant frequencies are predicted by Mersenne's Law, which you will explore in this lab.

Figure 1: The legendary Carlos Santana jamming on a guitar. Or is he just testing Mersenne's Law?

Theory: Mersenne's Law

Mersenne's Law states that the ***fundamental frequency*** (or ***first harmonic***) of a fixed-fixed string is:

$$f_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}} \tag{1}$$

where L is the length of the string *in vibrational motion*; T is the tension applied to the string, and μ is the mass per unit length: m_{string}/L_{string}.

Therefore, the the 2nd harmonic occurs at

$$2f_1 = f_2 = \frac{2}{2L}\sqrt{\frac{T}{\mu}}$$

The 3rd harmonic occurs at

$$3f_1 = f_3 = \frac{3}{2L}\sqrt{\frac{T}{\mu}}$$

and so on.

In general, the Nth harmonic occurs at

$$f_N = N f_1 = \frac{N}{2L}\sqrt{\frac{T}{\mu}} \tag{2}$$

Fig. 2 illustrates the fundamental mode and the first several harmonics of the harmonic series.

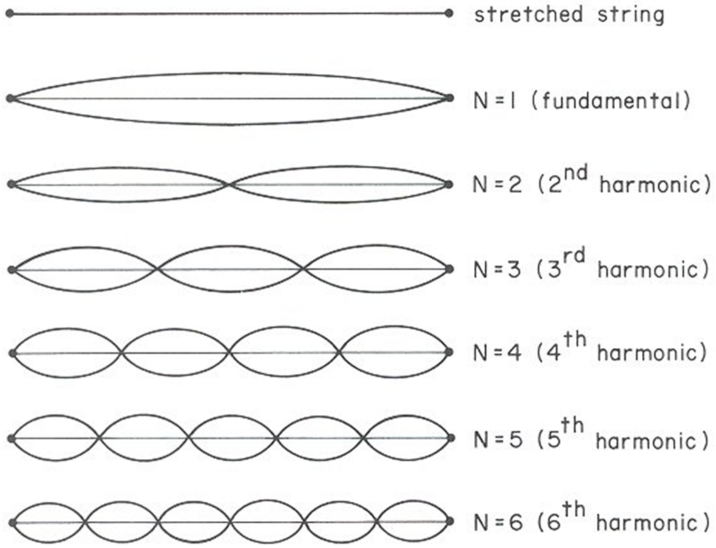

Figure 2: Harmonic Series for a fixed-fixed spring. The fixed-fixed string is a good model for stringed instruments: guitar, bass, violin, etc.

The Experiment

Equipment Setup

Before setting up the apparatus, measure the weight of the string using the scale provided. Measure the length, then compute and report the mass density, μ. Be careful to note that the mass density is the mass divided by the entire length of the string (assuming you put the entire length of the string on the scale).

Next, position the standing wave apparatus using the Pasco oscillator. There are various stands, pulleys, clamps, etc. for this arrangement. The instructor and/or TA will help illustrate the setup and usage of the equipment, which is also roughly diagrammed in Fig. 3 below.

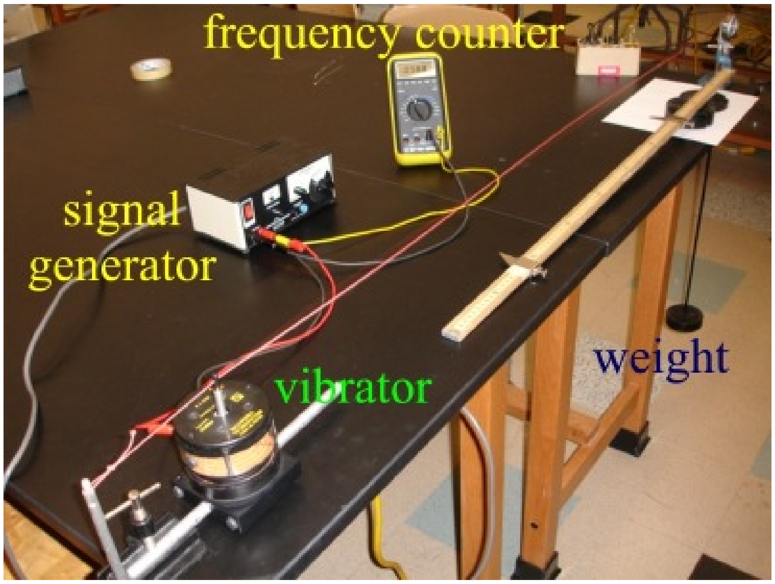

Figure 3: Basic setup for Mersenne's Law investigation. A wave generator drives a string, whose tension is set by hanging a weight over a pulley. Image credit: `http://faculty.colostate-pueblo.edu/stephen.wallin/P201L/expswave.htm`

Marsenne's Law and Harmonic Series

There are 3 main branches of this experiment, each of which investigates one aspect of Mersenne's Law.

Harmonic Series: 1, 2, 3,...

Your mission is to test the integer relationship (N) in Marsenne's Law generating the first 5 (or more) harmonics in the harmonic series. To start, set the length of the string such that it spans the entire lab table. Measure the length of string that is free to oscillate between the two fixed points. Set the tension by connecting about $m \approx 150$ grams total mass (the tension applied is $T = mg$, and g is the gravitational constant).

Before venturing on, compute the theoretical value for the fundamental frequency f_1 using Mersenne's Law. This value will help you begin a *directed* search to determine the experimental fundamental frequency. Then continue on to find the remainder of the harmonics, $f_2, f_3, \ldots$.

The first 5 harmonics should be sufficient. Record in a table the theoretical and experimentally determined values as you go. Be sure to compute % difference as well.

How did the harmonic series look in terms of frequency? Specifically, was each harmonic N times the fundamental frequency f_1? One great way to qauntitatively assess this is to create a plot of $f_{experimental}$ vs $f_{theoretical}$ and make a best fit line. What should the slope of a best fit line be if this case? What is the slope of this best fit line in reality? Therefore, what can you conclude?

Tuning up via Variable Tension

Next, you will find the fundamental frequency for various tensions applied. This is akin to tuning a stringed instrument! Find the fundamental frequency for five different tensions (leaving the length constant). Using masses in the tens-to-hundreds range should work well, for example: 50, 75, 100, 150, 200 g. Record the theoretical and experimental values for the fundamental frequency as you go. Also, compute % differences.

To assess Mersenne's Law, make a plot of $\log f$ versus the $\log T$. Doing so tells you how the frequency varied with the tension. By now, you know the drill: Make a best-fit line to the data, and find the slope. If Marsenne's Law holds exactly, what should the slope be? Compare/contrast theory vs. experiment. In your report, be sure to comment on the musical connection between increasing tension and fundamental frequencies and what you ultimately hear.

The Long and Short of It: Setting Pitch via String Length

Investigate how the fundamental frequency varies with the length of the string. You need to find a friend/classmate with a stringed instrument to do this part of the lab. It helps if one of you has some experience with playing a stringed instrument.

Using a microphone and Audacity software, determine the fundamental frequency of a plucked string of different lengths, as determined by where you fret the string/place it on a fingerboard. Don't use a piano for this experiment, because the strings have variable mass density and tension, which are unknown and not easy to measure. So use an instrument where you can isolate length as the only variable. Carefully record the length of the string and the fundamental frequency as you go. For musical context, write-down the name of the note you are playing and look up the "standard" fundamental pitch of that note (for instance A4 = 440 Hz). This standard pitch will be your basis for your comparison for measured frequency. In your results, include a table where you compute the % difference of the experimentally determined fundamental frequency relative to the pitch (frequency) you expected to play based on musical experience (the one you looked up).

Of course, the big issue you are also trying to address: Does Mersenne's Law accurately predict the relation between fundamental frequency and the length of the string (all other variables held constant). To assess this relation, perform a similar analysis as before: plot $\log f$ vs $\log L$. Find the slope of the best fit line, and compare to the predicted value for the slope if Marsenne's Law holds exactly.

The Report

Your report should include 3 of the 4 main pillars: Intro, Results, and Discussion (no Methods).

For each mini-mission be sure to analyze and interpret your data to address the main question at hand. (By now, you are practically an old pro at this!) Of course, include figures and tables, where appropriate (you should have, at the very least, a graph for each mini-mission). Perform an uncertainty analysis for each of your results to help draw conclusions as to the validity of Mersenne's Law (see, for example `http://home.wlu.edu/~ericksonj/phys102_s2016/other/UncertaintyReview_2015.pdf`).

As for the intro section, be sure to make sure to draw the link between the experimental setup for each "mission" above and the real-world musical context. Afterall, as musicians, we really do want to know why we are doing what we are doing in the physics lab!