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SUBNORMAL OPERATORS, SELF-COMMUTATORS, AND PSEUDOCONTINUATIONS

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We study pure subnormal operators whose self-commutators have zero as an eigenvalue. We show that various questions in this area are closely related to questions involving approximation by functions satisfying $\overline{\partial}^2 f = 0$ and to the study of *generalized quadrature domains*.

First some general results are given that apply to all subnormal operators within this class; then we consider characterizing the analytic Toeplitz operators, the Hardy operators and cyclic subnormal operators whose self-commutators have zero as an eigenvalue.

1 Introduction

The most well understood pure subnormal operator is, undoubtedly, the unilateral shift; in fact many of its properties are nearly characteristic. That is, other pure subnormal operators sharing certain properties with the shift are very closely related to it. For instance, up to a scaling and a translation the unilateral shift is the only pure subnormal operator with rank one self commutator, see [8].

Subnormal operators with finite rank self commutator have received considerable attention recently and the cyclic ones are close relatives of the unilateral shift (see [6], [7], [9], and [14]); in fact, if they have the same spectral properties as the shift, then they are simply finite dimensional extensions of the shift.

In this paper we take this one step further and consider pure subnormal operators whose self commutators have zero as an eigenvalue; or equivalently, operators whose self commutators do not have dense range. One may easily see that this is a larger class of operators than those with finite rank self-commutator. However one may still expect that such operators should be "close" to the unilateral shift.

The (pure) cyclic subnormal operators with finite rank self-commutators have a close connection with quadrature domains. A region G is a *quadrature domain* if there exists a meromorphic function S on G, called the *Schwarz function*, having only finitely many poles in G, continuous up to the boundary of G and agreeing with $\overline{z}$ on the boundary of G. For simply connected regions, an equivalent formulation is that the Riemann map from the unit disk onto G is a rational function.

One of the important ideas in this paper is the introduction of a new class of domains, called *generalized quadrature domains*. A region G is a generalized quadrature domain if there exists a Nevanlinna function (a quotient of bounded analytic functions) S on G, called the *generalized Schwarz function*, that has *boundary values* (think of the slit disk, the radial limits from above and below must agree along the slit) that agree with $\overline{z}$ a.e. on ∂G. We shall show that there exists (simply connected) generalized quadrature domains bounded by smooth Jordan curves that are not quadrature domains. Further, we shall see that the pure cyclic subnormal operators whose self-commutators have zero as an eigenvalue are closely related to these generalized quadrature domains.

An equivalent way to discuss generalized quadrature domains that avoids boundary values is in terms of the universal analytic covering maps. We show that a bounded region G is a generalized quadrature domain if and only if the analytic covering map from the unit disk onto G has a *pseudocontinuation* to the exterior of the unit disk. A pseudocontinuation of a Nevanlinna class function f on the unit disk is a (meromorphic) Nevanlinna class function F on the exterior of the unit disk such that the nontangential boundary values of f and F agree a.e. on the circle. This is a standard definition in complex analysis, and first made an appearance in operator theory in 1970 when Douglas, Shapiro and Shields [5] characterized the cyclic vectors for the backward shift.

This paper begins, in section 2, by considering general properties of pure subnormal operators having zero as an eigenvalue for its self commutator. We shall see that this implies that $R^\infty(\sigma(S),\mu)+\overline{z}R^\infty(\sigma(S),\mu)$ is not weak* dense in $L^\infty(\mu)$; where μ is a scalar valued spectral measure for S and $R^\infty(K,\mu)$ denotes the weak* closure, in $L^\infty(\mu)$, of the rational functions with poles off K.

In particular, it follows that if S is a pure subnormal operator and $0\in\sigma_p([S^*,S])$, then $\sigma(S)$ has nonempty interior.

We also characterize the scalar valued spectral measures (svsm) of pure subnormal operators S having $0\in\sigma_p([S^*,S])$. They are those measures μ such that $P^\infty(\mu)$ has no L^∞ summand and $P^\infty(\mu)+\overline{z}P^\infty(\mu)$ is not weak* dense in $L^\infty(\mu)$; where $P^\infty(\mu)$ denotes the weak* closure of the (analytic) polynomials in $L^\infty(\mu)$.

A *part* of an operator S is S restricted to an invariant subspace. We shall also consider pure subnormal operators S such that every part of S has zero as an eigenvalue for its self commutator. By comparison, this implies that for every positive integer n, $R^\infty(\sigma(S),\mu)+\overline{z}R^\infty(\sigma(S),\mu)+\cdots+\overline{z}^nR^\infty(\sigma(S),\mu)$ is not weak* dense in $L^\infty(\mu)$, where $\mu=svsmS$. For example, the unilateral shift has this property. We shall see that if every part of S has zero as an eigenvalue for its self commutator, then the normal spectrum of S, $\sigma_n(S)$, has no interior.

In the remainder of the paper we consider special classes of subnormal operators. In section 3 we consider analytic Toeplitz operators on H^2. For $f\in H^\infty(\mathbb{D})$, let T_f denote the corresponding Toeplitz operator on $H^2(\mathbb{D})$. It is well known that T_f has finite rank self commutator if and only if f is a rational function. We shall show that T_f has zero as an eigenvalue for its self commutator if and only if f has a *pseudocontinuation* to the exterior of the unit disk.

In section 4 we determine when the operators $S_G = M_z$ on $H^2(G)$ have zero as an eigenvalue for their self-commutators. We shall show that for an arbitrary bounded region G, zero is an eigenvalue for the self-commutator of S_G if and only if the universal covering map from the unit disk onto G has a pseudocontinuation; that is, precisely when G is a *generalized quadrature domain*.

We obtain partial results in section 5 on characterizing the irreducible cyclic subnormal operators having zero as an eigenvalue for their self commutator. In [6], [7], [9], and [14] it was shown that if S is an irreducible cyclic subnormal operator and G is the set of analytic bounded point evaluations for S, then S has finite rank self-commutator if and only if G is a quadrature domain and S is a finite dimensional extension of $S_G = M_z$ on $H^2(G)$. Thus the natural conjecture is that zero is an eigenvalue for the self-commutator of S if and only if G is a generalized quadrature domain and S is an arbitrary extension of $S_G = M_z$ on $H^2(G)$. Although these extensions are understood, the question remains unsolved in general.

However, it is shown that if $S = M_z$ on $P^2(\mu)$ is an irreducible cyclic subnormal operator, G is the set of analytic bounded point evaluations for S, and there is a non-zero Nevanlinna class function in $ker[S^*, S]$, then G is a generalized quadrature domain. Conversely, if G is a generalized quadrature domain and $\mu|G$ is a discrete measure whose atoms form an $H^\infty(G)$ zero set, then $ker[S^*, S]$ contains a bounded function.

In what follows all Hilbert spaces are separable and complex. If S is a subnormal operator, then $mneS$ denotes the minimal normal extension of S and $svsmS$ denotes the scalar valued spectral measure of S. The self-commutator of S, denoted by $[S^*, S]$, is the self-adjoint operator $S^*S - SS^*$. For other standard results, terminology and notation see Conway [4].

2 General Subnormal Operators

In this section we present some general results that apply to any pure subnormal operator S with $0 \in \sigma_p([S^*, S])$. The first result is well known and fundamental in what follows.

PROPOSITION 2.1. *If S is a subnormal operator on $\mathcal{H}$ and $N = mneS$ on $\mathcal{K}$, then*

$$ker[S^*, S] = \{h \in \mathcal{H} : N^*h \in \mathcal{H}\}.$$

In particular, $ker[S^*, S]$ is invariant for any operator on $\mathcal{H}$ that lifts to the commutant of N.

Proof. If $N = \begin{bmatrix} S & X \\ 0 & T^* \end{bmatrix}$ with respect to $\mathcal{H} \oplus \mathcal{H}^\perp$, then a matrix computation gives that $[S^*, S] = XX^*$. So $ker[S^*, S] = kerX^*$, but $X^* = P^\perp N^*|\mathcal{H}$ where P is the projection of $\mathcal{K}$ onto $\mathcal{H}$. It follows that $kerX^* = \{h \in \mathcal{H} : N^*h \in \mathcal{H}\}$. If A is an operator on $\mathcal{K}$ that commutes with N and leaves $\mathcal{H}$ invariant, then A also commutes with N^*. So if $h \in ker[S^*, S]$, then $N^*Ah = AN^*h \in A\mathcal{H} \subseteq \mathcal{H}$. Thus, $Ah \in ker[S^*, S]$. Hence A leaves $ker[S^*, S]$ invariant. □

COROLLARY 2.2. *If S is a subnormal operator, $\mathcal{M} \in LatS$ and $T = S|\mathcal{M}$, then $ker[T^*, T] \subseteq ker[S^*, S]$.*

Proof. Simply note that if $N = mneS$, then $mneT$ is N restricted to a reducing subspace. Thus $ker[T^*, T] = \{h \in \mathcal{M} : N^*h \in \mathcal{M}\} \subseteq \{h \in \mathcal{H} : N^*h \in \mathcal{H}\} = ker[S^*, S]$. □

It follows that if S is a subnormal *extension* of T and $0 \in \sigma_p([T^*, T])$, then also $0 \in \sigma_p([S^*, S])$. Let $P^2(\mu)$ be the closure of the polynomials in $L^2(\mu)$.

EXAMPLE 2.3. *The following operators have zero as an eigenvalue for their self commutators.*

(a) Operators with finite rank self commutators.

(b) Quasinormal Operators.

(c) $S_\mu = M_z$ on $P^2(\mu)$ where μ is arc length measure on $\partial\mathbb{D}$ plus a weighted sum of point masses at points in $\mathbb{D}$ that form a Blaschke sequence.

In section 4 we shall characterize the bounded regions G such that M_z on $H^2(G)$ has zero as an eigenvalue for its self commutator. They are those regions such that the universal covering map has a pseudocontinuation. However, in contrast, if G is a bounded region, then zero is not an eigenvalue for the self commutator of the Bergman operator over G. This follows easily from Proposition 2.1.

We now observe that the question of whether or not zero is an eigenvalue for $[S^*, S]$ is equivalent to an approximation problem.

PROPOSITION 2.4. *If S is a pure subnormal operator on $\mathcal{H}$ and $N = mneS$ on $\mathcal{K}$, then $[S^*, S]$ is one–to–one if and only if $N(\mathcal{H}^\perp) + \mathcal{H}^\perp$ is dense in $\mathcal{K}$.*

Proof. Simply observe that $\{N(\mathcal{H}^\perp) + \mathcal{H}^\perp\}^\perp = \{h \in \mathcal{H} : N^*h \in \mathcal{H}\}$ and apply Proposition 2.1. □

Let $R(K)$, $R^\infty(K, \mu)$ and $R^2(K, \mu)$ denote the uniform, weak*, and $L^2(\mu)$ closure of the rational functions with poles off K. If S is a subnormal operator on $\mathcal{H}$ and $N = mneS$, then $dualS = N^*|\mathcal{H}^\perp$.

COROLLARY 2.5. *If $S = M_z$ on $R^2(K, \mu)$ is a pure rationally cyclic subnormal operator and $T = dualS$, then the following hold:*

(a) $0 \in \sigma_p([S^*, S])$ if and only if $zR^2(K, \mu)^\perp + R^2(K, \mu)^\perp$ is not dense in $L^2(\mu)$;

(b) $0 \in \sigma_p([T^*, T])$ if and only if $\overline{z}R^2(K, \mu) + R^2(K, \mu)$ is not dense in $L^2(\mu)$.

PROPOSITION 2.6. *Let S be a pure subnormal operator on $\mathcal{H}$ and $\mu = svsmS$. If $0 \in \sigma_p([S^*, S])$, then $\overline{z}R^\infty(\sigma(S), \mu) + R^\infty(\sigma(S), \mu)$ is not weak* dense in $L^\infty(\mu)$.*

Proof. Let $N = mneS$. By Proposition 2.1, there is a vector $h \in \mathcal{H}$ such that $N^*h \in \mathcal{H}$. Now, if $f \in \overline{z}R^\infty(\sigma(S), \mu) + R^\infty(\sigma(S), \mu)$, then $f(N)h \in \mathcal{H}$. So, if $\overline{z}R^\infty(\sigma(S), \mu) + R^\infty(\sigma(S), \mu)$ is weak* dense in $L^\infty(\mu)$, then $cl\{f(N)h : f \in \overline{z}R^\infty(\sigma(S), \mu) + R^\infty(\sigma(S), \mu)\}$ would be a reducing subspace for N that is contained in $\mathcal{H}$, contradicting the purity of S. □

We shall see later that there are some rather natural measures, such as arc length measure on certain Jordan curves, for which the above space is weak* dense in L^∞. Although the above result is quite simple, it has the following surprising consequence.

COROLLARY 2.7. *If S is a pure subnormal operator and $0 \in \sigma_p([S^*, S])$, then $\sigma(S)$ has nonempty interior.*

Proof. By a result of Trent and Wang [13], if K is a compact set with no interior, then $\overline{z}R(K) + R(K)$ is uniformly dense in $C(K)$. □

For example if the spectrum of S is a swiss cheese, then $[S^*, S]$ is one–to–one. Notice though, that there are quasinormal operators S (they always have $0 \in \sigma_p([S^*, S])$) such that $\sigma(S) = \sigma_{ap}(S)$. Hence the interior of $\sigma(S)$ need not be residual spectrum.

We now characterize the scalar valued spectral measures of pure subnormal operators whose self commutators have zero as an eigenvalue. Let $P^\infty(\mu)$ denote the weak* closure of the polynomials in $L^\infty(\mu)$.

THEOREM 2.8. *If μ is a positive compactly supported Borel measure in $\mathbb{C}$, then there is a pure subnormal operator S such that $0 \in \sigma_p([S^*, S])$ and $svsmS = \mu$ if and only if $P^\infty(\mu)$ has no L^∞ summand and $\overline{z}P^\infty(\mu) + P^\infty(\mu)$ is not weak* dense in $L^\infty(\mu)$.*

Proof. One direction follows from Proposition 2.6. So, assume that $P^\infty(\mu)$ has no L^∞ summand and $\overline{z}P^\infty(\mu) + P^\infty(\mu)$ is not weak* dense in $L^\infty(\mu)$.

Special Case: Suppose $\overline{z}P^\infty(\mu) + P^\infty(\mu)$ has no L^∞ summand.

In this case, Chaumat's Lemma (see Conway [4], p. 246) implies that there is a function $f \in L^1(\mu)$ such that $|f| > 0$ μ a.e. and $f \perp \overline{z}P^\infty(\mu) + P^\infty(\mu)$. Write $f = g\overline{h}$ where $g, h \in L^2(\mu)$. Now let $S = M_z$ on $\mathcal{H}$ where $\mathcal{H} = cl[\{\overline{z}P^\infty(\mu) + P^\infty(\mu)\}g]$ (closure in $L^2(\mu)$). It follows that S is a subnormal operator and clearly the self commutator is not one–to–one (g is in the kernel). Furthermore, as g is "never zero" we have that $\mu = svsmS$. Also, since h is "never zero" and $h \perp \mathcal{H}$, S is pure.

General Case: Since $P^\infty(\mu)$ has no L^∞ summand there is an equivalent measure ν such that $S_\nu = M_z$ on $P^2(\nu)$ is pure (see Conway [4], p. 87). Again using Chaumat's Lemma, there exists an $f \in L^1(\mu)$ that annihilates $\overline{z}P^\infty(\mu) + P^\infty(\mu)$ and has the property that if $g \in L^1(\mu)$ and g annihilates $\overline{z}P^\infty(\mu) + P^\infty(\mu)$, then $|g|d\mu \ll |f|d\mu$. Let $\Delta = \{x : f(x) \neq 0\}$ and let $\mu_1 = \mu|\Delta$. Thus, $\overline{z}P^\infty(\mu_1) + P^\infty(\mu_1)$ has no L^∞ summand. Hence by the special case above, there is a pure subnormal operator S_1 such that svsm$S_1 = \mu_1$ and $0 \in \sigma_p([S_1^*, S_1])$. Now, let $S = S_1 \oplus S_\nu$. Clearly, S is a pure subnormal operator and svsm$S = \mu$. Furthermore it follows from Corollary 2.2 that $0 \in \sigma_p([S^*, S])$. □

EXAMPLE 2.9. *If $\mu = m|\partial\mathbb{D} + \nu$ where m is arc length measure on $\partial\mathbb{D}$ and ν is any finite Borel measure carried by $\mathbb{D}$, then there is a pure subnormal operator S such that $0 \in \sigma_p([S^*, S])$ and $svsmS = \mu$.*

Proof. We shall apply Theorem 2.8. It is clear that $P^\infty(\mu) = H^\infty(\mathbb{D})$. Also, $\overline{z}P^\infty(\mu) + P^\infty(\mu)$ is not weak* dense in $L^\infty(\mu)$, because if it is weak* dense in $L^\infty(\mu)$, then also $\{\overline{z}p + q : p, q$ are polynomials $\}$ is weak* dense in $L^\infty(\mu|\Delta)$ for any measurable set Δ. But for the measure μ, $\{\overline{z}p + q : p, q$ are polynomials $\}$ is not weak* dense in $L^\infty(\partial\mathbb{D})$. Thus the Theorem applies. □

The previous proof shows that if μ is any measure on $cl\mathbb{D}$ such that $P^\infty(\mu) = H^\infty(\mathbb{D})$ and $\mu|\partial\mathbb{D}$ is equivalent to arc length measure, then $\overline{z}P^\infty(\mu) + P^\infty(\mu)$ is not weak* dense in $L^\infty(\mu)$. This raises an interesting question. If $\overline{z}P^\infty(\mu) + P^\infty(\mu)$ is weak* dense in $L^\infty(\mu)$ for some measure μ, then does this place any restrictions on the Sarason Hull of $P^\infty(\mu)$? The answer is yes when μ is harmonic measure.

QUESTION 2.10. *Is there a measure μ on $cl\mathbb{D}$ such that $P^\infty(\mu) = H^\infty(\mathbb{D})$ and $\overline{z}P^\infty(\mu) + P^\infty(\mu)$ is weak* dense in $L^\infty(\mu)$?*

See Proposition 3.4 for some examples of measures where $\overline{z}P^\infty(\mu) + P^\infty(\mu)$ is weak* dense in $L^\infty(\mu)$.

We now consider subnormal operators S such that every *part* of S has zero as an eigenvalue for its self-commutator. That is, we want to consider those subnormal operators S, such that if $\mathcal{M}$ is any invariant subspace for S and $T = S|\mathcal{M}$, then zero is an eigenvalue for $[T^*, T]$.

If U is the Unilateral shift and B is the Bergman shift, then $S = U \oplus B$ is a subnormal operator such that zero is an eigenvalue for $[S^*, S]$, yet not every part of S has this property. We shall give an example of an irreducible subnormal operator with this property shortly; see Example 2.16.

EXAMPLE 2.11. *If $\mu = m + \sum_{k=1}^\infty w_k\delta_{a_k}$; where m is Lebesgue measure on $\partial\mathbb{D}$, $\{a_k\}$ is a Blaschke sequence in $\mathbb{D}$, $w_k > 0$ and $\sum_{k=1}^\infty w_k < \infty$, then $S_\mu = M_z$ on $P^2(\mu)$ is a pure subnormal operator such that every part of S_μ has zero as an eigenvalue for its self-commutator.*

Proof. If $B(z)$ is the Blaschke product associated with $\{a_k\}$ and $f(z) = zB(z)$, then $f \in ker[S_\mu^*, S_\mu]$ and $f \in P^2(\mu) \cap L^\infty(\mu) = H^\infty(\mathbb{D})$. If $\mathcal{M}$ is any non-trivial invariant subspace for S_μ and $g \in \mathcal{M}$ is a non-zero function, then $fg \in ker[T^*, T]$, where $T = S_\mu|\mathcal{M}$. This follows because both f and $\overline{z}f$ belong to $H^\infty(\mathbb{D})$, and $\mathcal{M}$ is invariant under multiplication by functions in $H^\infty(\mathbb{D})$. Thus, $fg \in \mathcal{M}$ and $\overline{z}fg \in \mathcal{M}$. □

PROPOSITION 2.12. *Let S be a pure subnormal operator on $\mathcal{H}$ with $mneS = N$. If every part of S has zero as an eigenvalue for its self-commutator, then for each positive integer n, there exists a vector $h \in \mathcal{H}$ such that $N^{*k}h \in \mathcal{H}$ for all k satisfying $0 \le k \le n$.*

Proof. Proceeding by induction, for $n = 1$ simply apply Proposition 2.1. Now suppose the result is true for $n = n_0$; we must show it to be true for $n = n_0 + 1$. So, suppose there is an $h \in \mathcal{H}$ with $N^{*k}h \in \mathcal{H}$ for all $0 \le k \le n_0$. Let $\mathcal{M}$ be the invariant subspace for S generated by h. Set $T = S|\mathcal{M}$. Since T is a part of S, $0 \in \sigma_p([T^*, T])$. So, by Proposition 2.1, there exists a (non-zero) vector $g \in \mathcal{M}$ with $N^*g \in \mathcal{M}$. Also notice that for every vector $x \in \mathcal{M}$, it follows that $N^{*k}x \in \mathcal{H}$ for all $0 \le k \le n_0$. So, as $N^*g \in \mathcal{M}$, we have that $N^{*k}g \in \mathcal{H}$ for all k with $0 \le k \le n_0 + 1$. Thus the result follows. □

This leads to a nice function theoretic property. It should be compared with Proposition 2.6; since its proof is similar we shall leave it to the reader.

COROLLARY 2.13. *Let S be a pure subnormal operator and $\mu = svsmS$. If every part of S has zero as an eigenvalue for its self-commutator, then for each positive integer n, $R^\infty(\sigma(S),\mu) + \overline{z}R^\infty(\sigma(S),\mu) + \cdots + \overline{z}^n R^\infty(\sigma(S),\mu)$ is not weak* dense in $L^\infty(\mu)$.*

We now prove a basic result for cyclic operators; cyclic operators will be considered in more detail in section 5. If S is a pure cyclic subnormal operator, then $S = M_z$ on $P^2(\mu)$, the closure of the polynomials in $L^2(\mu)$, for some compactly supported Borel measure μ on $\mathbb{C}$. Let G denote the set of analytic bounded point evaluations (abpe) for S. Also, let $supp(\mu)$ denote the support of the measure μ.

THEOREM 2.14. *If S is a pure cyclic subnormal operator, $G = abpe(S)$ and $0 \in \sigma_p([S^*, S])$, then:*

(a) Each compact set $K \subseteq supp(\mu) \cap G$ satisfies $P(K) = C(K)$;

(b) For each $r > 0$, $supp(\mu) \cap G \cap \{z : |z| = r\}$ is discrete in G.

Proof. (a) If $f \in ker[S^*, S]$, then by Proposition 2.1, $\overline{z}f \in P^2(\mu)$. Let $K \subseteq supp(\mu) \cap G$ be compact. Choose a simply connected region G_1 that contains K and whose closure in contained in G. This can be done since G is simply connected (use a Riemann map). If $\hat{f}$ denotes the analytic extension of f to G, then we have $\hat{(\overline{z}f)} = \overline{z}f = \overline{z}\hat{f}$ μ a.e. on G. Since $\hat{f}$ is analytic on G and $cl(G_1)$ is compactly contained in G, $\hat{f}$ has only a finite number of zeros on $cl(G_1)$. Further, each of these is a zero of $\hat{(\overline{z}f)}$. Hence $g = \hat{(\overline{z}f)}/\hat{f}$ is analytic on G_1. Thus, since G_1 is simply connected, $g \in P(K)$. But $g = \overline{z}$ on K, so $\overline{z} \in P(K)$. Thus, $P(K) = C(K)$.

(b) Keeping the same notation, notice that $zg(z)$ is meromorphic on G and constant on $supp(\mu) \cap G \cap \{z : |z| = r\}$ for each $r > 0$. Hence this set must be discrete in G; that is, every limit point must lie in ∂G. □

The previous result says that if zero is an eigenvalue for the self commutator of a pure cyclic subnormal operator S, then $supp(\mu) \cap G$ cannot be to large. For example, it cannot have any interior points. In fact (b) implies that the Hausdorff dimension of $supp(\mu) \cap G$ is at most one. In particular it has area zero. We now present a nice application of this result.

Let $\sigma_n(S)$ denote the normal spectrum of S; that is, the spectrum of $mneS$.

PROPOSITION 2.15. *If S is a pure subnormal operator and every part of S has zero as an eigenvalue for its self-commutator, then $\sigma_n(S)$ has no interior.*

Proof. Suppose that S acts on $\mathcal{H}$ and $N = mneS$. Let $h \in \mathcal{H}$ be a separating vector for N (see Conway [4], p. 249). Let $\mathcal{M}$ be the invariant subspace for S generated by h. Thus, $S|\mathcal{M}$ is a pure cyclic subnormal operator that has zero as an eigenvalue for its self-commutator. Hence Theorem 2.14 implies that $\sigma_n(S|\mathcal{M})$ has no interior. However, since h is a separating vector, $svsm(S|\mathcal{M}) \approx svsm(S)$. Thus $\sigma_n(S) = \sigma_n(S|\mathcal{M})$. So, $\sigma_n(S)$ has no interior. □

Observe that Proposition 2.15 applies to some quasinormal operators. Implying that while they all have zero as an eigenvalue for their self commutators. Some quasinormal operators do not have the property that each of its parts has zero as an eigenvalue for its self commutator. One easily checks that the unilateral shift does have this property.

EXAMPLE 2.16. *If B is the Bergman shift and $S = dual B$, then S is irreducible, $0 \in \sigma_p([S^*, S])$, but not every part of S has zero as an eigenvalue for its self-commutator.*

Proof. It is known that S is irreducible (see Conway [3]). It is easy to check that $\overline{z}L^2_a(\mathbb{D}) + L^2_a(\mathbb{D})$ is not dense in $L^2(\mathbb{D})$, since $\overline{z}^2$ is orthogonal to it. Thus, Corollary 2.5 implies that $0 \in \sigma_p([S^*, S])$. Since $\sigma_n(S) = cl\mathbb{D}$, Proposition 2.15 says that not every part of S has zero as an eigenvalue for its self-commutator. □

COROLLARY 2.17. *If S is a pure subnormal operator and every part of S has zero as an eigenvalue for its self-commutator, then $\sigma(S) \neq \sigma_{ap}(S)$.*

Proof. Since $0 \in \sigma_p([S^*, S])$, Corollary 2.7 says that $\sigma(S)$ has nonempty interior. However, $\sigma_{ap}(S) \subseteq \sigma_n(S)$ and $\sigma_n(S)$ has no interior by Proposition 2.15. Hence $\sigma(S) \neq \sigma_{ap}(S)$. □

3 Analytic Toeplitz Operators

In this section we shall consider the question, when does T_f, the analytic Toeplitz operator on $H^2(\mathbb{D})$, have zero as an eigenvalue for its self-commutator. We shall use the results of Section 2 to relate this to an approximation problem on the circle and also to the classical notion of *pseudocontinuation*.

Recall that if f is a Nevanlinna function on $\mathbb{D}$, then f has a *pseudocontinuation* (to the exterior of the disk) if there exists a (meromorophic) *Nevanlinna function* F on $\{z : |z| > 1\} \cup \{\infty\}$ such that the radial limits of f and F agree a.e. on the unit circle. If g is a function, then g^* is defined by $g^*(z) = \overline{g(1/\overline{z})}$. Thus, if F is defined on the exterior disk, then F is Nevanlinna function if and only if F^* is a Nevanlinna function on the disk. If K is a compact set, then $P(K)$ denotes the uniform closure of the polynomials on K. Thus, $P(\partial\mathbb{D})$ may be identified with the disk algebra. The following is our main theorem for this section.

THEOREM 3.1. *If $f \in H^\infty(\mathbb{D})$, then the following are equivalent:*

(a) $0 \in \sigma_p([T_f^*, T_f])$;

(b) $0 \in \sigma_p([S^*, S])$ *where* $S = dual(T_f)$;

(c) *f has a pseudocontinuation;*

(d) *$f \perp \mathcal{M}$ for some invariant subspace $\mathcal{M}$ of the Unilateral shift;*

(e) *$H^2 + \bar{f}H^2$ is not dense in $L^2(\partial\mathbb{D})$;*

(f) *$H^\infty + \bar{f}H^\infty$ is not weak* dense in $L^\infty(\partial\mathbb{D})$;*

If f is continuous on $\partial\mathbb{D}$, then the above are equivalent to:

(g) $P(\partial\mathbb{D}) + \overline{f}P(\partial\mathbb{D})$ is not uniformly dense in $C(\partial\mathbb{D})$.

Proof. We first show (a) $\Leftrightarrow$ (c). If $g \in ker[T_f^*, T_f]$ and $g \neq 0$, then by Proposition 2.1 we have that there is a function $h \in H^2$ such that $\overline{f}g = h$ a.e. on $\partial\mathbb{D}$. So, h/g is a Nevanlinna class function on $\mathbb{D}$ whose boundary values agree with those of $\overline{f}$. Hence h^*/g^* is a Nevanlinna function on $\mathbb{C}_\infty - cl\mathbb{D}$ whose boundary values agree a.e. with those of f. So, h^*/g^* is a pseudocontinuation of f.

Now suppose f has a pseudocontinuation F. If $F = g/h$ where g and h are bounded analytic functions on $\mathbb{C}_\infty - cl\mathbb{D}$, then g^*/h^* is a Nevanlinna function on $\mathbb{D}$ whose boundary values agree a.e. with those of $\overline{f}$ Hence, $\overline{f}h^* = g^*$ a.e. on the circle. Thus, $\overline{f}h^* \in H^2$. So, by Proposition 2.1 we see that $h \in ker[T_f^*, T_f]$.

(a) $\Leftrightarrow$ (b). It is known that $dual(T_f) \cong T_{f^\#}$, where $f^\#(z) = \overline{(f(\overline{z})}$ (see Conway [3]). Now simply observe that f has a pseudocontinuation if and only if $f^\#$ does also.

(b) $\Leftrightarrow$ (e). This follows immediately from Proposition 2.4.

(e) $\Rightarrow$ (f) $\Rightarrow$ (g). These are obvious.

(g) $\Rightarrow$ (c). Here we will assume that f is continuous on $cl\mathbb{D}$. If this is not the case, then a similar proof shows that (f) $\Rightarrow$ (c). If $P(\partial\mathbb{D}) + \overline{f}P(\partial\mathbb{D})$ is not uniformly dense in $C(\partial\mathbb{D})$, then there is a non-zero measure ν on $\partial\mathbb{D}$ such that $\nu \perp \{P(\partial\mathbb{D}) + \overline{f}P(\partial\mathbb{D})\}$. Since $\nu \perp P(\partial\mathbb{D})$, the F. & M. Riesz Theorem says that $\nu \ll m$, where m is arc length measure on the circle, and $d\nu/dm \in H^1$. So, there exists an $h \in H^1$ with $\nu = hdm$. Now also, $\nu \perp \overline{f}P(\partial\mathbb{D})$. So, $0 = \int z^n\overline{f}d\nu = \int z^n\overline{f}hdm$, for all $n \geq 0$. Thus, $\overline{f}h \in H^1$. So, let $g \in H^1$ be such that $\overline{f}h = g$ a.e. on $\partial\mathbb{D}$. Thus, g/h is a Nevanlinna function on $\mathbb{D}$ that agrees with $\overline{f}$ a.e. on $\partial\mathbb{D}$. Hence g^*/h^* is the required pseudocontinuation of f.

Thus (a) thru (c) and (e) thru (g) are all equivalent. The fact that (c) and (d) are equivalent first appeared in Douglas, Shapiro and Shields [5]. We include the following simple proof.

(a) $\Leftrightarrow$ (d). Simply observe that if $f \perp \mathcal{M}$ for some shift invariant subspace $\mathcal{M}$, then $\mathcal{M} \subseteq ker[T_f^*, T_f]$. Also a simple computation gives that $f \perp zker[T_f^*, T_f]$, and this latter space in invariant for the Unilateral shift by Proposition 2.1. □

It is an unsolved problem to characterize the compact sets K such that $\overline{z}P(K) + P(K)$ is uniformly dense in $C(K)$. The following result solves this problem for Jordan curves.

COROLLARY 3.2. *If Γ is a Jordan curve, $G = ins\Gamma$, and ω is harmonic measure for some point $a \in G$, then the following are equivalent:*

(a) $\overline{z}P(\Gamma) + P(\Gamma)$ is uniformly dense in $C(\Gamma)$;

(b) $\overline{z}P^\infty(\omega) + P^\infty(\omega)$ is weak dense in $L^\infty(\omega)$;*

(c) $\overline{z}P^2(\omega) + P^2(\omega)$ is dense in $L^2(\omega)$;

(d) The Riemann map of $\mathbb{D}$ onto G does not have a pseudocontinuation.

Proof. Let $f : \mathbb{D} \to G$ be a Riemann map with $f(0) = a$. Since $\Gamma = \partial G$ is a Jordan curve, f extends to a homeomorphism of $cl\mathbb{D}$ onto clG. Thus composition with f is an isometry from the appropriate function space on ∂G *onto* the corresponding space on $\partial\mathbb{D}$. Thus, we may apply Theorem 3.1. □

COROLLARY 3.3. *If $f \in P(\partial\mathbb{D})$, then the following are equivalent:*

(a) $P(\partial\mathbb{D}) + \overline{f}P(\partial\mathbb{D})$ *is uniformly dense in* $C(\partial\mathbb{D})$.

(b) $P(\partial\mathbb{D}) + \overline{f}P(\partial\mathbb{D}) + \cdots + \overline{f}^n P(\partial\mathbb{D})$ *is uniformly dense in* $C(\partial\mathbb{D})$ *for some* $n \geq 1$.

Proof. One direction is clear. So, suppose that $P(\partial\mathbb{D})+\overline{f}P(\partial\mathbb{D})+\cdots+\overline{f}^n P(\partial\mathbb{D})$ is uniformly dense in $C(\partial\mathbb{D})$ for some $n \geq 1$. Consider the analytic Toeplitz operator T_f. We claim that $[T_f^*, T_f]$ is one-to-one. If not, then $ker[T_f^*, T_f]$ is an invariant subspace for the Unilateral shift; see Proposition 2.1. Thus Beurling's Theorem implies that there exists a bounded function in $ker[T_f^*, T_f]$, call it g. Hence it follows from Proposition 2.1 that $g^n \in H^2$ and $\overline{f}^k g^n \in H^2$ for all $0 \leq k \leq n$. In particular, if $h \in P(\partial\mathbb{D}) + \overline{f}P(\partial\mathbb{D}) + \cdots + \overline{f}^n P(\partial\mathbb{D})$, then it follows easily that $hg^n \in H^2$. Thus, upon taking limits, we see that $hg^n \in H^2$ for all $h \in C(\partial\mathbb{D})$. Clearly this is not true. Hence our claim that $[T_f^*, T_f]$ is one-to-one is justified. Thus it follows from Theorem 3.1 that $P(\partial\mathbb{D}) + \overline{f}P(\partial\mathbb{D})$ is uniformly dense in $C(\partial\mathbb{D})$. □

It is easy to see that there are also weak* and L^2 versions of Corollary 3.3, where f is not necessarily continuous on the circle.

We now give some examples showing that $\overline{z}P^\infty(\mu) + P^\infty(\mu)$ can be weak* dense in $L^\infty(\mu)$. Recall, that a curve is called an *algebraic curve* if it is the zero set of a polynomial in z and $\overline{z}$. So, for example, circles, ellipses, lines, parabolas etc. are algebraic.

PROPOSITION 3.4. *If G is the inside of a Jordan curve and an arc I on ∂G is also an arc on an algebraic curve Γ, then $\overline{z}P(\partial G) + P(\partial G)$ is uniformly dense in $C(\partial G)$ provided that $\partial G \neq \Gamma$.*

Proof. Suppose that $\partial G \neq \Gamma$. We shall make use of Corollary 3.3. Let R be a polynomial in z and $\overline{z}$ such that the zero set of R equals Γ. Let $n \geq 1$ be the largest power of $\overline{z}$ that appears in R. We shall show that $P(\partial G) + \overline{z}P(\partial G) + \cdots + \overline{z}^n P(\partial G)$ is dense in $C(\partial G)$. This, together with Corollary 3.3 and a Riemann mapping argument will finish the proof.

So let μ be a measure on ∂G such that $\mu \perp P(\partial G)+\overline{z}P(\partial G)+\cdots+\overline{z}^n P(\partial G)$. Let I be the arc on Γ and on ∂G. So, $R = 0$ on I, but R is not zero on all of ∂G (because $\partial G \neq \Gamma$). Also, $R \in P(\partial G) + \overline{z}P(\partial G) + \cdots \overline{z}^n P(\partial G)$, so $\int pR d\mu = 0$ for all polynomials p. Observe that the measure $Rd\mu$ is supported on $K = cl(\partial G - I)$. Further, K is a polynomially convex set with no interior. It follows, by Lavrentiev's Theorem (see [4], p.232) that $P(K) = C(K)$. Thus $Rd\mu$ is the zero measure. That is, $|\mu|$ is zero on $\partial G \cap \{z : R(z) \neq 0\}$. Since $\partial G \neq \Gamma$, this latter set is an open subset of ∂G, hence contains an open arc J. Thus, μ is supported on the compact set $\partial G - J$ and annihilates the polynomials. Hence, again Lavrentiev's Theorem implies $|\mu| = 0$ on $\partial G - J$. Thus $\mu = 0$ and $P(\partial G) + \overline{z}P(\partial G) + \cdots + \overline{z}^n P(\partial G)$ is dense in $C(\partial G)$. □

The above technique even works for certain regions where ∂G is not a Jordan curve. For example, if G is the upper half of the unit disk with some slits.

The following result is well known, see Douglas, Shapiro and Shields [5], we shall state it for completeness.

PROPOSITION 3.5. *If f is analytic in a neighborhood of cl$\mathbb{D}$, then f has a pseudocontinuation if and only if f is a rational function.*

The proof essentially says that a pseudocontinuation of f must be a true analytic continuation of f. Thus together, f and its continuation define a meromorphic function on the Riemann sphere with only finitely many poles. Hence Liouville's Theorem implies that f is a rational function. The next result follows easily from the previous Proposition and Theorem 3.1.

COROLLARY 3.6. *If f is analytic on a neighborhood of cl$\mathbb{D}$, then the following are equivalent:*

(a) $0 \in \sigma_p([T_f^*, T_f])$*;*

(b) $[T_f^*, T_f]$ *is finite rank;*

(c) f *is a rational function.*

We now construct some bounded *univalent* functions with pseudocontinuations that are *not* rational functions. That such functions exist is not to surprising; however, in view of the previous corollary, it is not obvious.

LEMMA 3.7. *If $\{f_n\}$, f are analytic on $\{z : |z| < r_0\}$, f is univalent and $f_n \to f$ uniformly on compact subsets of $\{z : |z| < r_0\}$, then for each $r < r_0$, there exists an integer N such that f_n is univalent on $\{z : |z| < r\}$ for all $n \geq N$.*

Let $\{a_k\}$ be an infinite Blaschke sequence in $\mathbb{D}$ and $B(z)$ the corresponding Blaschke product with zeros at precisely the points $\{a_k\}$. Also, let $B_n(z)$ be the Blaschke product that corresponds to the first n terms of the sequence $\{a_k\}$ (set $B_0 = 1$).

Consider the shift invariant subspace BH^2. We want to construct a bounded univalent function orthogonal to this subspace that is not a rational function. Let

$$h_n(z) = \frac{B_n(z)}{(1 - \overline{a}_{n+1} z)}$$

for $n \geq 0$. So, h_n is the Blaschke product B_n times the reproducing kernel for the point a_{n+1}.

Using this, it is easy to see that $\{h_n/\|h_n\| : n \geq 0\}$ is a basis for $(BH^2)^{\perp}$. For more information on the orthogonal complement of shift invariant subspaces, see Ahern and Clark [2].

THEOREM 3.8. *If $\{a_j\}$ is any infinite Blaschke sequence in $\mathbb{D}$, B is the associated Blaschke product and $k \geq 1$ is an integer, then there is a univalent function ϕ that is in C^k(cl$\mathbb{D}$), orthogonal to BH^2 and is not a rational function.*

For convenience let $\|f\|_{\infty,n} = max_{1\leq k\leq n}\|f^{(k)}\|_{\infty}$.

Proof. Keeping the above notation, we are going to define inductively a sequence of scalars $\{c_n\}$ and set $\phi(z) = \sum_{n=0}^{\infty} c_n h_n(z)$. To begin, let $c_0 = 1/\|h_0\|_{\infty,k}$. For convenience, we may suppose that $a_1 \neq 0$. Now, let $f(z) = c_0 h_0$. Notice that f is univalent on the disk centered at the origin with radius $1/|a_1|$. If we let $f_n = f + (1/n)h_1$, then f and $\{f_n\}$ satisfy the hypothesis of Lemma 3.7. Hence, for all large n, f_n is univalent on some neighborhood of the closed unit disk. Now, choose n large enough such that f_n is univalent on a neighborhood of the closed unit disk, such that $(1/n) < 1/(2^1\|h_1\|_{\infty,k})$ and such that $(1/n) < 1/2^1$. Set $c_1 = 1/n$ for this value of n. For the next step, set $f = c_0 h_0 + c_1 h_1$ and $f_n = f + (1/n)h_2$. By construction, f is univalent on a neighborhood of the closed unit disk. So, choose n large enough such that f_n is also univalent on a neighborhood of the closed unit disk, such that $(1/n) < 1/(2^2\|h_2\|_{\infty,k})$ and such that $(1/n) < 1/2^2$. Set $c_2 = 1/n$.

Continue in this manner to construct a sequence $\{c_n\}$ such that for each N, if we set $S_N(z) = \sum_{n=0}^{N} c_n h_n(z)$, then $S_N(z)$ is univalent on a neighborhood of $cl\mathbb{D}$ and for each n we have $c_n\|h_n\|_{\infty,k} < 1/2^n$ and $c_n < 1/2^n$. Thus it follows that $\|S_N\|_{\infty,k} \leq 1$. Now define $\phi(z) = \sum_{n=0}^{\infty} c_n h_n(z)$.

By construction, the series and its first k derivatives converge uniformly on $cl\mathbb{D}$. Thus ϕ is an analytic function that is in $C^k(cl\mathbb{D})$. Also as the partial sums S_N are univalent in $\mathbb{D}$, either ϕ is constant or ϕ is univalent. However we shall see that ϕ is not constant. Also by the comments preceding the Theorem, $\phi \perp BH^2$. Finally, ϕ is not a rational function (hence not constant either) else ϕ would be analytic across the unit circle. However ϕ does not continue analytically past any point on the circle where the sequence $\{a_j\}$ clusters.

To see this, recall that the Blaschke product $B(z)$ does converge uniformly on compact subsets of $\Omega = \mathbb{C} - cl\{1/\overline{a}_j\}$. Thus the partial products $B_n(z)$ are uniformly bounded on compact subsets of Ω. Since $\sum_n c_n < \infty$, it follows that the series defining ϕ actually converges uniformly on compact subsets of Ω. Hence the series defining ϕ also defines an analytic extension of ϕ to Ω. Now it is clear that ϕ has poles at the points $\{1/\overline{a}_j\}$, hence cannot be analytic at any point where these poles cluster. □

EXAMPLE 3.9. *There exists one-to-one bounded analytic functions smooth up to $\partial\mathbb{D}$ with pseudocontinuations that are not rational functions.*

Notice that in Theorem 3.8, by choosing the $\{a_k\}$ appropriately, we may find *univalent* functions ϕ smooth up to $\partial\mathbb{D}$ with pseudocontinuations that do not continue analytically past any point on the circle. Or at the other extreme, we may have ϕ continue analytically past every point of the circle except one.

M. Putinar and H. Shapiro have shown that there exists a singular inner function S and a bounded univalent function ϕ such that $\phi \perp SH^2$ (private communication). Also, the author has shown that there exists singular inner functions S such that there are no bounded univalent functions orthogonal to SH^2. However, the following question remains unanswered.

QUESTION 3.10. *For which singular inner functions S, does there exist a bounded univalent function ϕ such that $\phi \perp SH^2$?*

4 The Hardy Operators

If G is a bounded region in $\mathbb{C}$, then let $S_G = M_z$ on $H^2(G)$. In this section we want to determine when $0 \in \sigma_p([S_G^*, S_G])$.

If G is simply connected, then S_G is unitarily equivalent to the analytic Toeplitz operator T_f where f is a Riemann map of $\mathbb{D}$ onto G (and $f(0)$ is the norming point for $H^2(G)$). Thus, when G is simply connected $0 \in \sigma_p([S_G^*, S_G])$ if and only if f has a pseudocontinuation.

We want to prove a similar result for arbitrary bounded regions where we replace f with the universal covering map. If G is a bounded region in $\mathbb{C}$, let $\phi : \mathbb{D} \to G$ be a universal analytic covering map. We want to discuss Nevanlinna functions on G and *boundary values* of functions on G. The following result is classical.

THEOREM 4.1. *Let G be a bounded region in $\mathbb{C}$ and $\phi : \mathbb{D} \to G$ an analytic covering map. If f is a meromorphic function on G, then the following are equivalent:*

- *(a) f is a quotient of bounded analytic functions on G;*
- *(b) $Log^+|f|$ has a harmonic majorant;*
- *(c) $f \circ \phi$ is a Nevanlinna function on $\mathbb{D}$.*

Thus if f is a meromorphic function on G that satisfies one of the above equivalent conditions, then we say that f is a *Nevanlinna function* on G.

If g is a Nevanlinna function on $\mathbb{D}$, then let g^* denote the radial limit function on $\partial\mathbb{D}$. We say that a Nevanlinna function f on G has *boundary values* if there exists a measurable function f^* defined a.e. on ∂G with respect to harmonic measure for G such that $f^* \circ \phi^* = (f \circ \phi)^*$.

Since the radial limits of ϕ all lie in the boundary of G, this is well defined. An alternative way to consider boundary values of Nevanlinna functions on G is as follows. Consider all those points $a \in \partial\mathbb{D}$ such that ϕ has a radial limit at a. For each such a, let $[0, a]$ be the radial line segment ending at a and consider the curves $\gamma_a = \phi([0, a])$. The curves $\{\gamma_a\}$ are hyperbolic geodesics in G and they end on ∂G. In fact a.e. point p (with respect to harmonic measure) on ∂G has a curve (or several curves) ending at p.

Furthermore, if f is a Nevanlinna function on G, then f has a limit along a.e. curve. This follows simply because $f \circ \phi$ has radial limits a.e. on $\partial\mathbb{D}$. Now, say that f has *boundary values* if for a.e. point $p \in \partial G$, f has the same limit along all the curves γ_a that end at p. Define this common limiting value to be $f^*(p)$. This defines a boundary function for f a.e. on ∂G.

Now, define a region G to be a *generalized quadrature domain* if there exists a Nevanlinna function R on G, called the *generalized Schwarz function*, such that R has boundary values, R^*, and $R^*(z) = \overline{z}$ a.e. on ∂G.

Recall that G is a quadrature domain if there exists a meromorphic function on G that is continuous up to ∂G and equals $\overline{z}$ on ∂G; see Aharonov and Shapiro [1] for more information on quadrature domains.

The following result is the main theorem in this section.

THEOREM 4.2. *If G is a bounded region in $\mathbb{C}$ and $\phi : \mathbb{D} \to G$ is an analytic covering map, then the following are equivalent:*

(a) $0 \in \sigma_p([S_G^*, S_G])$;

(b) *G is a generalized quadrature domain;*

(c) *ϕ has a pseudocontinuation.*

Proof. (a) $\Rightarrow$ (b). If $0 \in \sigma_p([S_G^*, S_G])$, then as is well known, S_G is unitarily equivalent to multiplication by ϕ, M_ϕ, on the subspace $\mathcal{M}$ consisting of all functions in $H^2(\mathbb{D})$ automorphic with respect to the group of covering transformations. Thus, if $0 \in \sigma_p([S_G^*, S_G])$, then there exists a functions g, $h \in \mathcal{M}$ such that $\overline{\phi}h = g$ a.e. on the circle. Thus, g/h is a Nevanlinna function on $\mathbb{D}$ whose boundary values agree a.e. with those of $\overline{\phi}$. Furthermore, since $g, h \in \mathcal{M}$, g/h is automorphic with respect to the group of covering transformations. Thus there is a Nevanlinna function R on G such that $R \circ \phi = g/h$. Thus, $(R \circ \phi)^* = (g/h)^* = \overline{\phi}^* = \overline{z} \circ \phi^*$. Thus, R has boundary values $\overline{z}$ a.e. on ∂G. Hence, G is a generalized quadrature domain.

(b) $\Rightarrow$ (c). If G is a generalized quadrature domain, then let R be the generalized Schwarz function. If we set $F = R \circ \phi$, then F is a Nevanlinna function on $\mathbb{D}$ whose boundary values agree a.e. with those of $\overline{\phi}$. Hence, the reflection of F, $\overline{F(1/\overline{z})}$, is the required pseudocontinuation of ϕ.

(c) $\Rightarrow$ (a). If ϕ has a pseudocontinuation, then reflecting the continuation across the circle gives a Nevanlinna function F on $\mathbb{D}$ such that the boundary values F agree a.e. with the boundary values of $\overline{\phi}$. Hence the boundary values of F are invariant under the covering transformations. It follows that F is also. Because if g is a covering transformation, then $F \circ g$ is a Nevanlinna function with the same boundary values as F. Hence $F \circ g = F$ on $\mathbb{D}$. Thus, there is a Nevanlinna function H on G such that $F = H \circ \phi$. So, by Theorem 4.1, H is a quotient of bounded analytic functions on G. Hence F is a quotient of two bounded analytic functions on $\mathbb{D}$, say $F = h/k$, each of which is invariant under the group of covering transformations. Thus, $h, k \in \mathcal{M}$ and on the circle $h/k = F = \overline{\phi}$. So, $\overline{\phi}k = h \in \mathcal{M}$. Thus, by Proposition 2.1, $S_G \cong M_\phi$ on $\mathcal{M}$ has zero as an eigenvalue for its self-commutator. □

EXAMPLE 4.3. *There exists a bounded simply connected region G such that the Riemann map $\phi : \mathbb{D} \to G$ extends smoothly up to $\partial\mathbb{D}$, $0 \in \sigma_p([S_G^*, S_G])$, and yet $[S_G^*, S_G]$ has infinite rank.*

Thus, G is a generalized quadrature domain but not a quadrature domain.

Proof. This follows immediately from Theorem 3.8 and Theorem 3.1. □

EXAMPLE 4.4. *If K is any compact subset of $\mathbb{D}$ with logarithmic capacity zero, then $G = \mathbb{D} - K$ is a generalized quadrature domain.*

Proof. Let ϕ be an analytic covering map of $\mathbb{D}$ onto G. Since the boundary function of a bounded analytic function cannot map a set of positive measure into a set of log capacity zero, we see that a.e. radial limit of ϕ lies in $\partial\mathbb{D}$. Thus ϕ is an inner function. But every inner function has a pseudocontinuation. Namely, the reflection of $1/\phi$ across the circle. Thus, by Theorem 4.2, G is a generalized quadrature domain. □

Remark. Notice that by the remarks preceding Question 3.10 there exists a bounded simply connected region that is a generalized quadrature domain, but whose Schwarz function has no poles.

5 Cyclic Operators

The general cyclic subnormal operator has the form $S_\mu = M_z$ on $P^2(\mu)$, the closure of the analytic polynomials in $L^2(\mu)$. In this section we shall investigate irreducible cyclic subnormal operators S_μ with $0 \in \sigma_p([S_\mu^*, S_\mu])$. The cyclic subnormal operators with finite rank self-commutator were characterized by Olin, Thomson and Trent [9] and independently by Xia [14]. Since Thomson's Theorem [12] appeared, another proof of the characterization of these operators was given by McCarthy and Yang [6]. We shall state these results so that they may be compared with the results obtained for cyclic subnormal operators with $0 \in \sigma_p([S_\mu^*, S_\mu])$. We shall assume the reader is familiar with Thomson's Theorem and analytic bounded point evaluations (abpe), see Conway [4] or Thomson [12].

THEOREM 5.1. *If S_μ is an irreducible cyclic subnormal operator, then S_μ has finite rank self-commutator if and only if $G = abpe(\mu)$ is a quadrature domain, the Riemann map of $\mathbb{D}$ onto G is a weak* generator of $H^\infty(\mathbb{D})$, and $\mu \approx \omega_G + \sum_{k=1}^n w_k \delta_{a_k}$, where ω_G is harmonic measure for G, $w_k > 0$ and $a_k \in G$.*

We shall make use of the following nice result allowing one to pull cyclic operators back to the disk. This is a standard technique in the theory of Hardy spaces, and Thomson's Theorem now makes it available for cyclic operators as well.

Let S_μ be an irreducible cyclic subnormal operator. Set $G = abpe(\mu)$ and let $\phi : \mathbb{D} \to G$ be a Riemann map. Also, let $\psi : G \to \mathbb{D}$ be the inverse of ϕ. Thomson's Theorem implies that the natural extension map $\hat{}: P^2(\mu) \cap L^\infty(\mu) \to H^\infty(G)$ is a dual algebra isomorphism. Let $\tilde{}: H^\infty(G) \to P^2(\mu) \cap L^\infty(\mu)$ be the inverse of $\hat{}$. So, $\tilde{\psi} \in P^2(\mu) \cap L^\infty(\mu)$. Define a measure ν on $cl\mathbb{D}$ by $\nu = \mu \circ \tilde{\psi}^{-1}$.

The following nice result is proven in Olin and Yang [10]. Keep the above notation.

THEOREM 5.2. *The operator S_ν is irreducible and $abpe(\nu) = \mathbb{D}$. Further, the densely defined map $U : P^2(\mu) \to P^2(\nu)$ given by $U(p) = p \circ \tilde{\phi}$ for polynomials p, extends to an onto isometry and $U^* \tilde{\phi}(S_\nu) U = S_\mu$.*

An equivalent way to state Theorem 5.1 is to pull the operator back to the disk.

THEOREM 5.3. *If S is an irreducible cyclic subnormal operator, then S has finite rank self–commutator if and only if S is unitarily equivalent to $r(S_\nu)$ where r is a rational function with poles off $cl\mathbb{D}$ that is a weak* generator of H^∞ and $\nu = m + \sum_{k=1}^n w_k \delta_{a_k}$, where m is Lebesgue measure on the circle, $w_k > 0$ and $a_k \in \mathbb{D}$.*

The idea for cyclic operators with $0 \in \sigma_p([S_\mu^*, S_\mu])$ is that they should have the form $\phi(S_\nu)$ where $abpe(\nu) = \mathbb{D}$, ϕ has a pseudocontinuation, and $\nu \approx m + \sum_{k=1}^\infty w_k \delta_{a_k}$; where $\{a_k\}$ is a Blaschke sequence in $\mathbb{D}$.

We now show that in fact, every such operator has a slightly stronger property than $0 \in \sigma_p([S_\mu^*, S_\mu])$. The measure m shall always denote Lebesgue measure on the unit circle.

THEOREM 5.4. *If S_ν is a pure cyclic subnormal operator such that $\nu \approx m + \sum_{k=1}^{\infty} w_k \delta_{a_k}$; where $\{a_k\}$ is a Blaschke sequence in $\mathbb{D}$ and ϕ is any bounded analytic function that has a pseudocontinuation, then there exists a nonzero bounded function in $ker[\phi(S_\nu)^*, \phi(S_\nu)]$.*

In contrast, notice that S_μ has finite rank self-commutator if and only if there is a polynomial in $ker[S_\mu^*, S_\mu]$.

Proof. Since ϕ has a pseudocontinuation, Theorem 3.1 implies that the analytic Toeplitz operator T_ϕ on H^2 has zero as an eigenvalue for its self-commutator. Hence by Proposition 2.1 we see that $ker[T_\phi^*, T_\phi]$ is a nonzero invariant subspace for the Unilateral shift. Hence by Beurling's Theorem there exists a nonzero bounded function $f \in ker[T_\phi^*, T_\phi]$. Thus, by Proposition 2.1 we have that $\overline{\phi} f \in H^2$. Since both ϕ and f are bounded, there exists a bounded analytic function g on $\mathbb{D}$ such that $\overline{\phi} f = g$ a.e. on $\partial\mathbb{D}$.

Now let B be a Blaschke product with zeros at the points $\{a_k\}$. It follows that $\overline{\phi} f B = gB$ ν a.e.. Thus, since $gB \in P^2(\nu)$ (by Thomson's Theorem), Proposition 2.1 implies that fB is a nonzero bounded function in $ker[\phi(S_\nu)^*, \phi(S_\nu)]$. □

COROLLARY 5.5. *If S_μ is an irreducible cyclic subnormal operator such that $G = abpe(\mu)$ is a generalized quadrature domain and $\mu \approx \omega_G + \sum_{k=1}^{\infty} w_k \delta_{a_k}$; where $\{a_k\}$ is an $H^\infty(G)$ zero set and ω_G is harmonic measure for G, then there exists a nonzero bounded function in $ker[S_\mu^*, S_\mu]$.*

Proof. It follows from Theorem 5.2 that $S_\mu \cong \phi(S_\nu)$ where $\phi : \mathbb{D} \to G$ is a Riemann map and ν is a measure such that S_ν is pure, $abpe(\nu) = \mathbb{D}$ and $\mu = \nu \circ \phi^{-1}$. Furthermore, the unitary is given by composition with ϕ.

It follows from Theorem 4.2 that ϕ has a pseudocontinuation. Also because $\mu = \nu \circ \phi^{-1}$ that $\nu \approx m + \sum_{k=1}^{\infty} w_k \delta_{a_k}$; where $\{a_k\}$ is a Blaschke sequence in $\mathbb{D}$. Thus Theorem 5.4 applies to say that there is a nonzero bounded function in $ker[\phi(S_\nu)^*, \phi(S_\nu)]$. Since the unitary conjugating $\phi(S_\nu)$ to S_μ is given by composition with ϕ, it preserves bounded functions. Thus there exists a nonzero bounded function in $ker[S_\mu^*, S_\mu]$. □

QUESTION 5.6. *Is the converse to Corollary 5.5 true?*

The author believe that the converse to Corollary 5.5 is true. We shall present some partial results illustrating this; although we do not have a full converse at this time.

It will be shown that if there exists a nonzero bounded function in the $ker[S_\mu^*, S_\mu]$, then $G = abpe(\mu)$ is a generalized quadrature domain. However it is unclear if the measure must have the above form. This involves a curious question about univalent functions with pseudocontinuations.

THEOREM 5.7. *If S_ν is a pure cyclic subnormal operator, $abpe(\nu) = \mathbb{D}$ and $\phi \in P^2(\nu) \cap L^\infty(\nu)$ is such that there exists a nonzero bounded function in $ker[\phi(S_\nu)^*, \phi(S_\nu)]$, then $\hat{\phi}$ has a pseudocontinuation.*

Since $abpe(\nu) = \mathbb{D}$ we shall consider ϕ as both a function in $P^2(\nu)$ and in $H^\infty(\mathbb{D})$.

Proof. If g is a nonzero bounded function in $ker[\phi(S_\nu)^*, \phi(S_\nu)]$, then by Proposition 2.1, we have that $\overline{\phi}g \in P^2(\nu)$. So there exists a function $h \in P^2(\nu) \cap L^\infty(\nu)$ such that $\overline{\phi}g = h$ ν a.e.. In particular, $\overline{\phi}g = h$ ν a.e. on $\partial\mathbb{D}$.

It is known (see Conway [4]) that $\nu|\partial\mathbb{D}$ is absolutely continuous with respect to Lebesgue measure. However, since S_ν is pure and $P^\infty(S_\nu) = H^\infty(\mathbb{D})$ it follows that for every measurable set $E \subseteq \partial\mathbb{D}$ with positive length, either $\nu(E) > 0$ or there exists a subsequence of $supp(\nu) \cap \mathbb{D}$ that clusters non-tangentially at almost every point of E.

Since $\overline{\phi}g = h$ a.e. ν and ϕ, g, and h are all bounded analytic functions on $\mathbb{D}$, we actually have that $\overline{\phi}g = h$ everywhere on $supp(\nu) \cap \mathbb{D}$. By taking non-tangential limits, if necessary, we have that $\overline{\phi}g = h$ a.e. on $\partial\mathbb{D}$ with respect to arc length measure. Thus, the reflection of h/g across the circle is the required pseudocontinuation of ϕ. □

COROLLARY 5.8. *If S_μ is an irreducible cyclic subnormal operator and there exists a nonzero bounded function in $ker[S_\mu^*, S_\mu]$, then $G = abpe(\mu)$ is a generalized quadrature domain.*

We now present a converse to Corollary 5.5 when G is assumed to be a quadrature domain. Notice in particular that the unit disk is a quadrature domain with Schwarz function $R(z) = 1/z$. Hence, Theorem 5.9 applies when $G = \mathbb{D}$.

THEOREM 5.9. *If S_μ is a pure cyclic subnormal operator such that $G = abpe(\mu)$ is a quadrature domain, then there exists a nonzero bounded function in $ker[S_\mu^*, S_\mu]$ if and only if $\mu \approx \omega_G + \sum_{k=1}^\infty w_k\delta_{a_k}$; where $\{a_k\}$ is an $H^\infty(G)$ zero set and ω_G is harmonic measure for G.*

We shall use the following Lemma from Olin, Thomson and Trent [9]. It was crucial in characterizing cyclic subnormal operators with finite rank self-commutator. A similar result was used in McCarthy and Yang [6]. Its proof uses facts about algebraic curves.

LEMMA 5.10. *If r is a rational function with poles off $cl\mathbb{D}$ and is univalent on $\mathbb{D}$, then the cardinality of $\{z \in \mathbb{D} : r(z) = r(1/\overline{z})\}$ is finite.*

Proof of Theorem 5.9. If μ has the required form, then Corollary 5.5 applies. Now suppose that there exists a nonzero bounded function in $ker[S_\mu^*, S_\mu]$. Since G is a quadrature domain, the Riemann map of $\mathbb{D}$ on G is a rational function (see [1]), say $r(z)$. Now Theorem 5.2 implies that there exists a measure ν on $cl\mathbb{D}$ such that $\mu = \nu \circ r^{-1}$, S_ν is pure, $abpe(\nu) = \mathbb{D}$ and S_μ is unitarily equivalent to $r(S_\nu)$ via composition with $r(z)$. Since there is a nonzero bounded function in $ker[S_\mu^*, S_\mu]$, we also get a nonzero bounded function in $ker[r(S_\nu)^*, r(S_\nu)]$. Thus by Proposition 2.1, there exists a nonzero bounded function $f \in P^2(\nu)$ such that $\overline{r}f \in P^2(\nu)$. Thus, there exists a bounded function $g \in P^2(\nu)$ such that $\overline{r}f = g$ a.e.-ν. Now it is easy to see that $\overline{r(z)}\hat{f}(z) = \hat{g}(z)$ everywhere on $supp(\nu) \cap \mathbb{D}$. Hence, it follows that $\Phi = \hat{g}/\hat{f}$ satisfies that $\Phi = \overline{r}$ a.e.-ν *except* at the poles of Φ. In particular since both Φ and r are Nevanlinna functions, arguing as above, we have that $\Phi = \overline{r}$ a.e. on $\partial\mathbb{D}$.

Thus Φ^* is a pseudocontinuation of r, where the $*$ represents reflection across the unit circle. However, r is its own pseudocontinuation. Hence $r(z) = \Phi^*(z)$ for all z with

$|z| > 1$. Hence $\overline{r(1/\overline{z})} = \Phi(z)$ everywhere on $\mathbb{D}$. But we also have that $\Phi = \overline{r}$ a.e.-ν except on the poles of Φ.

Hence it follows that $supp(\nu) \cap \mathbb{D} \subseteq \{z : r(z) = r(1/\overline{z})\} \cup \{z : z \text{ is a pole of } \Phi\}$.

Since Φ is a Nevanlinna function, and using Lemma 5.10, we see that $supp(\nu) \cap \mathbb{D}$ is a Blaschke sequence; call it $\{b_j\}$.

Now if we let $\mathcal{M} = \{h \in P^2(\nu) : \hat{h}(b_j) = 0 \text{ for all j }\}$, then $\mathcal{M}$ is a closed invariant subspace for S_ν and $svsm(S_\nu|\mathcal{M})$ is supported on $\partial\mathbb{D}$. Since S_ν is pure, we must have that $\nu|\partial\mathbb{D} \approx m$. Hence $\nu \approx m + \sum_{k=1}^{\infty} w_k \delta_{b_k}$. Since $\mu = \nu \circ r^{-1}$, μ also has the required form. □

Observe that the only place we used that G was a quadrature domain was to get the Riemann map onto G to be a rational function so that we could apply Lemma 5.10.

If we could answer affirmatively the following natural analogue of Lemma 5.10, then we would have a full converse to Corollary 5.5.

QUESTION 5.11. *If ϕ is a bounded univalent function on $\mathbb{D}$ that has a pseudocontinuation Φ, then is the set $\{z \in \mathbb{D} : \phi(z) = \Phi(1/\overline{z})\}$ a Blaschke sequence ?*

However "large" the above set can is exactly how "large" the measure $\mu|G$ can be. Also the following basic question remains open.

QUESTION 5.12. *If S_μ is an irreducible cyclic subnormal operator and $ker[S_\mu^*, S_\mu] \neq (0)$, then does there exist a nonzero bounded function in $ker[S_\mu^*, S_\mu]$?*

It is easy to see that if there exists a Nevanlinna function in $ker[S_\mu^*, S_\mu]$, then there also exists a bounded function in $ker[S_\mu^*, S_\mu]$. This relates the previous question to the next one.

QUESTION 5.13. *If S_μ is an irreducible cyclic subnormal operator and $ker[S_\mu^*, S_\mu] \neq (0)$, then is $\mu|\partial G \approx \omega_G$, where $G = abpe(\mu)$ and ω_G is harmonic measure for G ?*

As noted above, the next question has an affirmative answer for operators with finite rank self-commutator.

QUESTION 5.14. *If S is an irreducible cyclic subnormal operator and $0 \in \sigma_p([S^*, S])$, then does S have a full set of analytic bounded point evaluations ? That is, if $\phi : \mathbb{D} \to G$ is a Riemann map, then is ϕ a weak* generator of $H^\infty(\mathbb{D})$?*

We now want to consider some examples of cyclic subnormal operators where the measure on G is very sparse, yet there is no measure on ∂G.

We shall work on the disk and work with *sampling sequences* for the Bergman spaces.

We say that $\{a_n\}$ is a *sampling sequence* for the weighted Bergman space $L_a^2(\mathbb{D}, (1-|z|^2)^{2\alpha-1}dA)$, $\alpha > 0$, if the identity map on polynomials extends to give a similarity between the weighted Bergman operator and S_μ, where $\mu = \sum_{n=1}^{\infty}(1-|a_n|^2)^{2\alpha+1}\delta_{a_n}$. Sampling sequences always exist for these weighted Bergman spaces and have been characterized by Seip [11].

If $\alpha = 1/2$, then S_μ is similar to the Bergman operator. If $\alpha > 0$, then is $[S_\mu^*, S_\mu]$ one-to-one ?

Even though the self-commutator of the weighted Bergman operator is one-to-one (Theorem 2.14), it could still happen that $[S_\mu^*, S_\mu]$ is not one-to-one.

Observe that every sampling sequence is an H^∞ dominating sequence. This holds because S_μ is similar to a weighted Bergman operator, hence must satisfy $P^\infty(\mu) \approx H^\infty(\mathbb{D})$. But for such discrete measures, it is known that $P^\infty(\mu) \approx H^\infty(\mathbb{D})$ if and only if the atoms of μ form a dominating set (see Conway [4], p. 306).

For the first example, we shall take $\alpha = 1/2$.

EXAMPLE 5.15. *If $\{a_n\}$ is a sampling sequence for $L_a^2(\mathbb{D})$ and $\mu = \sum_{n=1}^\infty (1-|a_n|^2)^2 \delta_{a_n}$, then $[S_\mu^*, S_\mu]$ is one-to-one.*

Proof. If $f \in P^2(\mu)$ and $f \in ker[S_\mu^*, S_\mu]$, then Proposition 2.1 implies that $\overline{z}f \in P^2(\mu)$. Hence, also $(1-|z|^2)f \in P^2(\mu)$. But since $\hat{f} \in L_a^2(\mathbb{D})$, we know that $(1-|z|^2)f(z) \to 0$ as $|z| \to 1$ (because the normalized reproducing kernels go to zero weakly as $|z| \to 1$). So, $(1-|z|^2)f \in P^2(\mu) \cap L^\infty(\mu)$. That is, there is a function $g \in H^\infty(\mathbb{D})$ with $(1-|a_n|^2)f(a_n) = g(a_n)$. Thus, $g(a_n) \to 0$ as $n \to \infty$. Since $\{a_n\}$ is an H^∞ dominating set, we must have that $g = 0$ identically. Hence $f = 0$ μ - a.e.. Thus $[S_\mu^*, S_\mu]$ is one-to-one. □

The above proof actually works whenever $0 < \alpha \leq 1/2$. For in these cases, the functions in the weighted Bergman spaces satisfy the growth condition that $(1-|z|^2)|f(z)| \to 0$ as $|z| \to 1$. We now prove that $[S_\mu^*, S_\mu]$ is one-to-one whenever $\alpha \leq 1$. When $\alpha > 1/2$, the functions in the weighted Bergman spaces do not need to satisfy the above growth condition. Hence a different proof is needed.

EXAMPLE 5.16. *If $0 < \alpha \leq 1$, $\{a_n\}$ is a sampling sequence for $L_a^2(\mathbb{D}, (1-|z|^2)^{2\alpha-1} dA)$, and $\mu = \sum_{n=1}^\infty (1-|a_n|^2)^{2\alpha+1} \delta_{a_n}$, then $[S_\mu^*, S_\mu]$ is one-to-one.*

Proof. If $f \in ker[S_\mu^*, S_\mu]$, then $(1-|z|^2)f \in P^2(\mu)$. Suppose $g \in P^2(\mu)$ and $g(z) = (1-|z|^2)f(z)$ μ-a.e.. It follows that $f(z) = \frac{g(z)}{(1-|z|^2)}$ on the sequence $\{a_n\}$. Since $\alpha \leq 1$ we have that $2\alpha - 1 \leq 1$, so

$$\sum_{n=1}^\infty (1-|a_n|^2)|g(a_n)|^2 \leq \sum_{n=1}^\infty (1-|a_n|^2)^{2\alpha-1}|g(a_n)|^2 = \|f\|^2 < \infty$$

.

Now for $\varepsilon > 0$, Let $I_\varepsilon = \{a_k : |g(a_k)| > \varepsilon\}$. By the above inequalities, I_ε is a Blaschke sequence for each $\varepsilon > 0$.

So, if $\varepsilon = 1$, then g is bounded by 1 on $\{a_n\} - I_1$. If $B(z)$ is the Blaschke product with zeros at precisely the points in I_1, then Bg is bounded on the sampling sequence $\{a_n\}$. That is, $Bg \in P^2(\mu) \cap L^\infty(\mu)$. So, there exists an $h \in H^\infty(\mathbb{D})$ such that $h = Bg$ on $\{a_n\}$. Thus we have $B(z)f(z) = \frac{B(z)g(z)}{(1-|z|^2)} = \frac{h(z)}{(1-|z|^2)}$ on the sequence $\{a_n\}$ and h is bounded.

So, repeating the argument above, we see that for each $\varepsilon > 0$, the set $J_\varepsilon = \{a_k : |h(a_k)| > \varepsilon\}$ is a Blaschke sequence. However, if one removes a Blaschke sequence from a dominating sequence, then the new sequence is still dominating.

Thus, $\{a_n\} - J_\varepsilon$ is a dominating sequence, $h \in H^\infty$, and $|h| < \varepsilon$ on $\{a_n\} - J_\varepsilon$. Hence $\|h\|_\infty \le \varepsilon$. Since this holds for each $\varepsilon > 0$, we have that $h = 0$. Thus, $Bf = 0$ μ-a.e.. It follows easily that $f = 0$ μ-a.e.. Thus $[S_\mu^*, S_\mu]$ is one-to-one. □

QUESTION 5.17. *If $\alpha > 1$, $\{a_n\}$ is a sampling sequence for $L_a^2(\mathbb{D}, (1-|z|^2)^{2\alpha-1}dA)$, and $\mu = \sum_{n=1}^\infty (1-|a_n|^2)^{2\alpha+1}\delta_{a_n}$, then is $[S_\mu^*, S_\mu]$ one-to-one ?*

We finish by showing that if $\alpha > 0$, then not every part of S_μ has zero as an eigenvalue for its self-commutator.

EXAMPLE 5.18. *If $\alpha > 0$, $\{a_n\}$ is a sampling sequence for $L_a^2(\mathbb{D}, (1-|z|^2)^{2\alpha-1}dA)$, and $\mu = \sum_{n=1}^\infty (1-|a_n|^2)^{2\alpha+1}\delta_{a_n}$, then not every part of S_μ has zero as an eigenvalue for its self-commutator.*

Proof. If every part of S_μ has zero as an eigenvalue for its self-commutator, then by Proposition 2.12 we have that for each integer $m > 0$, there exists a nonzero function $f \in P^2(\mu)$ such that $\overline{z}^k f \in P^2(\mu)$ for all positive integers $k \le m$. In particular, we have that $(1-|z|^2)^m f \in P^2(\mu)$.

However, one easily checks that $(1-|a_n|^2)^{2\alpha+1}|f(a_n)|^2 \le \|f\|^2$. So, fix an $m > 2\alpha+1$. If $g \in P^2(\mu)$ is such that $(1-|z|^2)^m f = g$ on $\{a_n\}$, then $g \in P^2(\mu) \cap L^\infty(\mu)$ and $g(a_n) \to 0$ as $n \to \infty$. Since, $\{a_n\}$ is a dominating sequence, we must have $g = 0$ μ-a.e.. Hence $f = 0$, a contradiction. So not every part of S_μ can have zero as an eigenvalue for its self-commutator. □

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