

The Dynamics of Cohyponormal Operators

Nathan S. Feldman

ABSTRACT. In this expository article we survey some of the recent work on the dynamics of linear operators, and in particular for cohyponormal operators. We give several examples using the adjoints of Bergman operators and also state several open questions. The overall theme is that the spectral properties of a hyponormal operator S determine the dynamics of its adjoint S^* . The final section of the paper gives some examples of “complicated” orbits.

1. Introduction

In what follows X will denote a separable complex Banach space and \mathcal{H} , a separable complex Hilbert space; $\mathcal{B}(X)$ will denote the algebra of all bounded linear operators on X . Studying the dynamics of linear operators entails a study of the behavior of orbits. We shall see that it is natural and important to study the orbits of both points and of subsets. If $T \in \mathcal{B}(X)$ and $x \in X$, then the orbit of x under T is $\text{Orb}(T, x) = \{x, Tx, T^2x, \dots\}$. If $C \subseteq X$, then the orbit of C under T is $\text{Orb}(T, C) = \bigcup\{C, T(C), T^2(C), \dots\} = \bigcup_{x \in C} \text{Orb}(T, x)$.

While the term “chaotic” has a very precise meaning, the type of “chaotic behavior” we shall consider for an operator is that of having a “small” subset with dense orbit. Of course, the smallest (nonempty) subset would be a single point. If an operator has a point with dense orbit, it is called *hypercyclic*. An example where orbits of subsets play an important role is in the well known fact that an operator is hypercyclic if and only if every open set has dense orbit. An operator is called *chaotic* (Devaney’s definition) if it is hypercyclic and has a dense set of periodic points (x is a periodic point for T if $T^n x = x$ for some $n \geq 1$).

Other types of “small” subsets that we shall consider include finite sets, countable discrete sets, finite dimensional subspaces, and bounded sets. An operator is called *n-supercyclic* if it has an n -dimensional subspace with dense orbit and a 1-supercyclic operator is generally called *supercyclic*. Also an operator is *countably hypercyclic* if it has a bounded, countable, separated set with dense orbit.

It is surprising that a linear operator can actually be hypercyclic; the first example was constructed by Rolewicz [27] in 1969. He showed that if B is the backward shift on $\ell^2(\mathbb{N})$, then λB is hypercyclic if and only if $|\lambda| > 1$. Since that

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time, a ‘‘Hypercyclicity Criterion’’ has been developed independently by Kitai [24] and Gethner and Shapiro [19]. This criterion has been used to show that hypercyclic operators arise within the classes of composition operators [6], weighted shifts [29], adjoints of multiplication operators [20], and adjoints of subnormal and hyponormal operators [18].

An operator $S \in \mathcal{B}(\mathcal{H})$ is *subnormal* if it has a normal extension. An operator $T \in \mathcal{B}(\mathcal{H})$ is *hyponormal* if its self-commutator is positive, i.e. $[T^*, T] = T^*T - TT^* \geq 0$. This is equivalent to requiring that $\|T^*x\| \leq \|Tx\|$ for all $x \in \mathcal{H}$. An operator is called *cosubnormal* (resp. *cohyponormal*) if its adjoint is subnormal (resp. hyponormal). It is well known that every subnormal operator is hyponormal ([9]). A good example of a subnormal operator is the Bergman operator. For a bounded open set G in the complex plane, the Bergman space on G , denoted by $L_a^2(G)$, is the space of all analytic functions on G that are square integrable with respect to area measure on G . The operator S of multiplication by z on $L_a^2(G)$ is a subnormal operator, and hence also hyponormal. For further information on Bergman operators, subnormal operators and hyponormal operators, see Conway [9].

While a subnormal operator, or even a hyponormal operator, has very tame dynamics [4], their adjoints can be hypercyclic, and even chaotic. Furthermore, since pure cosubnormal operators are always cyclic (see Feldman [13]) it is natural to inquire about stronger forms of cyclicity for cosubnormal operators. The following theorem is a summary—and a special case—of the type of theorems discussed throughout the paper. Notice that the theme of the following theorem is that the spectral properties of S determine the dynamics of S^* .

In what follows a *region* is an open connected set.

THEOREM 1.1. *Suppose that $\{D_n\}$ is a bounded collection of bounded regions in \mathbb{C} . For each n , let $S_n = M_z$ on $L_a^2(D_n)$, and let $S = \bigoplus_n S_n$. Then the following hold:*

1. S^* is hypercyclic if and only if $D_n \cap \{z : |z| = 1\} \neq \emptyset$ for all n .
2. S^* is supercyclic if and only if there exists a circle $\Gamma_\rho = \{z : |z| = \rho\}$, $\rho \geq 0$, such that either:
 - (a) For every n , $\text{cl } D_n$ intersects Γ_ρ and $\text{int } \Gamma_\rho$, or
 - (b) For every n , $\text{cl } D_n$ intersects Γ_ρ and $\text{ext } \Gamma_\rho$.
3. S^* is n -supercyclic if and only if S^* can be expressed as the direct sum of n operators T_1, \dots, T_n where each T_k is supercyclic and has the form $T_k = \bigoplus_{n \in \mathcal{C}_k} S_n^*$ where $\mathcal{C}_k \subseteq \mathbb{N}$.
4. S^* has a bounded set with dense orbit if and only if $D_n \cap \{z : |z| > 1\} \neq \emptyset$ for all n .
5. S^* is countably hypercyclic if and only if $D_n \cap \{z : |z| > 1\} \neq \emptyset$ for all n and there exists an n such that $D_n \cap \{z : |z| < 1\} \neq \emptyset$.

It is natural to ask just how complicated can orbits of linear operators be? In section 6 of this paper, we present some examples of ‘‘interesting’’ orbits; for instance, orbits dense in Cantor sets, Julia sets and other geometrically complicated sets. In fact, we show that orbits for linear operators can be as complicated as the orbits of any continuous function on a compact metric space! We also define what a chaotic point is and show that several operators have a dense set of chaotic points. We also discuss examples of ϵ -dense orbits and weakly dense orbits that are not dense.

2. General Theory of Hypercyclic Operators

Here we outline the general theory of hypercyclic operators, summarize results that are known for all hypercyclic operators and state some open questions. We only give sketches of proofs, but provide a few important examples. We begin with some elementary necessary conditions for hypercyclicity, due to Carol Kitai [24].

PROPOSITION 2.1. *If $T \in \mathcal{B}(X)$ is hypercyclic, then the following hold:*

1. $\sup_n \|T^n\| = \infty$.
2. *Every component of $\sigma(T)$ intersects the unit circle.*
3. *If T is invertible, then T^{-1} is also hypercyclic.*
4. *T^* has no eigenvalues.*
5. *$\text{ind}(T - \lambda) \geq 0$ for all $\lambda \in \sigma(T) \setminus \sigma_e(T)$.*

The following two results are the main tools used to show that an operator is hypercyclic.

PROPOSITION 2.2. *If $T \in \mathcal{B}(X)$, then T is hypercyclic if and only if every open set in X has dense orbit under T . Equivalently, for any two open sets U, V in X , there exists an $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$. Furthermore, if T is hypercyclic, then T has a dense G_δ set of hypercyclic vectors.*

PROOF. Simply use the following fact: If $\{V_n\}_{n=1}^\infty$ is a countable basis for the topology of X , then the set of hypercyclic vectors for T is $\bigcap_{n=1}^\infty \bigcup_{k=0}^\infty (T^k)^{-1}(V_n)$, where $(T^k)^{-1}(V_n)$ means the inverse image of the set V_n under T^k . Thus, if every open set has dense orbit, then for each n , $\bigcup_{k=0}^\infty (T^k)^{-1}(V_n)$ is a dense open set, hence by the Baire Category Theorem $\bigcap_{n=1}^\infty \bigcup_{k=0}^\infty (T^k)^{-1}(V_n)$ is a dense G_δ . Hence, there exists a dense G_δ set of hypercyclic vectors. \square

The following criterion is due independently to Kitai [24] and Gethner and Shapiro [19].

THEOREM 2.3 (The Hypercyclicity Criterion). *Suppose that $T \in \mathcal{B}(X)$. If there exists two dense subsets Y and Z in X and a sequence $n_k \rightarrow \infty$ such that:*

1. $T^{n_k}x \rightarrow 0$ for every $x \in Y$, and
2. *There exists functions $B_{n_k} : Z \rightarrow X$ such that for every $x \in Z$, $B_{n_k}x \rightarrow 0$ and $T^{n_k}B_{n_k}x \rightarrow x$,*

then T is hypercyclic.

PROOF. Suppose that U and V are two nonempty open sets in X . Let $x \in Y \cap U$ and $y \in Z \cap V$. Consider the vectors $z_k = x + B_{n_k}y$. Then $z_k \rightarrow x$ and $T^{n_k}z_k \rightarrow y$ as $k \rightarrow \infty$. So, for all large values of k , $z_k \in U$ and $T^{n_k}z_k \in V$. Thus $T^{n_k}(U) \cap V \neq \emptyset$, for all large k ; thus by Proposition 2.2, T is hypercyclic \square

EXAMPLE 2.4 (Rolewicz, 1969 [27]). If $T = \lambda B$ where B denotes the Backward shift on $\ell^2(\mathbb{N})$ and $|\lambda| > 1$, then T is hypercyclic.

PROOF. We will use the Hypercyclicity Criterion. Let Y be the set of all vectors in $\ell^2(\mathbb{N})$ with only finitely many non-zero coordinates. Let $Z = \ell^2(\mathbb{N})$ and define $B_n : Z \rightarrow \ell^2(\mathbb{N})$ by $B_n(x_0, x_1, \dots) = \frac{1}{\lambda^n}(0, 0, \dots, 0, x_0, x_1, \dots)$, where there are n zeros in front of x_0 . Then the conditions of the Hypercyclicity Criterion are easily verified. So, T is hypercyclic. \square

The following result is of fundamental importance in the study of hypercyclicity. Recall that a set is said to be *somewhere dense* if its closure has nonempty interior, otherwise it is called *nowhere dense*. In 2001, Bourdon & Feldman [5] proved that somewhere dense orbits must be dense. This Theorem gives a unified approach to several important results about hypercyclicity.

THEOREM 2.5 (Somewhere Dense Theorem). *If $T \in \mathcal{B}(X)$, $x \in X$ and $\text{Orb}(T, x)$ is somewhere dense in X , then $\text{Orb}(T, x)$ is dense in X .*

In 1995 Ansari [1] answered a long standing question by giving a truly original proof that powers of hypercyclic operators are still hypercyclic. While her proof can now be simplified using the Somewhere Dense Theorem, she introduced important tools and ideas into hypercyclicity that played a role in proving the Somewhere Dense Theorem.

COROLLARY 2.6 (Ansari). *If T is hypercyclic, then T^n is hypercyclic. Furthermore, T and T^n have the same set of hypercyclic vectors.*

PROOF. We shall prove the $n = 2$ case. Suppose that $\text{Orb}(T, x)$ is dense. Now notice that $\text{Orb}(T, x) = \{T^{2n}x\} \cup \{T^{2n}(Tx)\} = \text{Orb}(T^2, x) \cup \text{Orb}(T^2, Tx)$. Hence taking closures we have that $X = \text{clOrb}(T^2, x) \cup \text{clOrb}(T^2, Tx)$. Thus either $\text{Orb}(T^2, x)$ or $\text{Orb}(T^2, Tx)$ is somewhere dense. Hence, by Theorem 2.5, either $\text{Orb}(T^2, x)$ or $\text{Orb}(T^2, Tx)$ is dense. Thus T^2 is hypercyclic. If $\text{Orb}(T^2, x)$ is dense, then x is a hypercyclic vector for T^2 . If $\text{Orb}(T^2, Tx)$ is dense, then since T has dense range, $T\text{Orb}(T^2, Tx)$ is also dense and $T\text{Orb}(T^2, Tx) \subseteq \text{Orb}(T^2, x)$. Thus $\text{Orb}(T^2, x)$ is dense, so in this case x is also a hypercyclic vector for T^2 . \square

An operator is said to be *finitely hypercyclic* if there exists a finite set with dense orbit. In 1992 Herrero conjectured that finitely hypercyclic operators were actually hypercyclic. His conjecture was proven independently by Costakis [10] and Peris [25]. It now follows easily from the Somewhere Dense Theorem.

COROLLARY 2.7 (Herrero's Conjecture). *If F is a finite set and $\text{Orb}(T, F)$ is dense, then T is hypercyclic. In fact, there exists an $x \in F$ such that $\text{Orb}(T, x)$ is dense.*

PROOF. If $F = \{x_1, \dots, x_n\}$. Then $\text{Orb}(T, F) = \bigcup_{k=1}^n \text{Orb}(T, x_k)$. Since $\text{Orb}(T, F)$ is dense, it follows that $\text{Orb}(T, x_k)$ is somewhere dense for some k . Hence $\text{Orb}(T, x_k)$ is actually dense for that value of k . In particular, T is hypercyclic. \square

We now state some general questions about hypercyclic operators. The first two are motivated by Ansari's Theorem that T^n is hypercyclic whenever T is hypercyclic.

QUESTION 2.8 (Bounded Steps). *If T is hypercyclic and $\{n_k\}$ is a sequence of natural numbers such that $0 < n_{k+1} - n_k \leq M$ for all k , then is there a vector x such that $\{T^{n_k}x\}$ is dense?*

In [11] it is proven that if T satisfies the Hypercyclicity criterion, then the above question has an affirmative answer. However, Alfredo Peris has shown (private communication) that there exists a vector x and a sequence $\{n_k\}$ with bounded steps such that if T is twice the Backward shift, then $\{T^{n_k}x\}$ is dense but $\{T^{m_k}x\}$ is not dense.

The next question asks which functions preserve hypercyclicity? It is known that the functions $f(z) = z^n$ and $f(z) = \frac{1}{z}$ preserve hypercyclicity. The fact that $f(z) = \frac{1}{z}$ preserves hypercyclicity, means that “if T is hypercyclic, then T^{-1} is also hypercyclic”. This is an elementary result, it follows from Proposition 2.2, and is more generally true for any homeomorphism on a complete separable metric space. However, the fact that $f(z) = z^n$ preserves hypercyclicity - namely “If T is hypercyclic, then T^n is also hypercyclic” is a nontrivial result due to Ansari (see Corollary 2.6), and was the first major result on hypercyclicity, and in general is not true for continuous maps on metric spaces.

QUESTION 2.9 (Functions of Hypercyclic Operators). If T is hypercyclic and f is a function analytic on an open set G satisfying $\sigma(T) \subseteq G$, $f(\mathbb{D} \cap G) \subseteq \mathbb{D}$ and $f((\mathbb{C} \setminus \mathbb{D}) \cap G) \subseteq (\mathbb{C} \setminus \mathbb{D})$, then is $f(T)$ also hypercyclic?

The answer to Question 2.9 is yes when T is cohyponormal, see Corollary 3.3, and when T has an ample supply of eigenvalues. See [2] and [21] for results along these lines. In [7], Bourdon & Shapiro considered functions of the Backward shift on the Bergman space - which is itself hypercyclic, unlike the backward shift on the Hardy space. They show that describing which functions preserve the hypercyclicity of the backward Bergman shift is non-trivial and involves some very nice function theory.

However, for general hypercyclic operators very little is known about Question 2.9. Only recently was it shown that $f(z) = e^{i\theta}z$, $\theta \in \mathbb{R}$, preserves hypercyclicity (see [12]). Some non-trivial examples to consider that are not understood in general include, $f(z) = \frac{z-a}{1-\bar{a}z}$, $a \in \mathbb{D}$, and $f(z) = e^{-\frac{1+z}{1-z}}$.

QUESTION 2.10 (Herrero’s Question). If T is hypercyclic, then is $T \oplus T$ hypercyclic?

Bes and Peris [3] proved that Herrero’s question is equivalent to the following question.

QUESTION 2.11 (Necessity of the Hypercyclicity Criterion). If T is hypercyclic, then does T satisfy the hypothesis of the Hypercyclicity Criterion?

It is easy to see that $T \oplus T$ is hypercyclic (and thus T satisfies the Hypercyclicity Criterion) if and only if for any four open sets U_1, V_1 and U_2, V_2 , there exists an integer $n \geq 0$ such that $T^n(U_i) \cap V_i \neq \emptyset$ for $i = 1, 2$ —this should be contrasted with Proposition 2.2. Every known hypercyclic operator—including those that are weighted shifts, composition operators, or cohyponormal operators—do satisfy the Hypercyclicity Criterion and thus satisfy that their inflation (direct sum with itself) is hypercyclic.

QUESTION 2.12 (Invariant Subset Question). If T is a bounded linear operator on a separable Hilbert space, then does T have a non-trivial invariant closed set? Equivalently, is there a non-zero vector $x \in \mathcal{H}$ such that the orbit of x is not dense?

It was shown in 1988 by Read [26] that there exists Banach space operators with no non-trivial invariant closed sets. In particular, such operators exist on ℓ^1 .

It is natural to ask if the invariant subspace problem is equivalent to the invariant subset problem.

QUESTION 2.13 (Invariant subspaces vs Invariant subsets). If every bounded linear operator on a separable Hilbert space \mathcal{H} has a non-trivial invariant closed set,

then does every bounded linear operator on \mathcal{H} also have a non-trivial invariant subspace? That is, are the Invariant subspace and Invariant subset problems equivalent?

If $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ where X and Y are Banach spaces, then A and B are *topologically conjugate* if there exists a homeomorphism $h : X \rightarrow Y$ such that $h^{-1} \circ B \circ h = A$. This is the standard equivalence used in dynamical systems theory to say that two continuous functions on metric spaces have the “same” dynamics.

QUESTION 2.14. When are two bounded linear operators topologically conjugate? What operator theory properties are preserved under topological conjugacy? Is every linear operator on ℓ^1 topologically conjugate to a linear operator on ℓ^2 ?

Notice that an affirmative answer to the last question would answer the invariant subset problem and thus the invariant subspace problem on Hilbert space.

3. Hypercyclic Cohyponormal Operators

If S is a bounded linear operator, then a part of S is an operator obtained by restricting S to an invariant subspace, say $S|_{\mathcal{M}}$. A part of the spectrum of S is the spectrum of a part of S , that is, $\sigma(S|_{\mathcal{M}})$, where \mathcal{M} is an invariant subspace for S . In 2000, Feldman, Miller and Miller [18] characterized the hyponormal operators S with hypercyclic adjoints in terms of the parts of the spectrum of S .

THEOREM 3.1. *If S is a hyponormal operator, then S^* is hypercyclic if and only if for every hyperinvariant subspace \mathcal{M} of S , $\sigma(S|_{\mathcal{M}}) \cap \{z : |z| < 1\} \neq \emptyset$ and $\sigma(S|_{\mathcal{M}}) \cap \{z : |z| > 1\} \neq \emptyset$.*

In [18] what was actually proven is that for any subdecomposable operator S , if every hyperinvariant part of the spectrum of S intersects both the interior and exterior of the unit circle, then S^* is hypercyclic. Furthermore, for hyponormal operators (which are subdecomposable), this condition becomes both necessary and sufficient.

EXAMPLE 3.2. Suppose that $\{D_n\}$ is a bounded collection of bounded regions in \mathbb{C} . For each n , let $S_n = M_z$ on $L_a^2(D_n)$, and let $S = \bigoplus_n S_n$. Then S is a pure subnormal operator and S^* is hypercyclic if and only if $D_n \cap \partial\mathbb{D} \neq \emptyset$ for all n .

COROLLARY 3.3.

1. *If $\{T_n\}$ is a bounded sequence of cohyponormal hypercyclic operators, then $\bigoplus_n T_n$ is also hypercyclic.*
2. *If T is a cohyponormal hypercyclic operator and f is a function analytic on an open set G satisfying $\sigma(T) \subseteq G$, $f(\mathbb{D} \cap G) \subseteq \mathbb{D}$ and $f((\mathbb{C} \setminus \mathbb{D}) \cap G) \subseteq (\mathbb{C} \setminus \mathbb{D})$, then $f(T)$ is also hypercyclic.*

4. n -Supercyclic Operators

An operator T is said to be n -supercyclic if there exists an n -dimensional subspace whose orbit is dense. A 1-supercyclic operator is simply called supercyclic. A simple example of a supercyclic operator is a multiple of a hypercyclic operator. Supercyclicity is a well-known property introduced in 1974 by Hilden and Wallen [23], where they proved that every backward unilateral weighted shift is

supercyclic. Their result gives examples of supercyclic operators for which no multiple is hypercyclic. The idea of n -supercyclicity was introduced by Feldman [14] in 2001.

In 1999 Salas [30] developed a Supercyclicity Criterion, similar to the Hypercyclicity Criterion, that could be used to show an operator is supercyclic.

THEOREM 4.1 (The Supercyclicity Criterion). *Suppose that $T \in \mathcal{B}(X)$. If there is a sequence $n_k \rightarrow \infty$ and dense sets Y and Z and functions $B_{n_k} : Z \rightarrow X$ such that:*

1. *If $z \in Z$, then $T^{n_k} B_{n_k} z \rightarrow z$ as $k \rightarrow \infty$, and*
2. *If $y \in Y$ and $z \in Z$, then $\|T^{n_k} y\| \|B_{n_k} z\| \rightarrow 0$ as $k \rightarrow \infty$;*

then T is supercyclic.

In [18], Feldman, Miller, and Miller used a refined (but equivalent) version of the Supercyclicity Criterion to characterize which cohyponormal operators are supercyclic. As with hypercyclicity the characterization involves the parts of the spectrum.

THEOREM 4.2 (Supercyclic Cohyponormal Operators). *If S is a pure hyponormal operator, then S^* is supercyclic if and only if there exists a circle $\Gamma_\rho = \{z : |z| = \rho\}$, $\rho \geq 0$, such that either:*

(a) For every hyperinvariant subspace \mathcal{M} of S , $\sigma(S|_{\mathcal{M}})$ intersects Γ_ρ and $\text{int } \Gamma_\rho$, or

(b) For every hyperinvariant subspace \mathcal{M} of S , $\sigma(S|_{\mathcal{M}})$ intersects Γ_ρ and $\text{ext } \Gamma_\rho$.

If condition (a) holds, then we say that S^* is ρ -inner and if (b) holds then we say that S^* is ρ -outer. The radius ρ , which is not necessarily unique, is called a supercyclicity radius for S^* .

COROLLARY 4.3 (Direct Sums of Supercyclic Cohyponormal Operators). *If $\{S_n\}$ is a bounded sequence of pure hyponormal operators such that S_n^* is supercyclic for every n , then $\bigoplus_n S_n^*$ is supercyclic if and only if the operators S_n^* all have a common supercyclicity radius, ρ , and all have the same type (ρ -inner or ρ -outer).*

EXAMPLE 4.4. Suppose that $\{D_n\}$ is a bounded collection of bounded regions in \mathbb{C} . For each n , let $S_n = M_z$ on $L_a^2(D_n)$, and let $S = \bigoplus_n S_n$. Then S^* is supercyclic if and only if there exists a circle $\Gamma_\rho = \{z : |z| = \rho\}$, $\rho \geq 0$, such that either:

- (a) For every n , $\text{cl}D_n$ intersects Γ_ρ and $\text{int } \Gamma_\rho$, or
- (b) For every n , $\text{cl}D_n$ intersects Γ_ρ and $\text{ext } \Gamma_\rho$.

Specific Examples

1. (An inner (resp. outer) example) If each D_n is a disk that is internally (resp. externally) tangent to the unit circle and their radii converge to zero, then S^* is supercyclic, inner (resp. outer) and has supercyclicity radius 1.
2. If $0 \in \text{cl}D_n$ for each n and $\text{diam}(D_n) \rightarrow 0$, then S^* is supercyclic, outer, and has supercyclicity radius 0.

We will now see how we can use supercyclic operators to construct n -supercyclic operators. The following theorem appeared in Feldman [14].

THEOREM 4.5 (Creating n -Supercyclic Operators). *If T_1, \dots, T_n are each operators that satisfy the hypothesis of the Supercyclicity Criterion with respect to the same sequence $\{n_k\}$, then $\bigoplus_{k=1}^n T_k$ is n -supercyclic.*

It was shown in [18] that supercyclic cohyponormal operators all satisfy the Supercyclicity Criterion with respect to the sequence $n_k = k$. Thus we have the following corollary from Feldman [14].

COROLLARY 4.6. *If T_1, \dots, T_n are each supercyclic cohyponormal operators, then $\bigoplus_{k=1}^n T_k$ is n -supercyclic.*

EXAMPLE 4.7. (a) If D_1, \dots, D_n are bounded regions and $S_k = M_z$ on $L_a^2(D_k)$, then $\bigoplus_{k=1}^n S_k^*$ is n -supercyclic.

(b) If $S = M_z$ on $L_a^2(D)$ where D is a bounded open set with a finite number of components, say n components, then S^* is n -supercyclic. (This is a special case of (a) where the regions are all disjoint.)

Now we need to know some necessary conditions for n -supercyclicity, so that we can say that certain operators in the previous example are not supercyclic. We present a necessary condition, due to Feldman [14] that is remarkably similar to the fact that every component of the spectrum of a hypercyclic operator must intersect the unit circle. However, since multiples of supercyclic operators are still supercyclic, one wouldn't expect their spectra to intersect the unit circle, but some circle centered at the origin.

THEOREM 4.8 (The Circle Theorem). *If $T \in \mathcal{B}(\mathcal{H})$ is n -supercyclic, then there are n circles $\Gamma_i = \{z : |z| = r_i\}$, $r_i \geq 0$, $i = 1, \dots, n$ such that for every invariant subspace \mathcal{M} of T^* , we have $\sigma(T^*|_{\mathcal{M}}) \cap (\bigcup_{i=1}^n \Gamma_i) \neq \emptyset$.*

In particular, every component of the spectrum of T intersects $\bigcup_{i=1}^n \Gamma_i$.

The above theorem says that if T is n -supercyclic, then there exists n circles such that every part of the spectrum of T^* intersects at least one of these n circles. With this Circle Theorem we can now construct operators that have n -dimensional subspaces with dense orbit, but no $(n-1)$ -dimensional subspaces with dense orbit.

EXAMPLE 4.9. Suppose that $\{D_k\}$ is a bounded collection of bounded regions and $S_k = M_z$ on $L_a^2(D_k)$. Let $S = \bigoplus_k S_k$. If S^* is n -supercyclic, then there exists n circles $\Gamma_1, \dots, \Gamma_n$ centered at the origin such that for each k , $\text{cl}D_k \cap (\bigcup_{i=1}^n \Gamma_i) \neq \emptyset$.

EXAMPLE 4.10. Suppose that D_k , $k = 1, \dots, n$ is the open disk with center at k and radius $1/4$. If $S_k = M_z$ on $L_a^2(D_k)$, then $\bigoplus_{k=1}^n S_k^*$ is n -supercyclic, but not $(n-1)$ -supercyclic.

THEOREM 4.11. *Suppose that $\{D_k\}_{k=1}^\infty$ is a bounded collection of bounded regions and $S_k = M_z$ on $L_a^2(D_k)$. Let $S = \bigoplus_{k=1}^\infty S_k$. Then S^* is n -supercyclic if and only if S^* can be expressed as the direct sum of n operators T_1, \dots, T_n where each T_k is supercyclic and has the form $T_k = \bigoplus_{j \in \mathcal{C}_k} S_j^*$ where $\mathcal{C}_k \subseteq \mathbb{N}$ and $\mathbb{N} = \bigcup_{k=1}^n \mathcal{C}_k$.*

The previous theorem says that for $S = \bigoplus_k S_k$, S^* is n -supercyclic if and only if we can group the operators $\{S_k\}$ into n different collections, namely $\{S_j : j \in \mathcal{C}_k\}$ for $k = 1, \dots, n$ in such a way that the direct sum of the operators in each collection is supercyclic. Then we may appeal to Corollary 4.6.

COROLLARY 4.12. *Suppose that $\{D_k\}$ is a bounded collection of bounded regions and $S_k = M_z$ on $L_a^2(D_k)$. Let $S = \bigoplus_k S_k$. If there exists n circles $\Gamma_1, \dots, \Gamma_n$*

centered at the origin such that for each k , $\text{cl}D_k \cap \bigcup_{i=1}^n \Gamma_i \neq \emptyset$, then S^* is $2n$ -supercyclic.

PROOF. For $1 \leq k \leq n$, let $\mathcal{C}_k = \{j : \text{cl}D_j \cap \Gamma_k \neq \emptyset \text{ and } \text{cl}D_j \cap \text{int}\Gamma_k \neq \emptyset\}$. Also for $1 \leq k \leq n$ let $\mathcal{C}'_k = \{j : \text{cl}D_j \cap \Gamma_k \neq \emptyset \text{ and } \text{cl}D_j \cap \text{ext}\Gamma_k \neq \emptyset\}$. Then by hypothesis, $\mathbb{N} = \bigcup_{k=1}^n \mathcal{C}_k \cup \bigcup_{k=1}^n \mathcal{C}'_k$. Let us also suppose that all the sets $\{\mathcal{C}_k\} \cup \{\mathcal{C}'_k\}$ are disjoint (note we may have to shrink some of the sets to achieve this, but we can disjointify them). Then for each k , $T_k := \bigoplus_{j \in \mathcal{C}_k} S_j^*$ and $T'_k := \bigoplus_{j \in \mathcal{C}'_k} S_j^*$ are both supercyclic (see Example 4.4). Thus by Corollary 4.6, $S^* = (\bigoplus_{k=1}^n T_k) \oplus (\bigoplus_{k=1}^n T'_k)$ is $2n$ -supercyclic. \square

EXAMPLE 4.13. Suppose that $\{D_k\}$ is a bounded collection of bounded regions and $S_k = M_z$ on $L_a^2(D_k)$. Let $S = \bigoplus_k S_k$. If there exists n circles $\Gamma_1, \dots, \Gamma_n$ centered at the origin such that for each region D_k , there exists a j such that $\text{cl}D_k \cap \Gamma_j \neq \emptyset$ and $\text{cl}D_k \cap \text{int}\Gamma_j \neq \emptyset$, then S^* is n -supercyclic.

QUESTION 4.14. If T is n -supercyclic, and T^* has no eigenvalues, then must T be cyclic?

QUESTION 4.15. If S is a pure cohyponormal operator and there exists n circles $\Gamma_1, \dots, \Gamma_n$ centered at the origin such that for each hyperinvariant subspace \mathcal{M} of S , $\sigma(S|_{\mathcal{M}}) \cap \bigcup_{i=1}^n \Gamma_i \neq \emptyset$, then is S^* $2n$ -supercyclic?

Feldman has answered the above question affirmatively for $n = 1$, see [14].

THEOREM 4.16. *Suppose that S is a pure hyponormal operator and there exists a circle $\Gamma = \{z : |z| = r\}$, $r > 0$, such that for every hyperinvariant subspace \mathcal{M} of S , $\sigma(S|_{\mathcal{M}}) \cap \Gamma \neq \emptyset$, then S^* is 2-supercyclic.*

QUESTION 4.17. Are there weighted shifts that are n -supercyclic and not supercyclic? If so, can we characterize the n -supercyclic weighted shifts?

Feldman [14] showed that a normal operator cannot be n -supercyclic, also an operator T such that T^* has an open set of eigenvalues cannot be n -supercyclic. It is therefore natural to expect a similar result for subnormal and hyponormal operators.

QUESTION 4.18. Can a subnormal or hyponormal operator be n -supercyclic?

Feldman [14] also defined ∞ -supercyclic operators and showed that there are ∞ -supercyclic operators that are not n -supercyclic for any $n < \infty$. There are subnormal operators that are ∞ -supercyclic because every bilateral weighted shift is ∞ -supercyclic. Thus, in fact, The bilateral weighted shift, a unitary operator, is ∞ -supercyclic.

5. Countably Hypercyclic Operators

We proved in Corollary 2.7 that if $T \in \mathcal{B}(X)$ and there is a finite set with dense orbit, then T must actually be hypercyclic. That is, a “finitely hypercyclic” operator is actually hypercyclic. Several people, including V. Miller and W. Wogen, asked whether there is a meaningful definition of “countably hypercyclic operators”, and if so, is a countably hypercyclic operator necessarily hypercyclic? Notice that every operator has a countable set with dense orbit, namely choose a countable dense set. Thus any reasonable definition of “countable hypercyclicity” should

require an operator to have a countable set with dense orbit, but the countable sets should be small in some sense.

In [16] Feldman gave a reasonable definition of countable hypercyclicity and proved that there are countably hypercyclic operators that are not hypercyclic. Thus the class of countably hypercyclic operators is a non-trivial class of operators; and as we shall see contains some very natural cohyponormal operators. Furthermore, Feldman was able to give a spectral characterization of the cohyponormal operators that are countably hypercyclic.

Following [16], we say that an operator $T \in \mathcal{B}(X)$ is *countably hypercyclic* if there exists a bounded, countable, separated set C with dense orbit. Recall that a set $C \subseteq X$ is separated if there exists an $\epsilon > 0$ such that $\|x - y\| \geq \epsilon$ for all $x, y \in C$ with $x \neq y$. The fact that C is required to be bounded and separated excludes certain trivial operators from being countably hypercyclic (see [16]). It also forces certain orbits of T to be unbounded, and others to cluster at zero, hence T must have some non-trivial dynamics. In particular we have the following theorem (see [16]).

THEOREM 5.1. *If T is a hyponormal operator, then T is not countably hypercyclic.*

Feldman [16] also proved the following result, which shows that for certain operators countable hypercyclicity does imply hypercyclicity.

THEOREM 5.2. *If T is a backward unilateral weighted shift that is countably hypercyclic, then T is hypercyclic.*

Paul Bourdon has proven (private communication) that a countably hypercyclic operator with some regularity must be hypercyclic.

THEOREM 5.3 (Bourdon). *If T is countably hypercyclic and there is a dense set of vectors whose orbits converge, then T is hypercyclic.*

Some simple necessary conditions for countable hypercyclicity are in the following proposition, see [16]. In fact, the proposition considers the more general situation of an operator that has a bounded set with dense orbit. Notice that twice the identity is such an operator!

PROPOSITION 5.4. *If $T \in \mathcal{B}(X)$ and T has a bounded set with dense orbit, then the following hold:*

1. $\sup_n \|T^n\| = \infty$.
2. Every component of $\sigma(T)$ must intersect $\{z : |z| \geq 1\}$.
3. T^* has no eigenvalues in $\{z : |z| \leq 1\}$.
4. T^* has no bounded orbits.
5. If T is countably hypercyclic, then $\sigma(T) \cap \partial\mathbb{D} \neq \emptyset$.

We shall see that every component of $\sigma(T)$ need not intersect the unit circle. The amazing discovery made in [16] is that there is a ‘‘Criterion’’ very similar to the Hypercyclicity Criterion—in fact, almost identical—that can be used to show that an operator is countably hypercyclic.

THEOREM 5.5 (The Countable Hypercyclicity Criterion). *Suppose that $T \in \mathcal{B}(X)$. If there exists two subsets Y and Z in X , with $\dim \text{span} Y = \infty$ and Z dense, and a sequence of integers $n_k \rightarrow \infty$ such that:*

1. $T^{n_k}x \rightarrow 0$ for every $x \in Y$, and
2. There exists functions $B_{n_k} : Z \rightarrow X$ such that for every $x \in Z$, $B_{n_k}x \rightarrow 0$ and $T^{n_k}B_{n_k}x \rightarrow x$,

then T is countably hypercyclic.

With this criterion we are able to characterize the cohyponormal operators that are countably hypercyclic. Notice that the only difference between the Countable Hypercyclicity Criterion and the Hypercyclicity Criterion is that here the set Y is not required to be dense, but only to span an infinite dimensional space. Another difference lies in the fact that while the proof of the Hypercyclicity Criterion relies on the Baire Category Theorem, the proof of the Countable Hypercyclicity Criterion does not.

THEOREM 5.6 (Countably Hypercyclic Cohyponormal Operators). *Suppose that S is a pure hyponormal operator, then S^* is countably hypercyclic if and only if $\sigma(S) \cap \{z : |z| < 1\} \neq \emptyset$ and for every hyperinvariant subspace \mathcal{M} of S , $\sigma(S|\mathcal{M}) \cap \{z : |z| > 1\} \neq \emptyset$.*

Thus for S^* to be countably hypercyclic, one needs every part of the spectrum of S to intersect $\{z : |z| > 1\}$ and at least one part of the spectrum of S should intersect $\{z : |z| < 1\}$. Since every part of the spectrum of S need not intersect $\{z : |z| < 1\}$, we see that S^* need not be hypercyclic.

EXAMPLE 5.7. Suppose that $\{D_n\}$ is a bounded collection of bounded regions in \mathbb{C} . For each n , let $S_n = M_z$ on $L_a^2(D_n)$, and let $S = \bigoplus_n S_n$. Then S^* is countably hypercyclic if and only if $D_n \cap \{z : |z| > 1\} \neq \emptyset$ for all n and there exists an n such that $D_n \cap \{z : |z| < 1\} \neq \emptyset$.

In [16] Feldman also characterized the cohyponormal operators that have bounded sets with dense orbit. One can easily check that an operator has a bounded set with dense orbit if and only if the unit ball in the space has dense orbit.

THEOREM 5.8 (Bounded sets with Dense Orbit). *If S is a hyponormal operator, then S^* has a bounded set with dense orbit if and only if for every hyperinvariant subspace \mathcal{M} of S , $\sigma(S|\mathcal{M}) \cap \{z : |z| > 1\} \neq \emptyset$.*

EXAMPLE 5.9. Suppose that $\{D_n\}$ is a bounded collection of bounded regions in \mathbb{C} . For each n , let $S_n = M_z$ on $L_a^2(D_n)$, and let $S = \bigoplus_n S_n$. Then S^* has a bounded set with dense orbit if and only if $D_n \cap \{z : |z| > 1\} \neq \emptyset$ for all n .

COROLLARY 5.10. *If T is an invertible cohyponormal operator and T and T^{-1} both have bounded sets with dense orbit, then T is hypercyclic.*

However, it was also shown in [16], that there is an invertible bilateral weighted shift such that T and T^{-1} have bounded sets with dense orbit, yet T is not hypercyclic. In fact Alfredo Peris has shown (private communication) that there is an invertible bilateral weighted shift T such that T and T^{-1} are both countably hypercyclic, and yet T is not hypercyclic.

The following corollary shows that if there is a set C that is bounded and bounded away from zero with dense orbit under a cohyponormal operator, then in fact there is a bounded, countable, separated set with dense orbit.

COROLLARY 5.11. *If S is a pure hyponormal operator, then S^* is countably hypercyclic if and only if S^* has a bounded set, which is also bounded away from zero, that has dense orbit.*

QUESTION 5.12. If $T \in \mathcal{B}(X)$ has a bounded set, that is also bounded away from zero, with dense orbit, then must T be countably hypercyclic?

EXAMPLE 5.13. Suppose that $T_1, T_2 \in \mathcal{B}(X)$ and T_1 satisfies the Countably Hypercyclic Criterion and the spectrum of T_2 is contained in $\{z : |z| > 1\}$, then $T_1 \oplus T_2$ also satisfies the Countably Hypercyclic Criterion.

It follows, for example, that if T_1 is twice the Backward shift on $\ell^2(\mathbb{N})$ and T_2 is any operator satisfying $\sigma(T_2) \subseteq \{z : |z| > 1\}$, then $T_1 \oplus T_2$ is countably hypercyclic. In particular, countably hypercyclic operators need not be cyclic, and their spectral properties in the exterior of the unit disk can be arbitrary.

6. Interesting Orbits

In this section we shall give some interesting and complicated examples of orbits of linear operators. Perhaps the most complicated orbits are those that are dense in the whole space. Here we will present some other complicated types of orbits. It follows from Theorem 2.5 that if an orbit is somewhere dense, then it must be everywhere dense. Thus, if we are looking for some interesting behavior of orbits that are not everywhere dense, they must be nowhere dense. For instance, one may ask if an orbit for a linear operator can be dense in a Cantor set (i.e. a compact, totally disconnected, perfect set), or some other geometrically complicated set. Can the closure of an orbit have any prescribed Hausdorff dimension? Surprisingly, the answers to these questions are YES!

In [17] Feldman proved that linear operators can have orbits that are just as complicated as the orbits of any continuous function. Recall that two continuous functions $f : X \rightarrow X$ and $g : K \rightarrow K$ on metric spaces are called topologically conjugate if there exists a homeomorphism $h : X \rightarrow K$ such that $h \circ f \circ h^{-1} = g$. Topological conjugacy is the standard equivalence used to say that two functions have the “same” dynamics.

THEOREM 6.1. *If $f : X \rightarrow X$ is a continuous function on a compact metric space X and T is twice the backward shift of infinite multiplicity, then there is an invariant compact set K for T such that $T|_K$ is topologically conjugate to f .*

It was also proven in [17] that under certain conditions, if f is Lipschitz, then the conjugating homeomorphism may also be chosen to be bi-Lipschitz. It follows then, for example, that one may show that linear operators can have orbits whose closures are bi-Lipschitz homeomorphic to Cantor sets or Julia sets and that there are orbits whose closures have any prescribed Hausdorff dimension, because such orbits exist for Lipschitz continuous functions.

6.1. Chaotic Points. While an orbit may be called predictable if it is convergent, periodic, or converging to a periodic orbit, or exhibits some other moderate behavior. What is a chaotic orbit? Recall that a continuous function $f : K \rightarrow K$ on a complete separable metric space K is said to be *transitive* if for any two open sets U, V in K , there exists an $n \geq 0$ such that $f^n(U) \cap V \neq \emptyset$, where f^n denotes the n^{th} iterate of f . Also, f is *chaotic* if f is transitive and has a dense set of periodic points.

We shall say that $x \in K$ is a *chaotic point* for f if x is not a periodic point and $f|_{\text{clOrb}(f, x)}$ is chaotic. We shall say that x is a *weakly chaotic point*, if x is not a periodic point and $f|_{\text{clOrb}(f, x)}$ is transitive.

The following proposition is an easy exercise that we leave to the reader.

PROPOSITION 6.2. *If $f : K \rightarrow K$ is a continuous function on a complete separable metric space K , then the following are equivalent for a point $x \in K$,*

1. x is a weakly chaotic point for f .
2. $\text{clOrb}(f, x)$ is a perfect set, that is, it has no isolated points.
3. For each $k \geq 0$, $\text{cl}\{f^n(x) : n \geq k\} = \text{clOrb}(f, x)$.

It follows that if $T \in \mathcal{B}(X)$, then an $x \in X$ is a chaotic point for T if $\text{clOrb}(T, x)$ is a perfect set and T has a dense set of periodic points in $\text{clOrb}(T, x)$.

Can an operator have a chaotic point that does not have a dense orbit? Yes!

EXAMPLE 6.3. There exists an operator T on a separable Hilbert space \mathcal{H} such that:

1. T has a dense set of periodic points,
2. T has a dense set of chaotic points,
3. T is cyclic,

but, T is not hypercyclic.

PROOF. The operator is the adjoint of a composition operator, $T = C_\phi^*$ acting on the Sobolev space $W^{1,2}(0, 1)$, where ϕ is the ‘‘Tent map’’, $\phi(x) = 1 - |x - (1/2)|$. One easily checks that ϕ induces a bounded composition operator on $W^{1,2}(0, 1)$ and using the well-known fact that $\phi : [0, 1] \rightarrow [0, 1]$ is chaotic gives the result. \square

A doubly infinite orbit for a linear operator T is a sequence $\{x_k\}_{k=-\infty}^\infty$ such that $Tx_k = x_{k+1}$ for all $k \in \mathbb{Z}$.

THEOREM 6.4. *If $T \in \mathcal{B}(X)$ and T has a doubly infinite orbit $\{x_k\}_{k=-\infty}^\infty$ such that $\sum_{k=-\infty}^\infty \|x_k\| < \infty$, then T has a chaotic point whose orbit has compact closure. Furthermore, if $\{x_k\}_{k=-\infty}^\infty$ has dense linear span in X , then T has a dense set of chaotic points whose orbits have compact closure.*

REMARK. It follows easily from the Hypercyclicity Criterion (Theorem 2.3) that if T has a doubly infinite orbit $\{x_k\}_{k=-\infty}^\infty$ with dense linear span such that $\|x_k\| \rightarrow 0$ as $|k| \rightarrow \infty$, then T is hypercyclic. Thus the above condition implies hypercyclicity.

EXAMPLE 6.5. (a) If T is a backward unilateral weighted shift with weight sequence $\{w_0, w_1, \dots\}$ and if $\sum_{n=1}^\infty \frac{1}{w_0 w_1 \dots w_n} < \infty$, then $x = e_0$ has a doubly infinite orbit that is norm summable and has dense linear span. Thus T has a dense set of chaotic points with orbits having compact closure. Recall that T is hypercyclic if and only if $\sup_n (w_0 w_1 \dots w_n) = \infty$.

(b) If $T = \lambda B$ where B is the Backward shift on $\ell^2(\mathbb{N})$ and $|\lambda| > 1$, then the above condition is satisfied. Thus T has a dense set of chaotic points having orbits with compact closure.

(c) If $S = M_z$ on $L_a^2(D)$ where $\{z : |z| \leq 1\} \subseteq D$, then S^* has a dense set of chaotic points. (consider the doubly infinite orbit generated by the reproducing kernel for the point 0.)

QUESTION 6.6. Can we characterize the cohyponormal operators (or weighted shifts, or other classes of operators) that have a dense set of chaotic points?

QUESTION 6.7. Does a chaotic linear operator have a chaotic point whose orbit is not dense?

6.2. ϵ -Dense Orbits. If $T \in \mathcal{B}(X)$ and $\epsilon > 0$, we shall say that $\text{Orb}(T, x)$ is ϵ -dense, if for every $y \in X$ there exists an n such that $\|y - T^n x\| \leq \epsilon$. In [15] Feldman proved the following results.

THEOREM 6.8. *If $T \in \mathcal{B}(X)$ and T has an ϵ -dense orbit for some $\epsilon > 0$, then T is hypercyclic.*

EXAMPLE 6.9. If T is twice the backward shift on $\ell^2(\mathbb{N})$, then for each $\epsilon > 0$, T has an orbit that is ϵ -dense, but is not dense. In fact, for each $\epsilon > 0$, there are vectors $x, y \in \ell^2(\mathbb{N})$ such that $\text{Orb}(T, x)$ is dense, and $\|T^n x - T^n y\| \leq \epsilon$ for all $n \geq 0$, yet $\text{Orb}(T, y)$ is not dense.

6.3. Weakly Dense Orbits. It is natural to ask if $T \in \mathcal{B}(X)$ and $x \in X$ is such that $\text{Orb}(T, x)$ is dense in X in the weak topology, then must $\text{Orb}(T, x)$ be dense in X in the norm topology? Or must T itself be hypercyclic, that is have some other vector with dense orbit? Feldman raised this question in [15].

EXAMPLE 6.10 (Feldman). If T is twice the backward shift on $\ell^2(\mathbb{N})$, then T has an orbit that is weakly dense but not norm dense.

Very recently Kit Chan and Rebecca Sanders [8] gave an example of a bilateral weighted shift T that is weakly hypercyclic - meaning T has a vector whose orbit is dense in the weak topology, but T is not hypercyclic. In fact the shift $T e_n = w_n e_{n-1}$ with weights $w_n = 1$ when $n \leq 0$ and $w_n = 2$ when $n > 0$ is weakly hypercyclic, but not hypercyclic. In fact, notice that T^{-1} has norm one! Hence T^{-1} is not even weakly hypercyclic (the Baire Category Theorem cannot be used with the weak topology so inverses of weakly hypercyclic operators need not be weakly hypercyclic). Notice that T is in fact cohyponormal! Thus there are cohyponormal operators that are weakly hypercyclic, but not hypercyclic.

QUESTION 6.11. Can we characterize the cohyponormal operators that are weakly hypercyclic?

Notice that having a weakly dense orbit is different than having an orbit that is weakly sequentially dense. In fact it follows easily from Theorem 3.1 that if a cohyponormal operator has a weakly sequentially dense orbit, then it must be hypercyclic.

7. Final Remarks

Another nice survey paper on the dynamics of linear operators, emphasizing the dynamics of composition operators, is given by J.H. Shapiro, *Notes on Dynamics of Linear Operators*. These are unpublished lecture notes available at <http://www.math.msu.edu/~shapiro>. Also, the author's papers are all available at <http://www.wlu.edu/~feldmanN>.

References

- [1] S. I. Ansari, 'Hypercyclic and cyclic vectors', *J. Funct. Anal.* **128** (1995), 374–383.
- [2] T. Bermdez and V.G. Miller, *On operators T such that $f(T)$ is hypercyclic*, *Integral Equations Operator Theory* **37** (2000), no. 3, 332–340.
- [3] J. Bes and A. Peris, *Hereditarily hypercyclic operators* *J. Funct. Anal.* **167** (1999), no. 1, 94–112.
- [4] P.S. Bourdon, *Orbits of Hyponormal Operators*, *Mich. Math. J.* **44** (1997), 345–353.
- [5] P.S. Bourdon and N. S. Feldman, *Somewhere dense orbits are everywhere dense*, To appear in *Indiana Univ. Math. J.*.

- [6] P. S. Bourdon and J. H. Shapiro, *Cyclic Phenomena for Composition Operators*, Memoirs of the AMS, 125, AMS, Providence, RI, 1997.
- [7] P. S. Bourdon and J. H. Shapiro, *Hypercyclic operators that commute with the Bergman backward shift*, Trans. Amer. Math. Soc. **352** (2000), no. 11, 5293–5316.
- [8] K. C. Chan and R. Sanders, *A weakly hypercyclic operator that is not norm hypercyclic*, preprint.
- [9] J.B. Conway, *The Theory of Subnormal Operators*, Amer. Math. Soc., Providence, RI, 1991.
- [10] G. Costakis, *On a conjecture of D. Herrero concerning hypercyclic operators*, C. R. Acad. Sci. Paris **330** (2000), 179–182.
- [11] F. León-Saavedra, *Notes about the Hypercyclicity Criterion*, preprint.
- [12] F. León-Saavedra and V. Müller, *Rotations of hypercyclic and supercyclic operators*, preprint.
- [13] N.S. Feldman, *Pure Subnormal Operators have Cyclic Adjoints*, J. Funct. Anal. **162** (1999), no. 2, 379 - 399.
- [14] N.S. Feldman *N-Supercyclic Operators*, Studia Math. **151** (2002), 141–159.
- [15] N.S. Feldman *Perturbations of Hypercyclic Vectors*, to appear in J. Math. Anal. Appl.
- [16] N.S. Feldman *Countably Hypercyclic Operators*, to appear in J. Operator Theory.
- [17] N.S. Feldman *Linear Chaos*, preprint.
- [18] N.S. Feldman, T.L. Miller, and V.G. Miller, *Hypercyclic and Supercyclic Cohypnormal Operators*, Acta Sci. Math. (Szeged) **68** (2002), 303–328.
- [19] R. M. Gethner and J. H. Shapiro, *Universal vectors for operators on spaces of holomorphic functions*, Proc. AMS, **100** (1987), 281–288.
- [20] G. Godefroy and J. H. Shapiro, *Operators with dense invariant cyclic manifolds*, J. Func. Anal., **98** (1991), 229–269.
- [21] G. Herzog and C. Schmoeger, *On operators T such that $f(T)$ is hypercyclic*, Studia Math. **108** (1994), no. 3, 209–216.
- [22] D. A. Herrero, ‘Hypercyclic operators and chaos’, *J. Operator Theory* 28 (1992), 93–103.
- [23] H. M. Hilden and L. J. Wallen, *Some cyclic and non-cyclic vectors of certain operators*, Ind. Univ. Math. J., **23** (1974), 557–565.
- [24] C. Kitai, *Invariant closed sets for linear operators*, Dissertation, Univ. of Toronto, 1982.
- [25] A. Peris, *Multihypercyclic operators are hypercyclic*, Math. Z. **236** (2001), 779–786.
- [26] C. Read, *The invariant subspace problem for a class of Banach spaces, 2: Hypercyclic operators*, Israel J. Math. **63** (1988), 1–40.
- [27] S. Rolewicz, *On orbits of elements*, Studia Math., **32** (1969), 17–22.
- [28] J.H. Shapiro, *Notes on Dynamics of Linear Operators*, unpublished lecture notes, available at <http://www.math.msu.edu/~shapiro>
- [29] H. N. Salas, *Hypercyclic weighted shifts*, Trans AMS, **347** (1995), 993–1004.
- [30] H. N. Salas, *Supercyclicity and weighted shifts*, Studia Math. **135** (1999), no. 1, 55–74.

DEPT. OF MATHEMATICS, WASHINGTON AND LEE UNIVERSITY, LEXINGTON VA 24450
 E-mail address: feldmanN@wlu.edu