HYPERCYCLIC AND SUPERCYCLIC COHYPONORMAL OPERATORS

NATHAN S. FELDMAN, VIVIEN GLASS MILLER, AND THOMAS L. MILLER

Abstract. We give a sufficient condition involving local spectra for an operator on a separable Banach space to be hypercyclic. Similar conditions are given for supercyclicity. These spectral conditions allows us to characterize the hyponormal operators with hypercyclic adjoints and those with supercyclic adjoints.

1. Introduction

An operator T on a Banach space X is *hypercyclic* if there is a vector $x \in X$ with dense orbit $\{x, Tx, T^2x, \dots\}$. For an example, consider the adjoint of the unilateral shift on $\ell^2(\mathbb{N})$. The operator $S^*(a_1, a_2, \dots) = (a_2, a_3, \dots)$ is not hypercyclic because $\|S^{*n}\| = 1$ for every $n \geq 1$, but, [33], λS^* is hypercyclic for all $|\lambda| > 1$. This leads naturally to the consideration of scaled orbits; an operator T on a Banach space X is said to be *supercyclic* if there is a vector $x \in X$ such that $\{cT^n x : n \geq 0, c \in \mathbb{C}\}$ is dense in X. Supercyclic operators were introduced by Hilden and Wallen in 1974, who showed that the adjoint of every unilateral weighted shift is supercyclic [22]. Many fundamental results regarding the theory of hypercyclic and supercyclic operators were established by C. Kitai in her thesis [23]. In particular, she showed that in order for an operator to be hypercyclic, every component of its spectrum must intersect the unit circle. This observation, along with the result of Hilden and Wallen, leads to an example of a supercyclic operator such that no multiple of it is hypercyclic. In Theorem 6.2 below, we establish a supercyclic analog of Kitai's unit circle theorem; specifically, we show that for every supercyclic operator T, there is a circle, $\{z : |z| = \rho\}$, that intersects not only every component of the spectrum of T, but in fact every part of the spectrum of T^*.

Another basic result of Kitai, formulated independently by Gethner and Shapiro [16], is a sufficient condition for hypercyclicity that has proven to be widely applicable. In Theorem 3.2, we use the well known "Hypercyclicity Criterion" of Kitai, Gethner and Shapiro to establish a sufficient condition for hypercyclicity in terms of the density of certain analytic spectral subspaces. We apply this to easily obtain as corollaries recent results regarding hypercyclicity of functions of operators, $f(T)$, [19, Theorem 1], [28, Theorem 1] and [4, Theorem 4], as well as the hypercyclicity of adjoints of multiplication operators on spaces of analytic functions [17, Theorem 4.5]. Moreover, in the case of a cohyponormal operator (the adjoint of a hyponormal operator), we show, in Theorem 4.3, that our sufficient condition is necessary as well.

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Hypercyclic Cohyponormal Operators (Theorem 4.3)**.** *If T is a hyponormal operator on a separable Hilbert space $\mathcal{H}$, then T^* is hypercyclic if and only if for every hyperinvariant subspace $\mathcal{M}$ of T, $\sigma(T|_{\mathcal{M}}) \cap \mathbb{D} \neq \emptyset$ and $\sigma(T|_{\mathcal{M}}) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$.*

A sufficient condition for supercyclicity reminiscent of the hypercyclicity criterion was established by Salas [35], who applied it to characterize supercyclic bilateral weighted shifts. In Section 5 below, we refine Salas's condition. Our results suggest that supercyclic cohyponormal operators come in two types, "inner" and "outer", and a criterion is given for each type. As in the hypercyclic case, these criteria give easy sufficient conditions for supercyclicity of functions $f(T)$. Also as in the hypercyclic case, the inner and outer supercyclicity criteria together characterize the cohyponormal supercyclic operators, Theorem 7.5. Consequently, we obtain a rich supply of examples. Because our criterion is a spectral condition, the construction of non-supercyclic and purely supercyclic operators is apparent, see Section 8. Throughout the paper, Γ_ρ will denote the circle in $\mathbb{C}$ centered at the origin with radius ρ and $\mathrm{int}\Gamma_\rho$ and $\mathrm{ext}\Gamma_\rho$ denote the interior and exterior regions bounded by Γ_ρ.

Supercyclic Cohyponormal Operators (Theorem 7.5)**.** *If T is a pure hyponormal operator, then T^* is supercyclic if and only if there exists a circle $\Gamma_\rho = \{z : |z| = \rho\}$, $\rho \geq 0$, such that either:*
(a) For every hyperinvariant subspace $\mathcal{M}$ of T, $\sigma(T|_{\mathcal{M}})$ intersects Γ_ρ and $\mathrm{int}\,\Gamma_\rho$.
or
(b) For every hyperinvariant subspace $\mathcal{M}$ of T, $\sigma(T|_{\mathcal{M}})$ intersects Γ_ρ and $\mathrm{ext}\,\Gamma_\rho$.

2. Local spectral theory preliminaries

Let X be a complex Banach space. For T a bounded linear operator on X, that is, for $T \in \mathcal{L}(X)$, we denote as usual the spectrum and the approximate point spectrum of T by $\sigma(T)$ and $\sigma_{ap}(T)$. The surjectivity spectrum of T is $\sigma_{su}(T) = \{\lambda \in \mathbb{C} : (\lambda - T)X \neq X\}$. Notice that if $T^* \in \mathcal{L}(X^*)$ is the adjoint of T, then $\sigma_{su}(T) = \sigma_{ap}(T^*)$ and $\sigma_{ap}(T) = \sigma_{su}(T^*)$. The complement of the surjectivity spectrum is $\rho_{su}(T)$, and so forth. If $T \in \mathcal{L}(X)$, let $\mathrm{Lat}(T)$ denote the lattice of closed T–invariant subspaces of X, and if $\mathcal{M} \in \mathrm{Lat}(T)$, then $T|_{\mathcal{M}} \in \mathcal{L}(\mathcal{M})$ is the restriction of T to $\mathcal{M}$. Following Halmos, we refer to $T|_{\mathcal{M}}$ as a part of T and to $\sigma(T|_{\mathcal{M}})$ as a part of the spectrum of T.

An operator T on a complex Banach space X is decomposable in the sense of Foiaş provided that whenever $\{U_1, U_2, \ldots, U_n\}$ is an open cover of $\mathbb{C}$, there exist $Y_1, Y_2, \ldots, Y_n \in \mathrm{Lat}(T)$ such that $X = Y_1 + Y_2 + \cdots + Y_n$ and $\sigma(T|_{Y_k}) \subset U_k$, for each k. This class of operators is quite large; for example, all normal operators on a Hilbert space, compact operators and generalized scalar operators on Banach spaces are decomposable. Although decomposable operators generally have no functional calculus beyond the basic analytic functional calculus of Riesz, these operators possess many of the spectral properties of normal operators. For the basic theory of decomposable operators, refer to [8], [25] or [36].

Let X be a complex Banach space. If U is an open subset of the complex plane $\mathbb{C}$, we denote by $\mathcal{O}(U, X)$ the Frèchet space of analytic X–valued functions on U. If $T \in \mathcal{L}(X)$, then T induces a continuous mapping T_U on every $\mathcal{O}(U, X)$ by $(T_U f)(\lambda) = (T - \lambda)f(\lambda)$ for every $f \in \mathcal{O}(U, X)$ and $\lambda \in U$. The operator T has Bishop's property (β) provided that for every open $U \subset \mathbb{C}$, the mapping T_U has closed range in $\mathcal{O}(U, X)$. That decomposable operators have property (β) is due to Albrecht [1].

For a closed subset F of $\mathbb{C}$, we will be concerned with the (glocal) analytic spectral subspaces $X_T(F)$. A vector $x \in X$ is in $X_T(F)$ provided that there exists an analytic resolvent function g so that $x = (T - \lambda)g(\lambda)$ for every $\lambda \in \mathbb{C} \setminus F$. The spaces $X_T(F)$ are T–invariant subspaces of X, but generally not closed. If, for every closed $F \subset \mathbb{C}$, the analytic spectral subspace $X_T(F)$ is closed, then the operator T is said to have Dunford's property (C). It is a theorem of Bishop [5] that an operator $T \in \mathcal{L}(X)$ with (β) has the property that the dual space decomposes as $X^* = X^*_{T^*}(\overline{U}) + X^*_{T^*}(\overline{V})$ whenever U and V are open with $\mathbb{C} \subset U \cup V$. This is the decomposition property (δ); specifically, $T \in \mathcal{L}(X)$ has property (δ) provided that for any open cover $\{U_1, U_2, \ldots, U_n\}$ of $\sigma(T)$, the space X can be written as the sum of the analytic subspaces: $X = X_T(\overline{U_1}) + X_T(\overline{U_2}) + \cdots + X_T(\overline{U_n})$. Thus an operator T is decomposable if and only if T has both properties (C) and (δ).

Albrecht and Eschmeier [2] have shown that the properties (β) and (δ) are completely dual; an operator T has one of these exactly when its adjoint has the other. Moreover, they characterize operators with Bishop's property (β) as those similar to the restriction of a decomposable operator, and operators with the decomposition property (δ) as those similar to a quotient of a decomposable operator. That hyponormal operators are subscalar and thus subdecomposable is due to Putinar [31]. Surjective Banach space isometries are generalized scalar [8, 5.1.4], and it is a result of Douglas that every isometry has a surjective extension [12].

Bishop's property (β) evidently implies (C), and property (C) in turn implies that T_U is injective for every open $U \subset \mathbb{C}$, [25, Proposition 3.3.4]; this last property is the single-valued extension property (SVEP). For operators T with (SVEP), we consider local spectra. If $x \in X$, the local resolvent of T at x is defined to be the set $\rho_T(x)$ of $\lambda \in \mathbb{C}$ for which there is a neighborhood U of λ and $f \in \mathcal{O}(U, X)$ so that $x = (T - z)f(z)$ for all $z \in U$. The local spectrum of T at x is then $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. We will need the following well known facts regarding the single-valued extension property, [25, Propositions 1.3.2 and 3.3.2].

Proposition 2.1. *If $T \in \mathcal{L}(X)$ has (SVEP), then*

1. *$\sigma_T(x)$ is nonempty for every $x \neq 0$.*
2. *$x \in X_T(F)$ if and only if $\sigma_T(x) \subset F$.*
3. *$\sigma(T) = \sigma_{su}(T)$.*
4. *The set of vectors x such that $\sigma_T(x) \neq \sigma(T)$ is meager.*
5. *If T has Dunford's property (C) and $F \subseteq \mathbb{C}$ is closed, then $\sigma(T|X_T(F)) \subseteq F \cap \sigma(T)$.*

Theorem 2.2. [25, Proposition 2.5.14] *If T is a bounded linear operator on a separable Banach space X having the decomposition property (δ), then for every open set $U \subseteq \mathbb{C}$ we have $X_T(U)^\perp = X^*_{T^*}(\mathbb{C} \setminus U) = \{x^* \in X^* : \sigma_{T^*}(x^*) \subseteq \mathbb{C} \setminus U\}$.*

There is a simple relation between parts of the spectrum and the local spectra of an operator with Dunford's Property (C).

Proposition 2.3. *If $T \in \mathcal{L}(X)$ has Dunford's property (C), then every local spectrum is a part of the spectrum of T, and every part of the spectrum contains a nonempty local spectrum.*

Proof. If F is any closed subset of $\mathbb{C}$, then $\sigma(T|_{X_T(F)}) \subset F$; indeed, suppose that $x \in X_T(F)$ and let f be a resolvent for x on $U = \mathbb{C} \setminus F$. The function $g : U \times U \to X$

defined by

$$g(\mu,\lambda)=\frac{f(\mu)-f(\lambda)}{\mu-\lambda}\quad\text{if }\mu\neq\lambda\text{ and}\quad g(\lambda,\lambda)=f'(\lambda)$$

is analytic, and $(T-\mu)g(\mu,\lambda)=f(\lambda)$ for every λ, $\mu\in U$, $\mu\neq\lambda$. By continuity, it follows that $(T-\lambda)g(\lambda,\lambda)=f(\lambda)$ as well, and so $f(\lambda)\in X_T(F)$ for each $\lambda\in U$. Thus $\mathbb{C}\setminus F\subset\rho_{su}(T|_{X_T(F)})=\rho(T|_{X_T(F)})$ since $T|_{X_T(F)}$ has (SVEP).

If $F=\sigma_T(x)$ for some nonzero x, then clearly $\rho(T|_{X_T(F)})\subset\rho_T(x)$, and therefore equality obtains. Finally, if $\mathcal{M}\in\operatorname{Lat}(T)\setminus\{0\}$, then by the preceding proposition, there is an $x\in\mathcal{M}$ so that $\sigma_{T|_\mathcal{M}}(x)=\sigma(T|_\mathcal{M})$, and as in the previous paragraph, it follows that $\sigma_T(x)\subset\sigma_{T|_\mathcal{M}}(x)=\sigma(T|_\mathcal{M})$. ☐

Not every part of the spectrum of an operator T with (C) need be a local spectrum. The operator $T=M_z$ on the Lebesgue space $L^2(\partial\mathbb{D})$ is normal and thus has property (β). The classical Hardy space H^2 is invariant under T, and $\sigma(T|_{H^2})=\overline{\mathbb{D}}$, the closed unit disk. If $f\in H^2\setminus\{0\}$, then f is analytic on the open unit disk $\mathbb{D}$, and so $\sigma_{T|_{H^2}}(f)=\sigma(T|_{H^2})=\overline{\mathbb{D}}$, whereas $\sigma_T(f)=\sigma(T)=\partial\mathbb{D}$ by Szegö's Theorem.

3. Hypercyclic operators

The principal tool in the proof of Theorem 3.2 below is a variation of the hypercyclicity criterion due to Gethner and Shapiro [16].

Theorem 3.1 (Hypercyclicity Criterion)**.** *Suppose that $T\in\mathcal{L}(X)$. If there exists two dense sets Y and Z in X such that:*

1. *$T^nx\to 0$ for every $x\in Y$, and*
2. *There exists functions $B_n:Z\to X$ such that $T^nB_n=I|Z$ and $B_nx\to 0$ for every $x\in Z$,*

then T is hypercyclic.

If U is an open subset of $\mathbb{C}$, let $X_T(U)=\bigcup\{X_T(F): F\text{ is a closed subset of }U\}$. Since $X_T(F)\subset X_T(K)$ whenever $F\subset K$, we see that $X_T(U)$ is a linear subspace of X.

Theorem 3.2. *Suppose that $T\in\mathcal{L}(X)$. If $X_T(\mathbb{D})$ and $X_T(\mathbb{C}\setminus\overline{\mathbb{D}})$ are each dense, then T is hypercyclic.*

Proof. To show that T is hypercyclic, we shall verify the hypercyclicity criterion with $Y=X_T(\mathbb{D})$ and $Z=X_T(\mathbb{C}\setminus\overline{\mathbb{D}})$.

If $x\in X_T(K)$ for some compact $K\subset\mathbb{D}$, choose ρ, $0<\rho<1$ so that $K\subset B(0,\rho)$. Let γ and Γ respectively denote the positively oriented circles $\{z:|z|=\rho\}$ and $\{z:|z|=\|T\|+1\}$. If g is any resolvent function for x on $\mathbb{C}\setminus K$, then Cauchy's formula and the Riesz functional calculus imply that for every $n\geq 0$

$$T^nx=-\frac{1}{2\pi i}\int_\Gamma z^ng(z)\,dz=-\frac{1}{2\pi i}\int_\gamma z^n\,g(z)\,dz.$$

In particular, for every $x\in X_T(\mathbb{D})$, it follows that

$$T^nx\to 0\text{ as }n\to\infty. \tag{1}$$

Thus the first condition in the hypercyclicity criterion is satisfied. Now for the second condition, if $x\in X_T(K)$ for some $K\subset\mathbb{C}\setminus\overline{\mathbb{D}}$, then choose $\rho_2>\rho_1>1$ so that K is contained in the annulus $\{z:\rho_1<|z|<\rho_2\}$. Let γ_1 and γ_2 be the inner and outer boundaries of the annulus respectively, each with counterclockwise orientation, and let $\gamma = \gamma_2 - \gamma_1$. Then the index of γ relative to K is $\operatorname{ind}(\gamma, K) = 1$ and $\operatorname{ind}(\gamma, \overline{\mathbb{D}}) = 0$. If g is any resolvent function for x on $\mathbb{C} \setminus K$, define

$$B_n x = -\frac{1}{2\pi i}\int_\gamma \frac{1}{z^n} g(z)\, dz$$

for every natural number n. Clearly,

$$B_n x \to 0 \text{ as } n \to \infty, \text{ and} \tag{2}$$

$$\begin{aligned}
T^n B_n x &= -\frac{1}{2\pi i}\int_\gamma \frac{(T^n - z^n) + z^n}{z^n} g(z)\, dz \\
&= -\frac{1}{2\pi i}\int_\gamma \left(\sum_{k=1}^{n} T^{k-1} z^{n-k}\right)(T - z) g(z)\, dz - \frac{1}{2\pi i}\int_\gamma g(z)\, dz \\
&= -\frac{1}{2\pi i}\int_\gamma \left(\sum_{k=1}^{n} T^{k-1} z^{n-k}\right) x\, dz + \frac{1}{2\pi i}\int_{\gamma_1} g(z)\, dz - \frac{1}{2\pi i}\int_{\gamma_2} g(z)\, dz \\
&= -\frac{1}{2\pi i}\int_{\gamma_2} g(z)\, dz
\end{aligned}$$

by Cauchy's formula. Since γ_2 is the outer boundary of the annulus, another application of Cauchy's theorem and the Riesz functional calculus implies that for $\Gamma = \{z : |z| = \|T\| + 1\}$

$$T^n B_n x = -\frac{1}{2\pi i}\int_{\gamma_2} g(z)\, dz = -\frac{1}{2\pi i}\int_\Gamma (z - T)^{-1} x\, dz = x. \tag{3}$$

Since $X_T(\mathbb{D})$ and $X_T(\mathbb{C}\setminus\overline{\mathbb{D}})$ are dense, (1), (2) and (3) along with the hypercyclicity criterion imply that T is hypercyclic. $\square$

Corollary 3.3. *Suppose that $T \in \mathcal{L}(X)$ and φ analytic in a neighborhood of $\sigma(T)$. If there exist open sets U, $V \subset \mathbb{C}$ so that each of the subspaces $X_T(U)$ and $X_T(V)$ is dense in X, then $\varphi(T)$ is hypercyclic if φ separates U and V is the sense that $\varphi(U) \subset \mathbb{D}$ and $\varphi(V) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$.*

Proof. If φ is analytic on $\sigma(T)$, then for every closed subset H of $\mathbb{C}$, we have that

$$X_{\varphi(T)}(H) = X_T(\varphi^{-1}(H)),$$

[25, Theorem 3.3.6]. Thus, $X_{\varphi(T)}(\mathbb{D}) = X_T(\varphi^{-1}(\mathbb{D})) \supseteq X_T(U)$. Similarly, we have that $X_{\varphi(T)}(\mathbb{C} \setminus \overline{\mathbb{D}}) = X_T(\varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}})) \supseteq X_T(V)$. Hence, both $X_{\varphi(T)}(\mathbb{D})$ and $X_{\varphi(T)}(\mathbb{C} \setminus \overline{\mathbb{D}})$ are dense. Thus $\varphi(T)$ is hypercyclic by Theorem 3.2. $\square$

Recall that the Kato resolvent of an operator T is the set $\rho_K(T)$ consisting of all $\lambda \in \mathbb{C}$ for which $(T - \lambda)$ has closed range and $\ker(T - \lambda) \subset \bigcap_{n=1}^\infty \operatorname{ran}(T - \lambda)^n$. Clearly, $\rho_K(T)$ contains $\rho_{su}(T)$. The significance of the Kato resolvent in our context is that $\rho_K(T)$ is open in $\mathbb{C}$, and the generalized kernel is invariant over every component of $\rho_K(T)$; see [25, Section 3.1] for details. As an application of Theorem 3.2, we obtain the following, [4, Theorem 4].

Corollary 3.4. *Suppose that* $\{G_n\}_n$ *is the set of components of* $\rho_K(T)$ *of an operator* $T \in \mathcal{L}(X)$*, and assume that* $\{\lambda_n\}_n$ *is a sequence of points,* $\lambda_n \in G_n$*, such that* $\operatorname{span}\left(\bigcup_{n,k} \ker(T-\lambda_n)^k\right)$ *is dense in* X*. If* φ *is analytic on* $\sigma(T)$*, nonconstant on each component of* $\sigma(T)$*, and if* $\varphi(G_n) \cap \partial\mathbb{D} \neq \emptyset$ *for each* n*, then* $\varphi(T)$ *is hypercyclic.*

Proof. Because generalized kernels are invariant over each component of $\rho_K(T)$, $\operatorname{span}\left(\bigcup_{n,k} \ker(T-\mu_n)^k\right)$ is dense in X whenever $\mu_n \in G_n$. Since, [25, Proposition 3.3.1], $\bigcup_k \ker(T-\lambda)^k \subset X_T(\{\lambda\})$ for all λ, it follows that $X_T(U)$ is dense in X whenever U is open and $\emptyset \neq U \cap G_n$ for every n. In particular, $U = \varphi^{-1}(\mathbb{D})$ and $V = \varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}})$ are disjoint, and both meet each component of $\rho_K(T)$, hence $X_T(U)$ and $X_T(V)$ are each dense. The result now follows from Corollary 3.3. ☐

As an additional example, in which there is no assumption of closed range, we can apply Theorem 3.2 to obtain [17, Theorem 4.5]. Let Ω be a domain in $\mathbb{C}^n$ and $\mathcal{H}$ a nontrivial Hilbert space of analytic functions on Ω such that for every $z \in \Omega$ the evaluation $f \mapsto f(z)$ is continuous on $\mathcal{H}$. If $\varphi : \Omega \to \mathbb{C}$ is a bounded analytic function such that $\varphi f \in \mathcal{H}$ for all $f \in \mathcal{H}$, define $M_\varphi : \mathcal{H} \to \mathcal{H}$ by $M_\varphi f = \varphi f$. The continuity of point evaluations and the Closed Graph Theorem show that M_φ is continuous.

Corollary 3.5. *If* $\varphi : \Omega \to \mathbb{C}$ *is nonconstant and such that* $\varphi f \in \mathcal{H}$ *for all* $f \in \mathcal{H}$*, then* M_φ^* *is hypercyclic whenever* $\varphi(\Omega) \cap \partial\mathbb{D} \neq \emptyset$*.*

Corollary 3.6. *Let* X *be a complex Banach space and suppose that* $S \in \mathcal{L}(X)$ *has the decomposition property* (δ)*. If*

$$\sigma_{S^*}(x^*) \cap \mathbb{D} \neq \emptyset \quad \textit{and} \quad \sigma_{S^*}(x^*) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset \tag{4}$$

for every nonzero $x^* \in X^*$*, then* S *is hypercyclic.*

Proof. Since S^* has Bishop's property (β), by Theorem 2.2, (4) is exactly the condition that $X_S(\mathbb{D})$ and $X_S(\mathbb{C}\setminus\overline{\mathbb{D}})$ are each dense; hence Theorem 3.2 applies. ☐

4. Hypercyclic Cohyponormal Operators

It was shown in Feldman [14] that the adjoint of every pure subnormal operator is cyclic. It is still unknown if the adjoint of every pure hyponormal operator is cyclic. Nevertheless, here we shall see that several hyponormal operators have hypercyclic or supercyclic adjoints. Recall though that hyponormal operators cannot be hypercyclic or supercyclic [6].

Here we observe that the sufficient condition in Corollary 3.6 is also necessary if we restrict our attention to the case of hyponormal operators on a Hilbert space. Recall that an operator $T \in \mathcal{L}(X)$ is hyponormal provided that $\|T^*x\| \leq \|Tx\|$ for every $x \in X$, equivalently, by the Closed Graph Theorem, if there is a contraction K on X so that $T^* = KT$. It is well known that the restriction $T|_{\mathcal{M}}$ of a hyponormal operator is hyponormal for every $\mathcal{M} \in \operatorname{Lat}(T)$, and if T is invertible, then T^{-1} is hyponormal. Also, $\|T^n\| = \|T\|^n$ for every natural number n; in particular, the spectral radius of T, $r(T)$, satisfies $r(T) = \|T\|$. See Martin & Putinar [26] for more on hyponormal operators.

Theorem 4.1. *If T is a hyponormal operator on a separable Hilbert space $\mathcal{H}$, then T^* is hypercyclic if and only if $\sigma_T(x) \cap \mathbb{D} \neq \emptyset$ and $\sigma_T(x) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$ for every nonzero $x \in \mathcal{H}$.*

Proof. Let T be hyponormal on a Hilbert space $\mathcal{H}$. It is known that a hyponormal operator has property (β), see Putinar [31] or [26], p.84, and thus T^* has property (δ). If the local spectra $\sigma_T(x) \cap \mathbb{D} \neq \emptyset$ and $\sigma_T(x) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$ for every nonzero $x \in \mathcal{H}$, then T^* is hypercyclic by Corollary 3.6.

Conversely, suppose that T^* is hypercyclic. By Proposition 2.3, it suffices to show that every part of the spectrum of T meets both $\mathbb{D}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$. Let $S = T|_{\mathcal{M}}$ for some $\mathcal{M} \in \mathrm{Lat}(T) \setminus \{0\}$. If x is a hypercyclic vector for T^*, then the projection $P_{\mathcal{M}}x$ is hypercyclic for $S^* = P_{\mathcal{M}}T^*|_{\mathcal{M}}$, and so $r(S) = \|S\| = \|S^*\| > 1$. Thus, $\sigma(S) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$. On the other hand, if $\sigma(S) \subset (\mathbb{C} \setminus \mathbb{D})$, then S^{-1} is hyponormal with $r(S^{-1}) \leq 1$ and hence $\|S^{-1}\| \leq 1$. But S^* hypercyclic and invertible which implies that $(S^*)^{-1}$ is hypercyclic, [3] or [27], and thus $\|(S^*)^{-1}\| > 1$, contradicting the fact that $\|S^{-1}\| \leq 1$. Thus we have that $\sigma(S) \cap \mathbb{D} \neq \emptyset$. ☐

Example 4.2. In the setting of Corollary 3.5, assume that $\mathcal{H}$ is a Hilbert space of analytic functions on a domain Ω in $\mathbb{C}$ such that $f \mapsto f(\omega)$ is continuous for every $\omega \in \Omega$ and such that $f \mapsto \varphi f$ is continuous for every bounded analytic function φ on Ω. This latter assumption guarantees that $\sigma(M_\varphi) \subset \overline{\varphi(\Omega)}$. If $\varphi \in H^\infty(\Omega)$ is nonconstant, then M_φ has property (C) on $\mathcal{H}$; in fact, if F is a closed subset of $\mathbb{C}$, then $\mathcal{H}_{M_\varphi}(F) = \{0\}$ or $\mathcal{H}_{M_\varphi}(F) = \mathcal{H}$. Indeed, suppose that F is a closed set such that $\varphi(\Omega) \not\subset F$, and let $f \in \mathcal{H}_{M_\varphi}(F)$. If $U = \mathbb{C} \setminus F$, then there is an analytic function $g : U \times \Omega \to \mathbb{C}$ such that $(\varphi(z) - \lambda)g(\lambda, z) = f(z)$ for every $\lambda \in U$ and $z \in \Omega$. It follows that $f = 0$ on $\varphi^{-1}(U) \cap \Omega$, and therefore that $f = 0$ on all of Ω. Thus $\mathcal{H}_{M_\varphi}(F) = \{0\}$ or, if $\varphi(\Omega) \subset F$, then $\mathcal{H}_{M_\varphi}(F) = \mathcal{H}$. Equivalently,

$$\sigma_{M_\varphi}(f) = \sigma(M_\varphi) = \overline{\varphi(\Omega)}$$

for every nonzero $f \in \mathcal{H}$. If we assume additionally that M_φ is hyponormal on $\mathcal{H}$, then the converse of Corollary 3.5 obtains: M_φ^* is hypercyclic if and only if $\varphi(\Omega) \cap \partial\mathbb{D} \neq \emptyset$.

In view of Proposition 2.3, an equivalent way to state Theorem 4.1 is as follows.

Theorem 4.3. *If T is a hyponormal operator on a separable Hilbert space $\mathcal{H}$, then T^* is hypercyclic if and only if $\sigma(T|_{\mathcal{M}}) \cap \mathbb{D} \neq \emptyset$ and $\sigma(T|_{\mathcal{M}}) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$, for every hyperinvariant subspace $\mathcal{M}$ of T.*

If μ is a positive compactly supported regular Borel measure on $\mathbb{C}$, then we denote by $P^2(\mu)$ the closure of the polynomials in $L^2(\mu)$. Let N_μ be the normal operator of multiplication by z on $L^2(\mu)$, and S_μ be the subnormal operator of multiplication by z on $P^2(\mu)$. Since N_μ leaves $P^2(\mu)$ invariant, N_μ^* will leave $P^2(\mu)^\perp$ invariant. Let $T = N_\mu^*|P^2(\mu)^\perp$, so $T = M_{\overline{z}}$ on $P^2(\mu)^\perp$. If S_μ is pure, then T is a pure subnormal operator, called the dual of S_μ (see Conway [9] or [10]).

Example 4.4. Suppose Δ is an open disk such that $\partial\Delta$ intersects both $\{z : |z| < 1\}$ and $\{z : |z| > 1\}$. Denote Lebesgue measure on $\partial\Delta$ by m and let dA be area measure on Δ. There exists a $w \in L^\infty(\Delta,\ dA)$, $w > 0$ a.e., such that if $T = M_{\overline{z}}$ on $P^2(m + w\, dA)^\perp$, then T is an irreducible subnormal operator such that T^* is hypercyclic.

Remark. Notice that if $w = 1$, then T is an irreducible subnormal operator, but T^* is not hypercyclic.

Proof of Example 4.4. Let $\Delta = \{z : |z-a| < R\}$ and define $G(z) = \exp(-\exp((R-|z-a|)^{-1}))$ for every $z \in \Delta$. By [10, Theorem 14.21], M_z on $L^2(G\,dA)$ has a bounded cyclic vector, φ; let $w = |\varphi|^2 G$. Notice that $P^2(w\,dA) = L^2(w\,dA)$. Let $\mu = m + w\,dA$.

Claim: If $f \in P^2(\mu)^\perp$ and $f|I = 0$ for an interval $I \subseteq \partial\Delta$, then $f = 0$ μ-a.e.

Given the Claim, let's finish the proof. We want to apply Theorem 4.3. Let $\mathcal{M}$ be an invariant subspace for T. If $f \in \mathcal{M}$ and $f \neq 0$, then $\partial\Delta \subseteq \operatorname{support}(f) \subseteq \sigma(T|_{\mathcal{M}})$. It follows that $\sigma(T|_{\mathcal{M}})$ intersects both $\{z : |z| < 1\}$ and $\{z : |z| > 1\}$. Thus Theorem 4.3 applies to give that T^* is hypercyclic.

In order to establish the Claim, suppose that $f \in P^2(\mu)^\perp$ and $f = 0$ μ-a.e. on an interval $I \subseteq \partial\Delta$. If $E = \overline{\Delta} \setminus I$, then by [24, Corollary 2],

$$P^2(\mu|_E) = L^2(m|_{\partial\Delta\setminus I}) \oplus L^2(|\varphi|^2 G\,dA) = L^2(\mu|_E).$$

Since $f \in L^2(\mu|E)$ and $f \perp P^2(\mu|E))$, it follows that $f = 0$ μ-a.e. □

Some immediate corollaries of Theorem 4.3 are as follows.

Corollary 4.5. *(a) If T_n is a bounded sequence of cohyponormal operators each of which is hypercyclic, then $\bigoplus_n T_n$ is also hypercyclic.*

(b) If T is a hypercyclic cohyponormal operator and $f(z)$ is an inner function that is analytic on $\sigma(T)$, then $f(T)$ is also hypercyclic.

Example 4.6. Let $\{\Delta_n\}_{n=1}^\infty$ be a sequence of open disks in the plane; for each n let $L^2_a(\Delta_n)$ be the Bergman space on Δ_n, and let $\mathcal{H} = \bigoplus_n L^2_a(\Delta_n)$. Let $\{\varphi_n\}_n$ be a uniformly bounded sequence of nonconstant analytic functions $\varphi_n : \Delta_n \to \mathbb{C}$, and let $T = \bigoplus_n M_{\varphi_n}$. Then T is a pure subnormal operator. By Example 4.2 and Proposition 2.3, the spectrum of M_{φ_n} has only one part, and the parts of the spectrum of T are of the form $\operatorname{cl}\big[\bigcup_k \varphi_{n_k}(\Delta_{n_k})\big]$. Therefore, by Theorem 4.3, T^* is hypercyclic if and only if $\varphi_n(\Delta_n) \cap \partial\mathbb{D} \neq \emptyset$ for every n.

We will use direct sums of Bergman spaces to construct additional examples in Section 8 below.

5. Supercyclic Operators

As we shall see, supercyclic cohyponormal operators fall into two categories: inner and outer, and a slightly different criterion is used in the two different cases. The following is a slight refinement of the (outer) *supercyclicity criterion* due to Hector Salas [35] used to show that an operator is supercyclic, as well as an "inner" formulation of his criterion. One can also see that if T is invertible and satisfies one of the criteria, then T^{-1} satisfies the other criteria. In what follows, X will denote a separable Banach space.

Theorem 5.1 (An Outer Supercyclicity Criterion)**.** *Suppose that $T \in \mathcal{L}(X)$. If there exists a dense linear subspace Y and for every $y \in Y$ there is a dense linear subspace X_y such that:*

1. *There exists functions $B_n : Y \to X$ such that $T^n B_n y = y$ for all $y \in Y$, and*
2. *if $y \in Y$ and $x \in X_y$, then $\|T^n x\|\|B_n y\| \to 0$ as $n \to \infty$;*

then T is supercyclic.

Theorem 5.2 (An Inner Supercyclicity Criterion)**.** *Suppose that $T \in \mathcal{L}(X)$. If there exists a dense linear subspace Y and for every $y \in Y$ there is a dense linear subspace X_y such that:*

1. *There exists functions $B_{y,n} : X_y \to X$ such that $T^n B_{y,n} x = x$ for all $x \in X_y$, and*
2. *if $y \in Y$ and $x \in X_y$, then $\|T^n y\| \|B_{y,n} x\| \to 0$ as $n \to \infty$;*

then T is supercyclic.

Note that the functions B_n and $B_{y,n}$, which are right inverses of T, are nothing more than well defined functions; they may be, and usually are, discontinuous. The proof of Theorem 5.1 is essentially due to Salas [35] and is similar to the proof of Theorem 5.2 given below. Both proofs use the following characterization of supercyclicity, see [15].

Proposition 5.3. *If $T \in \mathcal{L}(X)$, then T is supercyclic if and only if*

$$\{(x, \alpha T^n x) : x \in X, n \geq 0, \alpha \in \mathbb{C}\}$$

is dense in $X \oplus X$.

Proof of Theorem 5.2. We will apply Proposition 5.3. So let $g, h \in X$ and let $\epsilon > 0$, then we must find $z \in X$, $c \in \mathbb{C}$, and $n \geq 0$ such that $\|g - z\| \leq \epsilon$ and $\|h - cT^n z\| \leq \epsilon$.

First choose $y \in Y$ such that $\|g - y\| < \epsilon/2$. Now that $y \in Y$ has been chosen, our hypothesis guarantees that there is a dense set X_y with various properties. So, choose $x \in X_y$ such that $\|h - x\| \leq \epsilon/2$.

We want to let $z = y + (1/c)B_{y,n} x$ for some choice of $c > 0$ and $n \geq 1$. First choose $n \geq 1$ such that $\|T^n y\| \|B_{y,n} x\| \leq \epsilon^2/4$. Next choose $c > 0$ such that $(1/c)\|B_{y,n} x\| = \epsilon/2$. With these choices we claim that z has the required properties.

First, $\|g - z\| = \|g - y - (1/c)B_{y,n} x\| \leq \|g - y\| + (1/c)\|B_{y,n} x\| \leq \epsilon/2 + \epsilon/2 = \epsilon$.

Second, using the fact that $T^n B_{y,n} = I|_{X_y}$, we have

$$\begin{aligned} \|h - cT^n z\| &= \|h - cT^n y - x\| \leq \|h - x\| + c\|T^n y\| \\ &= \|h - x\| + (2/\epsilon)\|B_{y,n} x\| \|T^n y\| \\ &\leq \epsilon/2 + (2/\epsilon)(\epsilon^2/4) = \epsilon. \end{aligned}$$

Thus by Proposition 5.3, T is supercyclic. □

If $\rho \geq 0$, we denote the circle $\{z : |z| = \rho\}$ by Γ_ρ. The interior and exterior of Γ_ρ are the regions $\operatorname{int} \Gamma_\rho = \{z : |z| < \rho\}$ and $\operatorname{ext} \Gamma_\rho = \{z : |z| > \rho\}$.

Corollary 5.4. *Let $T \in \mathcal{L}(X)$. If there exists a number $\rho \geq 0$, such that for every $\epsilon > 0$, $\operatorname{span}\{\ker(T - \lambda) : \rho < |\lambda| < \rho + \epsilon\}$ is dense in X, then T is supercyclic.*

Proof. We will show how to apply Theorem 5.1. Let Y be the linear span (finite linear combinations only) of $\{\ker(T - \lambda) : |\lambda| > \rho\}$. Then Y is a dense linear subspace of $\mathcal{H}$. We now define $B : Y \to Y$ as follows: If $v \in \ker(T - \lambda)$, $|\lambda| > \rho$, let $Bv = \frac{1}{\lambda} v$. Now extend B linearly to all of Y. It follows easily that B is well defined and $TB|_Y = I$.

Now, for $y \in Y$, we need to define X_y. Suppose that $y = \alpha_1 x_1 + \cdots + \alpha_n x_n$ where $x_i \in \ker(T - \lambda_i)$ and $|\lambda_i| > \rho$. Let $m = \min\{|\lambda_i| : 1 \leq i \leq n\}$. Notice that $m > \rho$. Thus, if we let X_y be the set of all finite linear combinations of $\{\ker(T - \lambda) : \rho < |\lambda| < m\}$, then by our assumption, X_y is dense.

Finally, one easily sees that if $y \in Y$ and $x \in X_y$, then $\|T^n x\| \|B^n y\| \to 0$ as $n \to \infty$. Hence the criterion is satisfied and T is supercyclic. □

Similarly, one has the following result. Its proof is the same other than using Theorem 5.2 instead of Theorem 5.1.

Corollary 5.5. *Supppose that* $T \in \mathcal{L}(X)$ *and that there exists a number* $\rho > 0$*, such that for every* $\epsilon > 0$*,* $\text{span}\{\ker(T - \lambda) : \rho - \epsilon < |\lambda| < \rho\}$ *is dense in* X*, then* T *is supercyclic.*

Proposition 5.6. *If* S *is an operator on a Hilbert space with a supercyclic adjoint, then for every invariant subspace* $\mathcal{M}$ *of* S*, we have* $S|\mathcal{M}$ *has a supercyclic adjoint.*

Proof. If x is a supercyclic vector for S^* and P is the projection onto $\mathcal{M}$, then Px is a supercyclic vector for $(S|\mathcal{M})^*$. ☐

6. The Supercyclicity Circle for General Operators

If C is a compact set in the plane and $\epsilon > 0$, then let $B(C, \epsilon)$ denote the ϵ-neighborhood of C, that is $B(C, \epsilon) = \{z : dist(z, C) < \epsilon\}$. For the proof of the following classic result see Newman [30], corollary 1, p.83.

Lemma 6.1. *If* K *is any compact set in the complex plane,* C *is a component of* K*, and* $\epsilon > 0$*, then there exists disjoint open sets* U*,* V *such that* $K \subseteq U \cup V$ *and* $C \subseteq U \subseteq B(C, \epsilon)$*.*

The following Theorem is a stronger form of a result due to Herrero [20, Proposition 3.1].

Theorem 6.2. *If* $T \in \mathcal{L}(X)$ *is a supercyclic operator on a separable Banach space* X*, then there exists a circle* Γ_ρ*,* $\rho \geq 0$*, such that* $\sigma(T^*|_{\mathcal{M}}) \cap \Gamma_\rho \neq \emptyset$ *for every nonzero weak* closed* T^**-invariant subspace* $\mathcal{M}$ *of* X^**.*

In particular, every component of the spectrum of T *intersects* Γ_ρ*.*

If T is a supercyclic operator, then any circle as in Theorem 6.2 will be called a **supercyclicity circle** for T and its radius is a **supercyclicity radius** of T. Notice that a supercyclicity circle may or may not be unique (but see Corollary 7.7). However, the set of supercyclicity radii form a closed interval. Also, notice that ρ may be zero, see the examples in Section 8.

The proof of Theorem 6.2 follows from the following lemmas

Lemma 6.3. *Let* $T_i \in \mathcal{L}(X_i)$*,* $i = 1, 2$*. If* $T = T_1 \oplus T_2$ *and* $\|T_1^n x\| \to 0$ *as* $n \to \infty$ *for every* $x \in X_1$ *and* $\varliminf\limits_{n\to\infty} \|T_2^n x\| > 0$ *for every nonzero* $x \in X_2$*, then* T *is not supercyclic.*

Proof. Suppose that T is supercyclic and $(x, y) \in X_1 \oplus X_2$, is a supercyclic vector for T. If $(z, w) \in X_1 \oplus X_2$ with $z \notin \{cT_1^n x : c \in \mathbb{C}, n \geq 0\}$, then there exists scalars $\{c_k\}$, and a subsequence $\{n_k\}$ such that $c_k T^{n_k}(x, y) \to (z, w)$. Thus, $(c_k T_1^{n_k} x, c_k T_2^{n_k} y) \to (z, w)$. Since $z \notin \{cT_1^n x : c \in \mathbb{C}, n \geq 0\}$, it follows that $n_k \to \infty$.

However, since $T_1^{n_k} x \to 0$ and $c_k T_1^{n_k} x \to z \neq 0$, we must have that $c_k \to \infty$. On the other hand, $c_k T_2^{n_k} y \to w$, and since $c_k \to \infty$, we must have that $\|T_2^{n_k} y\| \to 0$. However this is a contradiction, so T is not supercyclic. ☐

Lemma 6.4. *(a) If* $T \in \mathcal{L}(X)$ *is an invertible operator such that* $\|T^{-1}\| \leq 1$*, then* $\|T^n x\| \geq \|x\|$ *for every* $x \in X$ *and for every* $n \geq 0$*.*

(b) If $\sigma(T)$ *is contained in* $\{z : |z| > 1\}$*, then* $\|T^n x\| \to \infty$ *as* $n \to \infty$ *for every nonzero vector* $x \in X$*.*

(c) If $\sigma(T)$ is contained in $\{z : |z| < 1\}$, then $\|T^n x\| \to 0$ as $n \to \infty$ for every vector $x \in X$.

Proof. (a) If $x \in X$, then $\|x\| = \|T^{-n}T^n x\| \leq \|T^{-n}\|\|T^n x\| \leq \|T^n x\|$.

(b) Notice that T is invertible and the spectral radius of T^{-1} is strictly less than one, $r(T^{-1}) < 1$. Now, if $x \neq 0$, then

$$\|x\| = \|T^{-n}T^n x\| \leq \|T^{-n}\|\|T^n x\|$$

thus,

$$\|x\|^{1/n} \leq \|T^{-n}\|^{1/n}\|T^n x\|^{1/n}.$$

Now taking limits we have,

$$1 \leq r(T^{-1}) \varliminf \|T^n x\|^{1/n}.$$

Thus, $\varliminf \|T^n x\|^{1/n} \geq r(T^{-1})^{-1} > 1$. So, $\|T^n x\|^{1/n} \geq \eta > 1$ for all large n, thus $\|T^n x\| \to \infty$.

(c) This is well known and follows for example from the continuity of the Riesz functional calculus. □

Proof of Theorem 6.2. Let wk*–Lat(T^*) denote the set of weak* closed subspaces in Lat(T^*). First notice that every component of $\sigma(T)$ arises as the decreasing intersection of spectra of the form $\sigma(T^*|_{\mathcal{M}})$, for some $\mathcal{M} \in$ wk*–Lat(T^*). This follows easily from Lemma 6.1 and the Riesz functional calculus; we shall briefly sketch the argument. Let U, V be a separation of $\sigma(T)$, so $\sigma(T) \subseteq U \cup V$ and $U \cap V = \emptyset$, and let E be the Riesz idempotent corresponding to U for the operator T. Thus, if $\mathcal{K} = \operatorname{ran}(E)$, then $\sigma(T|_{\mathcal{K}}) = \sigma(T) \cap U$. If $\mathcal{M} = \ker(E)^{\perp}$, then $\mathcal{M}$ is a weak* closed invariant subspace for T^* and the predual of $T^*|_{\mathcal{M}}$ is similar to $T|_{\mathcal{K}}$. (Notice that if $S^* = T^*|_{\mathcal{M}}$, then S acts on $X/\mathcal{M}_{\perp} = X/\ker(E) \approx \mathcal{K}$.) So, $\sigma(T) \cap U = \sigma(T|_{\mathcal{K}}) = \sigma(T^*|_{\mathcal{M}})$. Now if a component C of $\sigma(T)$ is given then for any $\epsilon > 0$, using Lemma 6.1 we may choose a separation U, V of $\sigma(T)$ such that $C \subseteq U \subset B(C, \epsilon)$. This together with the above argument gives the desired result.

Letting $f(z) = |z|$, we may restate the conclusion of the Theorem as

$$\bigcap\{f(\sigma(T^*|_{\mathcal{M}})) : \mathcal{M} \in \text{wk*–Lat}(T^*) \setminus \{0\}\} \neq \emptyset.$$

Let $\mathcal{P}$ be the collection of all components of all sets of the form $\sigma(T^*|_{\mathcal{M}})$ where $\mathcal{M} \in$ wk*–Lat$(T^*) \setminus \{0\}$. Then $\mathcal{P}$ is a collection of connected sets, each arising as the component of a part of $\sigma(T)$. Notice that

$$\bigcap\{f(C) : C \in \mathcal{P}\} \subset \bigcap\{f(\sigma(T^*|_{\mathcal{M}})) : \mathcal{M} \in \text{wk*–Lat}(T^*) \setminus \{0\}\}$$

We shall show that $\bigcap\{f(C) : C \in \mathcal{P}\} \neq \emptyset$. Suppose by way of contradiction that $\bigcap\{f(C) : C \in \mathcal{P}\} = \emptyset$, since $\{f(C) : C \in \mathcal{P}\}$ is a collection of closed intervals, possibly degenerate ones, with empty intersection, there must be two members with empty intersection. That is, there must exist $C_1, C_2 \in \mathcal{P}$ such that $f(C_1) \cap f(C_2) = \emptyset$. Since $f(C_i)$ is a closed interval, we may choose an $r > 0$ strictly between the two intervals $f(C_1)$ and $f(C_2)$; by rescaling we may assume that $r = 1$. Thus we have $C_1 \subset \mathbb{D}$ and $C_2 \subset \mathbb{C} \setminus \overline{\mathbb{D}}$. By definition of $\mathcal{P}$, C_i is a component of a set of the form $\sigma(T^*|_{\mathcal{N}_i})$, $i = 1, 2$. Now using Lemma 6.1 and the Riesz functional calculus we may find $\mathcal{M}_i \subseteq \mathcal{N}_i$, $\mathcal{M}_i \in$ wk*–Lat$(T^*) \setminus \{0\}$, such that

$$\sigma(T^*|_{\mathcal{M}_1}) \subset \mathbb{D} \text{ and } \sigma(T^*|_{\mathcal{M}_2}) \subset \mathbb{C} \setminus \overline{\mathbb{D}}.$$

By Lemma 6.4 we have that $\|T^{*n}x\| \to 0$ for all $x \in \mathcal{M}_1$ and $\|T^{*n}x\| \to \infty$ for all nonzero $x \in \mathcal{M}_2$. Thus, $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$. If $\mathcal{M}_3$ is the wk* closure of $\mathcal{M}_1 + \mathcal{M}_2$, then $\mathcal{M}_3 \in$ wk*–Lat(T^*). Let T_i be the quotient operator acting on the space $X/\mathcal{M}_{i\perp}$, and define $A : \mathcal{M}_1 \oplus \mathcal{M}_2 \to \mathcal{M}_3$ by $A(x, y) = x+y$. Notice that A is weak* continuous and thus there exists an operator $B : X/\mathcal{M}_{3\perp} \to (X/\mathcal{M}_{1\perp}) \oplus (X/\mathcal{M}_{2\perp})$ such that $A = B^*$. Since A is one-to-one and $A[(T^*|_{\mathcal{M}_1}) \oplus (T^*|_{\mathcal{M}_2})] = (T^*|_{\mathcal{M}})A$, it follows that B has dense range and $BT_3 = (T_1 \oplus T_2)B$.

Since T_3 is supercyclic and the map B intertwines and has dense range, we get that $T_1 \oplus T_2$ is supercyclic. However, since $\sigma(T_1) \subseteq \{z : |z| < 1\}$ and $\sigma(T_2) \subseteq \{z : |z| > 1\}$, Lemmas 6.3 and 6.4 give that $T_1 \oplus T_2$ is not supercyclic. Thus we have a contradiction; hence $\bigcap\{f(C) : C \in \mathcal{P}\} \neq \emptyset$, and thus it follows that

$$\bigcap\{f(\sigma(T|_{\mathcal{M}})) : \mathcal{M} \in \text{wk*–Lat}(T^*) \setminus \{0\}\} \neq \emptyset$$

as required. It also follows that if $\rho \geq 0$ is such that $\sigma(T|_{\mathcal{M}}) \cap \{z : |z| = \rho\} \neq \emptyset$ for every $\mathcal{M} \in$ wk*–Lat$(T^*) \setminus \{0\}$, then every component of $\sigma(T)$ will intersect $\{z : |z| = \rho\}$ as well. ☐

Corollary 6.5. *If $T \in \mathcal{L}(X)$ is a supercyclic operator on a separable Banach space X such that T^* has Dunford's property (C), then there is a circle Γ_ρ such that for every nonzero $x^* \in X^*$, $\sigma_{T^*}(x^*) \cap \Gamma_\rho \neq \emptyset$.*

Proof. Since T^* has Dunford's property (C), (the proof of) Proposition 2.3 shows that if $x^* \in X^*$, and $F = \sigma_{T^*}(x^*)$, then $\sigma(T^*|X^*_{T^*}(F)) = F$. Since $X^*_{T^*}(F)$ is norm closed and T^* has (SVEP), it is also weak* closed, [13, Lemma 6.1.5] or [25, Proposition 2.5.6]. Thus, Theorem 6.2 applies. ☐

Remark. See Corollary 6.10, where a converse is shown to be true under a slightly stronger assumption.

The next corollary was obtained for subdecomposable operators in [28].

Corollary 6.6. *If T is a supercyclic decomposable operator, then $\sigma(T)$ is contained in a circle centered at the origin.*

Proof. Since T is decomposable, then T^* is decomposable, thus T^* has properties (C) and (δ). By Corollary 6.5 there is a ρ such that $\sigma_{T^*}(x^*) \cap \Gamma_\rho \neq \emptyset$ for every nonzero $x^* \in X^*$. If $\epsilon > 0$, let $U_\epsilon = \{z : |z| < \rho - \epsilon/2\}$, $V_\epsilon = \{z : \rho - \epsilon < |z| < \rho + \epsilon\}$ and $W_\epsilon = \{z : |z| > \rho + \epsilon/2\}$. By property (δ) and Proposition 2.3,

$$\begin{aligned} X^* &= X^*_{T^*}(\overline{U_\epsilon}) + X^*_{T^*}(\overline{V_\epsilon}) + X^*_{T^*}(\overline{W_\epsilon}) \\ &= X^*_{T^*}(\overline{V_\epsilon}). \end{aligned}$$

Thus, $X^* = \bigcap_{\epsilon>0} X^*_{T^*}(\overline{V_\epsilon})$; in particular, $\sigma(T^*) = \sigma_{su}(T^*) \subset \bigcap_{\epsilon>0} \overline{V_\epsilon} = \Gamma_\rho$. ☐

Next we give the supercyclic analog of Theorem 3.2.

Theorem 6.7. *Let T be an operator on a separable Banach space X and suppose that there is a number $\rho \geq 0$ satisfying either:*
(a) $X_T(\operatorname{ext}\Gamma_\rho)$ is dense and for every $\epsilon > 0$, $X_T(\operatorname{int}\Gamma_{\rho+\epsilon})$ is dense, or
(b) $X_T(\operatorname{int}\Gamma_\rho)$ is dense and for every $\epsilon > 0$, $X_T(\operatorname{ext}\Gamma_{\rho-\epsilon})$ is dense.
Then T is supercyclic.

If (a) holds, we say T is ρ-outer or outer with respect to Γ_ρ and if (b) holds, then we say T is ρ-inner or inner with respect to Γ_ρ. The proof of Theorem 6.7 follows from the inner/outer supercyclicity criteria.

Recall an operator has the *decomposition property* (δ) if it is the adjoint of a subdecomposable operator. Thus, every cohyponormal operator has property (δ). The following proposition follows easily from Theorem 2.2.

Proposition 6.8. *Let T be an operator on a separable Banach space with the decomposition property (δ) and let $\rho \geq 0$. Then the following are equivalent:*

(a) For every $\epsilon > 0$, $X_T(\{z \in \mathbb{C} : \rho < |z| < \rho + \epsilon\})$ is dense in X.

(b) $X_T(\{z : |z| > \rho\})$ is dense in X and for every $\epsilon > 0$, $X_T(\{z : |z| < \rho + \epsilon\})$ is dense in X.

If $\rho > 0$, then the following two conditions are equivalent:

(c) For every $\epsilon > 0$, $X_T(\{z \in \mathbb{C} : \rho - \epsilon < |z| < \rho\})$ is dense in X.

(d) $X_T(\{z : |z| < \rho\})$ is dense in X and for every $\epsilon > 0$, $X_T(\{z : |z| > \rho - \epsilon\})$ is dense in X.

The previous proposition implies that Theorem 6.7 is equivalent to Corollary 6.9 when T has the decomposition property (δ).

Corollary 6.9. *Let T be an operator on a separable Banach space X and suppose that there is a number $\rho \geq 0$ satisfying either:*

(a) For every $\epsilon > 0$, $X_T(\{z \in \mathbb{C} : \rho < |z| < \rho + \epsilon\})$ is dense in X, or

(b) For every $\epsilon > 0$, $X_T(\{z \in \mathbb{C} : \rho - \epsilon < |z| < \rho\})$ is dense in X.

Then T is supercyclic.

Proof of Theorem 6.7. Suppose (a) holds. Let $Y = X_T(\operatorname{ext}\Gamma_\rho)$. If we let $B_n : Y \to X$ be given by

$$B_n y = \frac{-1}{2\pi i}\int_\gamma \frac{1}{z^n} g(z)dz$$

where $y = (T - z)g(z)$ and g is analytic off a compact set $K \subseteq \operatorname{ext}\Gamma_\rho$ and where γ surrounds K in $\operatorname{ext}\Gamma_\rho$. Then $T^n B_n y = y$ for all $y \in Y$, as in the proof Theorem 3.2.

Now, given a $y \in Y$, say $y = (T - z)g(z)$ and g is analytic off a compact set $K \subseteq \{|z| > \rho\}$. Since K is compact there is an $\epsilon > 0$ such that $K \subseteq \{|z| > \rho + 2\epsilon\}$. For such an epsilon, let $X_y = X_T(\{z \in \mathbb{C} : |z| < \rho + \epsilon\})$.

Now if $y \in Y$ and $x \in X_y$, then $\|T^n x\| \leq C(\rho + \epsilon)^n$ for some constant $C > 0$ independent of n and similarly, $\|B_n y\| \leq C' \frac{1}{(\rho+2\epsilon)^n}$. (The estimates make use of the integral representations of $B_n y$ and $T^n x = \frac{1}{2\pi i}\int_{\gamma_1} z^n h(z)dz$, where $x = (T - z)h(z)$ and γ_1 is in $\{z : |z| < (\rho + \epsilon)\}$). Thus, $\|T^n x\|\|B_n y\| \leq C''[\frac{(\rho+\epsilon)}{(\rho+2\epsilon)}]^n \to 0$, as $n \to \infty$. So, T is supercyclic. If (b) holds, then the proof is similar. □

Corollary 6.10. *Let X be a complex Banach space and assume that $T \in \mathcal{L}(X)$ has the decomposition property (δ). If there exists a circle Γ_ρ, $\rho \geq 0$, satisfying either:*

(a) For every nonzero $x^* \in X^*$, $\sigma_{T^*}(x^*)$ intersects both Γ_ρ and $\operatorname{int}\Gamma_\rho$, or

(b) For every nonzero $x^* \in X^*$, $\sigma_{T^*}(x^*)$ intersects both Γ_ρ and $\operatorname{ext}\Gamma_\rho$.

Then T is supercyclic.

Proof. Simply apply Theorem 2.2 and Theorem 6.7. □

7. The Supercyclicity Circle for Cohyponormal Operators

For supercyclic cohyponormal operators we have a stronger version of Theorem 6.2. We recall that a pure hyponormal operator on a Hilbert space $\mathcal{H}$ is a hyponormal operator T such that there is no reducing subspace $\mathcal{M}$ such that $T|_{\mathcal{M}}$ is normal.

Lemma 7.1. *If T is a pure hyponormal operator and T has an invariant subspace $\mathcal{M}$ such that $T|_{\mathcal{M}}$ is an isometry, then $\sigma(T) \cap \mathbb{D} \neq \emptyset$.*

Proof. Suppose that $T \in \mathcal{L}(\mathcal{H})$ is a pure hyponormal operator and $\mathcal{M} \in \operatorname{Lat}(T)$ is such that $T|_{\mathcal{M}}$ is an isometry. Since T is pure, $T|_{\mathcal{M}}$ is a pure isometry and thus $\sigma(T|_{\mathcal{M}}) = \overline{\mathbb{D}}$. Choose any nonzero $x \in \mathcal{M}$. Then $K := \sigma_T(x) \subseteq \sigma(T|_{\mathcal{M}}) = \overline{\mathbb{D}}$. Consider the spectral subspace $X_T(K)$. Since T is hyponormal, $X_T(K)$ is a closed hyperinvariant subspace for T and $\sigma(T|_{X_T(K)}) = K$. Now either $K \subseteq \partial\mathbb{D}$ or $K \cap \mathbb{D} \neq \emptyset$. If $K \subseteq \partial\mathbb{D}$, then Putnam's Inequality implies that $T|_{X_T(K)}$ is normal; contradicting the purity of T. Thus $K \cap \mathbb{D} \neq \emptyset$. Since $X_T(K)$ is hyperinvariant, it is also rationally invariant, and thus $K = \sigma(T|_{X_T(K)}) \subseteq \sigma(T)$. Thus, $\sigma(T) \cap \mathbb{D} \neq \emptyset$. ☐

Remark. In Lemma 7.1 if T is actually subnormal, then it can be shown that $\mathbb{D} \subset \sigma(T)$, however this is not true for hyponormal operators. In fact there exists an invertible pure hyponormal operator (a bilateral weighted shift) that is an extension of an isometry.

Recall that a part of the operator S is any operator of the form $S|\mathcal{M}$ and a part of the spectrum of S is any set of the form $\sigma(S|_{\mathcal{M}})$ where $\mathcal{M}$ is an invariant subspace for S.

Proposition 7.2. *If S is a pure hyponormal operator and S^* is supercyclic, then there is a circle Γ_ρ, $\rho \geq 0$, such that either:*
(a) Every part of $\sigma(S)$ intersects both Γ_ρ and $\operatorname{int}\Gamma_\rho$, or
(b) Every part of $\sigma(S)$ intersects both Γ_ρ and $\operatorname{ext}\Gamma_\rho$.

Proof. Theorem 6.2 gives a circle Γ_ρ such that for every part T of S, $\sigma(T) \cap \Gamma_\rho \neq \emptyset$. If $\rho = 0$, then (b) holds. Thus we may assume that $\rho > 0$, and, by normalizing, that $\rho = 1$. Suppose that neither (a) nor (b) are true; that is, assume that there exist $\mathcal{M}_1$, $\mathcal{M}_2 \in \operatorname{Lat}(S) \setminus \{0\}$ such that, if $T_i = S|_{\mathcal{M}_i}$, $i = 1, 2$, then $\sigma(T_1) \subseteq \overline{\mathbb{D}}$ and $\sigma(T_2) \subseteq \mathbb{C} \setminus \mathbb{D}$.

Let $\mathcal{M} = \operatorname{cl}(\mathcal{M}_1 + \mathcal{M}_2)$ and set $T = S|_{\mathcal{M}}$. Notice that $\mathcal{M}_1 \cap \mathcal{M}_2$ is an invariant subspace for T_2 and $T_2|(\mathcal{M}_1 \cap \mathcal{M}_2)$ is an isometry; hence if $\mathcal{M}_1 \cap \mathcal{M}_2 \neq (0)$, then Lemma 7.1 implies that $\sigma(T_2) \cap \mathbb{D} \neq \emptyset$, contradicting our assumption that $\sigma(T_2) \subseteq \mathbb{C} \setminus \mathbb{D}$. Since $\mathcal{M}_1 \cap \mathcal{M}_2 = (0)$, the natural map from $\mathcal{M}_1 \oplus \mathcal{M}_2 \to \mathcal{M}$ that sends $(x, y) \mapsto x + y$ is a one-to-one map that intertwines $T_1 \oplus T_2$ with T. Since, T^* is supercyclic, it follows that $T_1^* \oplus T_2^*$ is supercyclic. Although, since $\sigma(T_1^*) \subseteq \{z : |z| \leq 1\}$, and T_1 is a pure hyponormal operator, it follows that $\|T_1^{*n} x\| \to 0$ for every x (see [32]). Also, since $\sigma(T_2^*) \subseteq \{z : |z| \geq 1\}$, we have that $\|T_2^{*-1}\| \leq 1$ and thus $\underline{\lim}\, \|T_2^{*n} x\| \geq \|x\| > 0$ for every nonzero $x \in \mathcal{M}_2$. But by Lemma 6.3 this implies that $T_1^* \oplus T_2^*$ is not supercyclic, a contradiction. Thus either (a) or (b) must hold. ☐

Recall that an operator T of is said to be ρ–outer or ρ–inner provided that T satisfies conditions (a) or (b) of Theorem 6.7, respectively. If S is a pure hyponormal operator then conditions (a) and (b) in Proposition 7.2 determine whether S^* is ρ–inner or ρ–outer. Notice that if conditions (a) or (b) above hold, then S is necessarily pure.

Example 7.3 (A non supercyclic operator)**.** If S_1 and S_2 are hyponormal operators such that $\sigma(S_1) \subseteq \{z : |z| \leq 1\}$ and $\sigma(S_2) \subseteq \{z : |z| \geq 1\}$, and $S = S_1 \oplus S_2$, then S^* is not supercyclic.

For cohyponormal operators, Corollary 6.10 becomes necessary and sufficient.

Theorem 7.4. *If $S \in \mathcal{L}(\mathcal{H})$ is a pure hyponormal operator, then S^* is supercyclic if and only if there exists a circle $\Gamma_\rho = \{z : |z| = \rho\}$, $\rho \geq 0$, such that either:*

(a) For every nonzero $x \in \mathcal{H}$, $\sigma_S(x)$ intersects both Γ_ρ and $\operatorname{int}\Gamma_\rho$, or

(b) For every nonzero $x \in \mathcal{H}$, $\sigma_S(x)$ intersects both Γ_ρ and $\operatorname{ext}\Gamma_\rho$.

As in the hypercyclic case, the above result may be equivalently stated in terms of parts of the spectrum of S.

Theorem 7.5. *If S is a pure hyponormal operator, then S^* is supercyclic if and only if there exists a circle $\Gamma_\rho = \{z : |z| = \rho\}$, $\rho \geq 0$, such that either:*

(a) For every hyperinvariant subspace $\mathcal{M}$ of S, $\sigma(S|_{\mathcal{M}})$ intersects Γ_ρ and $\operatorname{int}\Gamma_\rho$, or

(b) For every hyperinvariant subspace $\mathcal{M}$ of S, $\sigma(S|_{\mathcal{M}})$ intersects Γ_ρ and $\operatorname{ext}\Gamma_\rho$.

Proof. Since $\sigma_S(x) = \sigma(S|_{X_S(K)})$ where $K = \sigma_S(x)$, and since spectral subspaces are hyperinvariant, the result follows. ☐

One way to get a supercyclic operator is to take a multiple of a hypercyclic operator, but, as mentioned in the introduction, there are supercyclic operators that do not arise in this way. In order to distinguish between these, we shall say that an operator is **purely supercyclic** if it is supercyclic with no normal summand and no multiple of it is hypercyclic.

Corollary 7.6 (Characterizing the Purely Supercyclic Operators)**.** *If S is a hyponormal operator, then a multiple of S^* is hypercyclic if and only if S^* is both inner and outer with respect to a circle Γ_ρ, $\rho > 0$.*

Proof. If S^* is both inner and outer with respect to Γ_ρ, then $\frac{1}{\rho}S^*$ will be hypercyclic by Theorem 4.3. Conversely, if αS^* is hypercyclic, then S^* is both ρ-inner and ρ-outer where $\rho = \frac{1}{|\alpha|} > 0$. ☐

Corollary 7.7 (Uniqueness of the Supercyclicity Circle)**.** *If S is a hyponormal operator and S^* is purely supercyclic, then S^* has a unique supercyclicity circle.*

Proof. If there are two supercyclicity circles, Γ_{ρ_1} and Γ_{ρ_2} with $0 \leq \rho_1 < \rho_2$, then every part of the spectrum of S intersects both Γ_{ρ_1} and Γ_{ρ_2}. Now choose a ρ such that $\rho_1 < \rho < \rho_2$. An application of Lemma 6.1 and the fact that every part of $\sigma(S)$ must intersect both Γ_{ρ_1} and Γ_{ρ_2}, imply that every part of $\sigma(S)$ will intersect Γ_ρ, as well as the interior and exterior of Γ_ρ. Thus, S^* is both ρ-inner and ρ-outer, and the previous result implies that a multiple of S^* is hypercyclic, contrary to our assumption. ☐

Notice that if there exists a sequence of parts of the spectrum of S that have diameters that tend to zero, then the supercyclicity circle will be unique. However, it is not necessary for the diameters of the parts to go to zero for the supercyclicity circle to be unique, see the examples in Section 8

Corollary 7.8 (Direct Sums of Supercyclic Cohyponormal Operators)**.** *If $\{S_n\}$ is a bounded sequence of pure hyponormal operators such that for every n, S_n^* is supercyclic, then $\bigoplus_n S_n^*$ is supercyclic if and only if there is a common supercyclicity circle, Γ_ρ, $\rho \geq 0$, and S_n^* is ρ-inner for every n or S_n^* is ρ-outer for every n.*

Proof. Let $S = \bigoplus_n S_n$. If S^* is supercyclic, then a supercyclicity circle for S^* will be a supercyclicity circle for each S_n^*. Similarly, if S^* is ρ-inner (or ρ-outer), then S_n^* is ρ-inner (ρ-outer) for every n. Conversely, suppose Γ_ρ, is a supercyclicity circle for each S_n^* and each S_n^* is ρ-outer. We need to check that if $\mathcal{M}$ is a hyperinvariant subspace for S, then $\sigma(S|_{\mathcal{M}})$ intersects both Γ_ρ and $\operatorname{ext}\Gamma_\rho$. However, since $\mathcal{M}$ is hyperinvariant, it must be invariant under every coordinate projection. Thus $\mathcal{M} = \bigoplus_n \mathcal{M}_n$ where $\mathcal{M}_n$ is a hyperinvariant subspace of S_n. Thus, $\sigma(S|_{\mathcal{M}}) \supseteq \sigma(S_n|_{\mathcal{M}_n})$ for each n. So, if n is such that $\mathcal{M}_n \neq \{0\}$, then by assumption $\sigma(S_n|_{\mathcal{M}_n})$ intersects both Γ_ρ and $ext\Gamma_\rho$. Thus $\sigma(S|_{\mathcal{M}})$ also intersects both Γ_ρ and $ext\Gamma_\rho$. So, Theorem 7.5 implies that S^* is supercyclic. If each S_n^* is ρ-inner, then the proof is similar. ☐

One can easily form conditions for $f(T)$ to be supercyclic, given that T is supercyclic, as was done for hypercyclicity, here is a sample result for the case of supercyclicity radius 0. Notice in this case, that almost any analytic function preserves the supercyclicity.

Corollary 7.9. *Let T be a cohyponormal operator. If T is supercyclic and has supercyclicity radius 0, and $f(z)$ is any function analytic on $\sigma(T)$ that satisfies $f(0) = 0$, then $f(T)$ is also supercyclic.*

Suppose that S is a hyponormal operator and S^* is supercyclic, but S is not pure. That is, $S = T \oplus N$ on $\mathcal{H}_1 \oplus \mathcal{H}_2$ where T is a pure hyponormal operator and N is a normal operator and $\mathcal{H}_2 \neq (0)$. Since S^* is supercyclic it follows that N must be supercyclic, and thus we must have that $dim\mathcal{H}_2 = 1$ (see [6]). Thus S must have the form $T \oplus \lambda I_1$ where I_1 is the identity operator on a one-dimensional space. Also, λ must be nonzero, otherwise $S^* = T^* \oplus \overline{\lambda} I_1$ would not have dense range and thus could not be supercyclic.

Theorem 7.10 (Non pure Supercyclic Cohyponormal Operators)**.** *Suppose that $S = T \oplus \lambda I_1$ on $\mathcal{H}_1 \oplus \mathbb{C}$ where T is a pure hyponormal operator, $\lambda \in \mathbb{C}$ is nonzero, and I_1 is the identity operator on a one-dimensional space. Then S^* is supercyclic if and only if $\frac{1}{\lambda}T^*$ is hypercyclic.*

This is a special case of a more general result where T is not assumed to be hyponormal, see [18].

8. Examples

Let $S_n = M_z$ on $L_a^2(\Delta_n)$ where $\{\Delta_n : 1 \leq n \leq N\}$, $N \in \mathbb{Z}^+ \cup \{\infty\}$, is a bounded collection of open disks with radii $\{r_n\}$. If we let $S = \bigoplus_{n=1}^N S_n$, then S is a pure subnormal operator. We shall consider various examples below where S^* is or is not supercyclic depending on the arrangement of the disks. Recall that, as in Example 4.6, the spectrum of S_n has only one part, and the parts of the spectrum of S are of the form $\operatorname{cl}\big[\bigcup_k \Delta_{n_k}\big]$. Hence when verifying the conditions in Theorem 7.5, it suffices to check that each disk Δ_n intersects the circle in the appropriate manner.

Example 8.1. If $N < \infty$ and S^* is supercyclic, then S^* has more than one supercyclicity circle. Thus a multiple of S^* is hypercyclic (so S^* is not purely supercyclic).

Example 8.2 (An inner (outer) purely supercyclic operator)**.** If $N = \infty$ and each disk Δ_n is internally (externally) tangent to the unit circle and the radii $r_n \to 0$ as $n \to \infty$, then S^* is an inner (outer) purely supercyclic operator with supercyclicity radius one.

Example 8.3 (A purely supercyclic operator with supercyclicity radius zero)**.** If $0 \in \operatorname{cl}\Delta_n$ for all n and a subsequence of the radii $r_{n_k} \to 0$ as $k \to \infty$, then S^* is purely supercyclic (necessarily outer) with supercyclicity radius zero.

Example 8.4 (Another Purely Supercyclic Example)**.** Here we show how to arrange the disks so that S^* is a purely supercyclic operator, but each of the disks have radius equal to one. In the previous examples, the operators were purely supercyclic because the radii of the disks tended to zero. For this example, keeping the notation as above, simply let $N = \infty$, $r_n = 1$ for every $n \geq 1$, and if a_n is the center of the disk Δ_n, then let $a_1 = 2$ and $a_n = \frac{1}{n}$ for $n \geq 2$. Then S^* is an outer purely supercyclic operator with supercyclicity radius one.

Example 8.5 (General Examples with Supercyclicity Radius Zero)**.**

(a) If S is a pure subnormal operator on $\mathcal{H}$ such that for every $\epsilon > 0$ we have that $span\{ker(S-\lambda)^* : |\lambda| < \epsilon\} = \mathcal{H}$, then S^* is supercyclic.

(b) If $S = M_z$ on $\mathcal{H} \subseteq L^2(\mu)$ is pure, $0 \in supp(\mu)$ and for all $\epsilon > 0$ no non-zero function in $\mathcal{H}$ vanishes μ-a.e. on $\{z : |z| < \epsilon\}$, then S^* is supercyclic.

Example 8.6. If $\mathcal{H} \subseteq L^2(\mu)$ is a closed subspace such that no non-zero function in $\mathcal{H}$ vanishes on a set of positive μ measure and ϕ is non-constant multiplier of $\mathcal{H}$, then a multiple of M_ϕ^* is hypercyclic. In particular, M_ϕ^* is supercyclic.

Example 8.7. If $S = U \otimes A$ is a quasinormal operator (U is the unilateral shift, and A is a one-to-one positive operator), then S^* is supercyclic and any number ρ such that $0 \leq \rho \leq inf\{x : x \in \sigma(A)\}$ is a supercyclicity radius for S^*. Furthermore, S^* is hypercyclic if and only if the spectral measure for A is carried by $\{x \in \mathbb{R} : x > 1\}$. Thus a multiple of S^* is hypercyclic if and only if A is invertible. So, S^* is purely supercyclic if and only if $0 \in \sigma(A)$.

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References

[1] E. Albrecht, *On decomposable operators*, Integral Equations and Operator Theory, **2** (1979), 1–10.

[2] E. Albrecht and J. Eschmeier, *Analytic functional models and local spectral theory*, Proc. London Math. Soc. (3), **75** no 2 (1997), 323–348.

[3] S. Ansari and P. Bourdon, *Some properties of cyclic vectors*, Acta Sci. Math. (Szeged), **63** (1997), 195–207.

[4] T. Bermúdez and V. G. Miller, *On operators T such that $f(T)$ is hypercyclic*, Integral Equations and Opererator Theory, **37** (2000), no. 3, 332–340.

[5] E. Bishop, *A duality theory for an arbitrary operator*, Pacific J. Math., **9** (1959), 379–397.

[6] P.S. Bourdon, *Orbits of hyponormal operators*, Michigan Math. J. **44** (1997), no. 2, 345–353.

[7] P. S. Bourdon and J. H. Shapiro, *Cyclic Phenomena for Composition Operators*, Mem. Amer. Math. Soc., **125**, Providence, RI, 1997.

[8] I. Colojoară and C. Foiaş *Theory of Generalized Spectral Operators*, Gordon and Breach, New York, 1968.

[9] J.B. Conway, *The Dual of a Subnormal Operator*, J. Operator Theory **5** (1981), 195-211.

[10] J.B. Conway, *The Theory of Subnormal Operators*, Amer. Math. Soc., Providence, RI, 1991.

[11] G. Costakis, *On a conjecture of D. Herrero concerning hypercyclic operators*, C. R. Acad. Sci. Paris 330 (2000), 179–182.

[12] R. G. Douglas, *On extending semigroups of isometries*, Bull. London Math. Soc., **1** (1969), 157-159.

[13] J. Eschmeier and M. Putinar *Spectral Decomposition and Sheaf Theory*, London Math. Soc. Monographs New Series, Clarendon Press, Oxford, 1996.

[14] N.S. Feldman, *Pure Subnormal Operators have Cyclic Adjoints*, J. Funct. Analysis **162** (1999), no. 2, 379 - 399.

[15] K-G. Grosse-Erdmann, *Universal families and hypercyclic operators*, Bull. Amer. Math. Soc., **36** no 3 (1999), 345–381.

[16] R. M. Gethner and J. H. Shapiro, *Universal vectors for operators on spaces of holomorphic functions*, Proc. Amer. Math. Soc., **100** (1987), 281–288.

[17] G. Godefroy and J. H. Shapiro, *Operators with dense invariant cyclic manifolds*, J. Funct. Anal., **98** (1991), 229–269.

[18] M. Gonzlez, F. Len-Saavedra and A. Montes-Rodrguez *Semi-Fredholm theory: hypercyclic and supercyclic subspaces*, Proc. London Math. Soc. (3) **81** (2000), no. 1, 169–189.

[19] G. Herzog and C. Schmoeger, *On operators such that $f(T)$ is hypercyclic*, Studia Math., **108** (1994), 209–216.

[20] D. Herrero, *Limits of Hypercyclic and Supercyclic Operators*, J. Funct. Anal. **99** (1991), 179–190.

[21] D. Herrero, *Hypercyclic operators and chaos*, J. Operator Th., **28** (1992), 93–103.

[22] H. M. Hilden and L. J. Wallen, *Some cyclic and non–cyclic vectors of certain operators*, Ind. Univ. Math. J., **23** (1974), 557–565.

[23] C. Kitai, *Invariant closed sets for linear operators*, Dissertation, Univ. of Toronto, 1982.

[24] T. L. Kriete, *On the structure of certain $H^2(\mu)$ spaces*, Ind. Univ. Math. J., **28** (1979), 757–773.

[25] K. B. Laursen and M. M. Neumann, *Introduction to Local Spectral Theory*, London Math. Soc. Monographs New Series, Clarendon Press, Oxford, 2000.

[26] M. Martin & M. Putinar, *Lectures on Hyponormal Operators*, Birkhauser, 1989.

[27] V. G. Miller, *Remarks on finitely–hypercyclic and finitely–supercyclic operators*, Integral Equations Oper. Theory, **29** (1997), 110–115.

[28] T. L. Miller and V. G. Miller, *Local spectral theory and orbits of operators*, Proc. Amer. Math. Soc., **127** (1999), 1029–1037.

[29] M. M. Neumann, *On local spectral properties of operators on Banach spaces*, Rend. Circ. Mat. Palermo (2), Supp. **56** (1998), 15–25.

[30] M.H.A. Newman, *Elements of the Topology of Plane Sets of Points*, Dover 1992

[31] M. Putinar, *Hyponormal operators are subscalar*, J. Operator Th., **12** (1984), 385–395.

[32] C. R. Putnam, *Hyponormal contractions and strong power convergence*, Pacific J. Math. **57** (1975), no. 2, 531–538.

[33] S. Rolewicz, *On orbits of elements*, Studia Math., **32** (1969), 17–22.

[34] H. N. Salas, *Hypercyclic weighted shifts*, Trans Amer. Math. Soc., **347** (1995), 993–1004.

[35] H. N. Salas, *Supercyclicity and weighted shifts*, Studia Math. **135** (1999), no. 1, 55–74.

[36] F-H. Vasilescu *Analytic Functional Calculus and Spectral Decomposition*, Editura Academiei and D. Reidel Publishing Company, Bucharest and Dordrecht, 1982.

Dept. of Mathematics, Washington and Lee Univ. Lexington VA 24450
E-mail address: `feldmanN@wlu.edu`

Dept. of Mathematics, Mississippi State Univ, Drawer MA, Mississippi State, MS 39762
E-mail address: `vivien@math.msstate.edu`

Dept. of Mathematics, Mississippi State Univ, Drawer MA, Mississippi State, MS 39762
E-mail address: `miller@math.msstate.edu`