HYPERCYCLIC PAIRS OF COANALYTIC TOEPLITZ OPERATORS

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Abstract. A pair of commuting operators, (A, B), on a Hilbert space $\mathcal{H}$ is said to be hypercyclic if there exists a vector $x \in \mathcal{H}$ such that $\{A^n B^k x : n, k \geq 0\}$ is dense in $\mathcal{H}$. If $f, g \in H^\infty(G)$ where G is an open set with finitely many components in the complex plane, then we show that the pair (M_f^*, M_g^*) of adjoints of multiplcation operators on a Hilbert space of analytic functions on G is hypercyclic if and only if the semigroup they generate contains a hypercyclic operator. However, if G has infinitely many components, then we show that there exists $f, g \in H^\infty(G)$ such that the pair (M_f^*, M_g^*) is hypercyclic but the semigroup they generate does not contain a hypercyclic operator. We also consider hypercyclic n-tuples.

1. Introduction

Let $\mathcal{H}$ denote a separable complex Hilbert space and let A be a bounded linear operator on $\mathcal{H}$. We say that A is hypercyclic if there exists a vector $x \in \mathcal{H}$ such that the orbit of x under A, $Orb(A, x) := \{A^n x : n \geq 0\}$ is dense in $\mathcal{H}$. We say that A is supercyclic if if there exists a vector $x \in \mathcal{H}$ such that $\{\alpha A^n x : n \geq 0, \alpha \in \mathbb{C}\}$ is dense in $\mathcal{H}$.

There has been much work done on hypercyclic and supercyclic linear operators. The first example of a hypercyclic operator constructed on a Banach space was by Rolewicz [16] in 1969. He showed that if B is the backward shift on $\ell^p(\mathbb{N})$, then λB is hypercyclic if and only if $|\lambda| > 1$. Since that time, a "Hypercyclicity Criterion" has been developed independently by Kitai [15] and Gethner and Shapiro [12]. This criterion has been used to show that hypercyclic operators arise within the classes of composition operators [6], weighted shifts [17], adjoints of multiplication operators [13], and adjoints of subnormal and hyponormal operators [11].

If $\mathbb{A} := (A_1, A_2, \ldots, A_n)$ is an $n-$tuple of commuting operators on $\mathcal{H}$, then let $\mathcal{F} = \mathcal{F}_{\mathbb{A}} = \{A_1^{k_1} A_2^{k_2} \cdots A_n^{k_n} : k_i \geq 0\}$ be the semigroup generated by $\mathbb{A}$. Since $\mathbb{A}$ is a commuting tuple, then $\mathcal{F}$ is a finitely generated Abelian semigroup. If $x \in \mathcal{H}$, then the orbit of x under the tuple $\mathbb{A}$ or under $\mathcal{F}$ is $Orb(\mathbb{A}, x) := Orb(\mathcal{F}, x) := \{Ax : A \in \mathcal{F}\}$. We say that $\mathbb{A}$ (or $\mathcal{F}$) is hypercyclic on $\mathcal{H}$ if there exists an $x \in \mathcal{H}$ such that $Orb(\mathbb{A}, x)$ is dense in $\mathcal{H}$.

There is a growing literature on strongly continuous hypercyclic semigroups of linear operators, see for instance [4], [5], and [10]. However these are one-parameter families of operators and we are considering multi-parameter families of operators. Recently, Kérchy [14] has studied supercyclic properties of discrete abelian semigroups of operators.

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There are simple examples of hypercyclic semigroups, namely any semigroup that contains a hypercyclic operator. An easy example of this goes as follows: if B is the backward shift on $\ell^2(\mathbb{N})$ and I denotes the identity operator on $\ell^2(\mathbb{N})$, then the semigroup generated by the pair $(B, 2I)$ will be hypercyclic because it will contain the hypercyclic operator $2B$. In fact, if A is any supercyclic operator, then one can easily see that the semigroup generated by the tuple $(A, 2I, \frac{1}{3}I, e^{i\theta}I)$ is hyperyclic whenever $\theta \in \mathbb{R}$ is an irrational multiple of π; since in that case $\{\frac{2^i}{3^j}e^{ik\theta} : i, j, k \geq 0\}$ is dense in $\mathbb{C}$.

In this last example, if A is chosen to be a supercyclic operator such that no multiple of A is hypercyclic (see [17] or [11]), then the semigroup generated by $(A, 2I, \frac{1}{3}I, e^{i\theta}I)$ will be hypercyclic yet contain no hypercyclic operator. We also see from above that the study of (discrete Abelian) hypercyclic semigroups includes the study of supercyclic operators.

This paper mainly focuses on pairs or tuples of adjoints of multiplication operators on spaces of analytic functions (often called adjoint multiplication operators).

If G is an open set in the complex plane, $\mathbb{C}$, then let $Hol(G)$ denote the space of all analytic functions on G. Also let $H^\infty(G)$ denote the Banach space of all bounded analytic functions on G and we will use $\mathcal{H}(G)$ to denote a "Hilbert space of analytic functions on G" which will be carefully defined below, but will include such spaces as the Hardy space and Bergman space over G.

The following two results are samples of our main theorems. In what follows if $f \in H^\infty(G)$, then M_f will denote the operator of multiplication by f on $\mathcal{H}(G)$.

Theorem. *Let $f, g \in H^\infty(G)$ where G is an open set with finitely many components and let $\mathcal{H}(G)$ be a Hilbert space of analytic functions on G. If $\mathcal{F} = \{M^*_{f^n}M^*_{g^k} : n, k \geq 0\}$, then the following are equivalent:*

1. *The pair (M^*_f, M^*_g) is hypercyclic on $\mathcal{H}(G)$.*
2. *The semigroup $\mathcal{F}$ generated by (M^*_f, M^*_g) contains a hypercyclic operator.*
3. *There exists integers $n, k \geq 0$ such that $f^n g^k$ is non-constant on every component of G and $(f^n g^k)(G_i) \cap \partial\mathbb{D} \neq \emptyset$ for every $i \in \{1, \ldots, N\}$.*

If G is connected and say $|f(z)| > 1$ and $|g(z)| < 1$ for all $z \in G$, then one may also add the following equivalent condition:
(4) There does not exist a $p > 0$ such that $|f(z)|^p = 1/|g(z)|$ for all $z \in G$.

If G has infinitely many components, then the pair (M^*_f, M^*_g) is hypercyclic on $\mathcal{H}(G)$ if and only if (M^*_f, M^*_g) is hypercyclic on $\mathcal{H}(\Omega_N)$ for each $N \geq 1$, where $\Omega_N = \bigcup_{i=1}^N G_i$ and $\{G_i\}_{i=1}^\infty$ are the components of G.

This latter characterization of when the pair (M^*_f, M^*_g) is hypercyclic on $\mathcal{H}(G)$ where G has infinitely many components allows us to give an example of a pair of adjoint multiplication operators which is hypercyclic yet the semigroup they generate does not contain a hypercyclic operator.

Theorem. *If G is a bounded open set with infinitely many components and $\mathcal{H}(G)$ is a Hilbert space of analytic functions on G, then there exists $f, g \in H^\infty(G)$ such that the pair (M^*_f, M^*_g) is hypercyclic on $\mathcal{H}(G)$, but the semigroup $\mathcal{F}$ generated by (M^*_f, M^*_g) contains no hypercyclic operator.*

2. Preliminaries

If G is an open set and $f \in H^\infty(G)$, then let $\|f\|_\infty = \sup\{|f(z)| : z \in G\}$ and let $\|f\|_{inf} = \inf\{|f(z)| : z \in G\}$. By a region in $\mathbb{C}$ we will mean an open connected set, however we are also interested in working on open sets that are not connected. This will correspond to working with direct sums of multiplication operators.

Definition 2.1. *If G is a open set (not necessarily connected) and $\mathcal{H}(G) \subseteq Hol(G)$, then $\mathcal{H}(G)$ is said to be a Hilbert space of analytic functions on G if the following conditions are satisfied:*

1. *$\mathcal{H}(G)$ is a vector subspace of $Hol(G)$.*
2. *$\mathcal{H}(G)$ is complete with respect to an inner product on $\mathcal{H}(G)$.*
3. *For each point $a \in G$ the point evaluation functional $f \mapsto f(a)$ is continuous on $\mathcal{H}(G)$.*
4. *$H^\infty(G) \subseteq \mathcal{H}(G)$.*
5. *$\mathcal{H}(G)$ is invariant under multiplication by f for all $f \in H^\infty(G)$.*
6. *$\|M_f\|_{\mathcal{H}(G)} = \|f\|_\infty$ for all $f \in H^\infty(G)$.*

Remark. If G is an open set and G_1 is a component of G, and f is the characteristic function of G_1, then by property (6) multiplication by f has norm one, hence is a norm one idempotent, thus a (self-adjoint) projection. We will let $\mathcal{H}(G_1)$ denote the range of this projection. It follows that if $\{G_i\}$ are all the components of G, then $\mathcal{H}(G)$ is naturally isomorphic to $\oplus_i \mathcal{H}(G_i)$.

Lemma 2.2 (Expansive Inequality)**.** *If $\mathcal{H}(G)$ is a Hilbert space of analytic functions on an open set G as in Definition 2.1, $h \in H^\infty(G)$ and $|h| \geq 1$ on G, then $\|M_h^* f\| \geq \|f\|$ for all $f \in \mathcal{H}(G)$.*

Proof. To prove this fact, note that h is invertible and thus M_h^* is invertible and the inequality above simply says that the inverse is a contraction. Since $|h^{-1}| \leq 1$ on G, property (6) of Definition 2.1 implies that $\|((M_h)^{-1})^*\| = \|(M_h)^{-1}\| = \|h^{-1}\|_\infty \leq 1$. So the Expansive inequality is true. □

Examples of spaces that fit into the above definition include the Hardy space $H^2(G)$, the Bergman space $L^2_a(G)$, weighted Bergman spaces, pure $P^2(\mu)$ spaces, representing the closure of the polynomials in $L^2(\mu)$ and certain (but not all) pure $R^2(K, \mu)$ spaces representing the closure of the rational functions with poles off K in $L^2(\mu)$. The Dirichlet space does not satisfy condition (6) above.

The following result is a small variation of one due to Godefroy & Shapiro; see [13, Theorem 4.9].

Theorem 2.3 (Godefroy & Shapiro)**.** *If G is an open set in $\mathbb{C}$ with components $\{G_i\}$ and $\mathcal{H}(G)$ is a Hilbert space of analytic functions on G as in Definition 2.1, and $f \in H^\infty(G)$, then M_f^* is hypercyclic on $\mathcal{H}(G)$ if and only if $f|G_i$ is nonconstant for each i and $f(G_i) \cap \partial\mathbb{D} \neq \emptyset$ for all i.*

We will need the following function theoretic result.

Proposition 2.4. *If G is a region in $\mathbb{C}$, $f, g \in Hol(G)$, f has no zeros in G, and p is an irrational real number such that $|f(z)|^p = |g(z)|$ for all $z \in G$, then f has an analytic logarithm on G; that is, there is an $h \in Hol(G)$ such that $f = e^h$. In particular, then f^r is a well-defined analytic function on G for any $r \in \mathbb{C}$, $f^r = e^{rh}$.*

The author would like to thank Paul Bourdon for the following proof.

Proof. Recall that f has an analytic logarithm on G if and only if $\text{int}_\gamma \frac{f'}{f} dz = 0$ for all rectifiable simple closed curves γ contained in G. Since we make take an exhaustion of G by a sequence of regions $\{G_n\}_{n=1}^\infty$ each of which is bounded by a finite number of disjoint smooth Jordan curves and each simple closed curve γ in G will be contained in some G_n. Thus it suffices to assume (which we will now do) that G itself is bounded by a finite number of disjoint smooth Jordan curves. Say $\mathbb{C} \setminus clG$ has n bounded components and choose a point a_k from each of the bounded components such that $Im(a_k) \neq Im(a_j)$ if $k \neq j$ (where $Im(z)$ denotes the imaginary part of the complex number z). By the Logarithmic Conjugation Theorem (see [2] or [3, p. 203]), there are real constants $\{b_k\}_{k=1}^n$ and an analytic function h on G such that

$$\ln|f(z)| = Re(h(z)) + \sum_{k=1}^n b_k \ln|z - a_k| \text{ for all } z \in G. \tag{1}$$

It follows that

$$|f(z)| = e^{Re(h(z))} \prod_{k=1}^n e^{b_k \ln|z-a_k|} \text{ for all } z \in G. \tag{2}$$

Let $\Gamma_k = \{z \in \mathbb{C} : Im(z) = Im(a_k) \text{ and } Re(z) \leq Re(a_k)\}$ and $\Gamma = \bigcup_{k=1}^n \Gamma_k$. If we let $\log_\pi(z)$ denote the principal branch of the logarithm on $\mathbb{C} \setminus (-\infty, 0]$, then $\log_\pi(z - a_k)$ is analytic on $G \setminus \Gamma_k$ and so $\log_\pi(z - a_k)$ is analytic on $G \setminus \Gamma$ for all k. It follows from equation (2) that there exists a unimodular constant c such that

$$f(z) = ce^{h(z)} \prod_{k=1}^n e^{b_k \log_\pi(z-a_k)} \text{ for all } z \in G \setminus \Gamma. \tag{3}$$

Also taking p^{th} powers of equation (2) we have

$$|g(z)| = |f(z)|^p = e^{pRe(h(z))} \prod_{k=1}^n e^{pb_k \ln|z-a_k|} \text{ for all } z \in G. \tag{4}$$

and thus there is a unimodular constant d such that

$$g(z) = de^{ph(z)} \prod_{k=1}^n e^{pb_k \log_\pi(z-a_k)} \text{ for all } z \in G \setminus \Gamma. \tag{5}$$

Now consider equation (3). The left hand side is continuous on G, whereas the right hand side is continuous on G if and only if b_k is an integer for each k. Likewise, considering equation (5), since the left hand side is continuous on G, then the right hand side must also be, which happens if and only if pb_k is an integer for all k. However, since p is irrational, the only way that b_k and pb_k can be an integer is is $b_k = 0$ and this is true for all k. Thus equation (2) becomes

$$|f(z)| = e^{Re(h(z))} \text{ for all } z \in G. \tag{6}$$

Thus there is a unimodular constant α such that $f = \alpha e^h$. It follows that f has an analytic logarithm on G. □

3. Pairs of multiplication operators on connected open sets

In this section we consider the case where G is a *region*. We also begin by considering pairs of operators. We will see that we can easily derive an analogous result for $n-$tuples from the result for pairs.

Theorem 3.1. *Let $\mathcal{H}(G)$ be a Hilbert space of analytic functions as in Definition 2.1 on a region G. Also, let $f, g \in H^\infty(G)$. If $\mathcal{F} = \{M_{f^n}^* M_{g^k}^* : n, k \geq 0\}$, then the following are equivalent:*

(1) *The pair (M_f^*, M_g^*) is hypercyclic on $\mathcal{H}(G)$.*
(2) *The semigroup $\mathcal{F}$ generated by (M_f^*, M_g^*) contains a hypercyclic operator.*
(3) *There exists integers $n, k \geq 0$ such that $f^n g^k$ is non-constant on G and $(f^n g^k)(G) \cap \partial\mathbb{D} \neq \emptyset$.*
(4) *One of the following holds:*
   (a) *f is non-constant and $f(G) \cap \partial\mathbb{D} \neq \emptyset$.*
   (b) *g is non-constant and $g(G) \cap \partial\mathbb{D} \neq \emptyset$.*
   (c) *$|f(z)| > 1$ and $|g(z)| < 1$ for all $z \in G$ and there does not exist a $p > 0$ such that $|f(z)|^p = \frac{1}{|g(z)|}$ for all $z \in G$.*
   (d) *$|f(z)| < 1$ and $|g(z)| > 1$ for all $z \in G$ and there does not exist a $p > 0$ such that $|f(z)|^p = \frac{1}{|g(z)|}$ for all $z \in G$.*

Proof of Theorem 3.1. It follows from Theorem 2.3 of Godefroy and Shapiro that (2) holds if and only if (3) holds. Also clearly (2) $\Rightarrow$ (1). We will prove that (3) $\Leftrightarrow$ (4) holds and that (1) $\Rightarrow$ (4). This will prove the theorem.

It is easy to see that if $|f(z)| < 1$ and $|g(z)| < 1$ for all $z \in G$ or if $|f(z)| > 1$ and $|g(z)| > 1$ for all $z \in G$, then none of the conditions (1), (2), (3) or (4) hold. Also if either f or g maps G onto an open set that intersects the unit circle, then by Theorem 2.3 all four conditions are satisfied. Finally if, say, g is a unimodular constant, then (3) is satisfied if and only if f is nonconstant and $f(G) \cap \partial\mathbb{D} \neq \emptyset$ which happens if and only if (4) is satisfied. Also in this case where g is unimodular, conditions (1) and (2) hold if and only if f is nonconstant and $f(G) \cap \partial\mathbb{D} \neq \emptyset$. And finally, if f or g is identically zero, then all four conditions are equivalent, by Theorem 2.3.

Let's assume that

$$(*) \qquad |f(z)| > 1 \text{ for all } z \in G \text{ and } |g(z)| < 1 \text{ for all } z \in G.$$

Assuming (*) we now show that (3) $\Leftrightarrow$ $(4c)$.

There exists integers $n, k \geq 0$ such that $f^n g^k$ is non-constant and $(f^n g^k)(G) \cap \partial\mathbb{D} \neq \emptyset$ if and only if there exist $a, b \in G$ and integers $n, k \geq 0$ such that

$$(**) \qquad |f(a)|^n |g(a)|^k < 1 \text{ and } |f(b)|^n |g(b)|^k > 1.$$

Since g is not identically zero, we may assume that $g(a) \neq 0$, otherwise replace a by a' where $a' \in G$, $g(a') \neq 0$ and a' is sufficiently close to a to preserve the above inequality. Taking logarithms of (**) gives

$$n \ln |f(a)| + k \ln |g(a)| < 0 \text{ and } n \ln |f(b)| + k \ln |g(b)| > 0$$

which (using (*)) may be rewritten as

$$\frac{\ln |f(a)|}{-\ln |g(a)|} < \frac{k}{n} \text{ and } \frac{k}{n} < \frac{\ln |f(b)|}{-\ln |g(b)|}.$$

or equivalently

$$\frac{\ln|f(a)|}{-\ln|g(a)|} < \frac{k}{n} < \frac{\ln|f(b)|}{-\ln|g(b)|}.$$

Thus there exists an $a, b \in G$ and integers $n, k \geq 0$ such that (**) holds if and only if the positive (extended real-valued) function

$$w(z) := \frac{\ln|f(z)|}{-\ln|g(z)|}$$

which is defined on G is non-constant. Further, one easily checks that w is constant if and only if there exists a $p > 0$ such that $|f|^p = \frac{1}{|g|}$ on G. Thus we have established (3) $\Leftrightarrow$ (4c) assuming that (*) holds. The same argument will also show (3) $\Leftrightarrow$ (4d) assuming $|f(z)| < 1$ for all $z \in G$ and $|g(z)| > 1$ for all $z \in G$.

It remains to show that (1) $\Rightarrow$ (4c) (we are still assuming that (*) holds). So, assume that $\mathcal{F}$ is hypercyclic on $\mathcal{H}(G)$ and that (4c) does not hold. Then there exists a $p > 0$ such that $|f(z)|^p = 1/|g(z)|$ for all $z \in G$.

There are now two cases, either p is rational or irrational.

Case I: p is rational.

Suppose that $p = a/b$ where $a, b \in \mathbb{N}$ and $gcd(a, b) = 1$. Then since $|f(z)|^p = 1/|g(z)|$ for all $z \in G$, we have $|f(z)|^a |g(z)|^b = 1$ for all $z \in G$. Which implies that $|f(z)^a g(z)^b| = 1$ for all $z \in G$. Thus, there is a unimodular constant c such that $f^a g^b = c$ on G or

$$g^b = c/f^a.$$

So, if $n, k \geq 0$ and $n = aq_1 + r_1$ and $k = bq_2 + r_2$ where $q_i \geq 0$ and $0 \leq r_1 < a$ and $0 \leq r_2 < b$, then

$$f^n g^k = f^{r_1} g^{r_2} f^{aq_1} g^{bq_2} = f^{r_1} g^{r_2} f^{aq_1} (c/f^a)^{q_2} = c^{q_2} f^{r_1} g^{r_2} f^{a(q_1 - q_2)} \text{ on } G.$$

Let $\phi \in \mathcal{H}(G)$ be a vector with dense orbit under $\mathcal{F}$. Also let

$$\mathcal{F}' := \{M_h^* : h = \alpha f^n, \alpha \in \mathbb{C}, |\alpha| = 1, n \in \mathbb{Z}\}$$

and notice that $Orb(\mathcal{F}, \phi) \subseteq \bigcup\{M^*_{f^r g^s}(Orb(\mathcal{F}', \phi)) : 0 \leq r < a, 0 \leq s < b\}$.

Since $|f(z)|^p |g(z)| = 1$ for all $z \in G$ and since f is bounded, then g is bounded away from zero (and f is also, since $|f| > 1$ on G). So, f and g are invertible in $H^\infty(G)$, thus $M^*_{f^r g^s}$ is an invertible linear operator. So, if we can show that $Orb(\mathcal{F}', \phi)$ is nowhere dense in $\mathcal{H}(G)$, then $Orb(\mathcal{F}, \phi)$ will be contained in a finite union of nowhere dense sets, hence it will also be nowhere dense (meaning its closure has empty interior), contradicting the definition of ϕ.

Claim: $Orb(\mathcal{F}', \phi)$ is nowhere dense.

Suppose that $int[\text{cl}Orb(\mathcal{F}', \phi)] \neq \emptyset$. Then either (i) there exists

$$z, w \in int[\text{cl}Orb(\mathcal{F}', \phi)] \setminus Orb(\mathcal{F}', \phi) \text{ such that } \|z\| > \|w\| > \|\phi\|$$

or (ii) there exists

$$z, w \in int[\text{cl}Orb(\mathcal{F}', \phi)] \setminus Orb(\mathcal{F}', \phi) \text{ such that } \|z\| < \|w\| < \|\phi\|.$$

We will consider case (i); case (ii) is similar. Let $\epsilon > 0$ be chosen such that $\epsilon < (1/3)(\|z\| - \|w\|)$. Since $z \in clOrb(\mathcal{F}', \phi)$, then there exists an $n \in \mathbb{N}$ and an $\alpha \in \mathbb{C}$, $|\alpha| = 1$ such that $\|\alpha M^*_{f^n}\phi - z\| < \epsilon$. Hence, $\|M^*_{f^n}\phi\| = \|\alpha M^*_{f^n}\phi\| \geq \|z\| - \epsilon$. Now since $w \in int[\text{cl}Orb(\mathcal{F}', \phi)] \setminus Orb(\mathcal{F}', \phi)$, then there exists a $k > n$ and a $\beta \in \mathbb{C}$, $|\beta| = 1$ such that $\|\beta M^*_{f^k}\phi - w\| < \epsilon$. It follows as above that $\|M^*_{f^k}\phi\| < \|w\| + \epsilon$. Thus we have that $\|M^*_{f^n}\phi\| > \|M^*_{f^k}\phi\|$. However, since $k > n$ this contradicts the fact that $\{\|M^*_{f^n}\phi\|\}_{n=0}^{\infty}$ is an increasing sequence. To see that $\{\|M^*_{f^n}\phi\|\}_{n=0}^{\infty}$ is an increasing sequence notice that $|f| > 1$ on G that M^*_f is an expansive operator by Lemma 2.2. This contradiction implies that $int[\mathrm{cl}Orb(\mathcal{F}', \phi)] = \emptyset$ and so the claim follows. Thus we have that (1) $\Rightarrow$ (4c) when p is rational.

Case II: p is irrational.

Let $\phi \in \mathcal{H}(G)$ be a vector with dense orbit under $\mathcal{F}$. Since f is never zero (recall $|f| > 1$ on G) and since $|f(z)|^p = 1/|g(z)|$ for all $z \in G$ (which implies that g is never zero, hence $1/g$ is analytic on G) and since p is irrational, then Proposition 2.4 implies that f has an analytic logarithm on G. It follows that for every $t \in \mathbb{R}$, f^t is a well defined bounded analytic function on G. Thus $|f(z)|^p = 1/|g(z)|$ may be written as $|f(z)^p g(z)| = 1$ for all $z \in G$. Hence there is a unimodular constant c such that $f(z)^p g(z) = c$ for all $z \in G$. Thus we have that $f^n g^k = c^k f^{n-kp}$. Hence $Orb(\mathcal{F}, \phi) \subseteq Orb(\mathcal{F}', \phi)$ where $\mathcal{F}' = \{M^*_h : h = \alpha f^t, \alpha \in \mathbb{C}, |\alpha| = 1, t \in \mathbb{R}\}$.

Notice that the map $\varphi : [0, \infty) \to \mathcal{H}(G)$ defined by $\varphi(t) := M^*_{f^t}\phi$ is continuous and $\varphi(0) = \phi$.

Claim: The function $t \mapsto \|\varphi(t)\|$ is continuous and increasing on $[0, \infty)$.

We will leave the continuity to the reader, for the increasing part we use the Expansive inequality in Lemma 2.2.

To see that $t \mapsto \|\varphi(t)\|$ is increasing, suppose that $0 < s < t$. Since $|f| > 1$ on G, then $|f|^{(t-s)} > 1$ on G. So, $\|M^*_{f^t}\phi\| = \|M^*_{f^{(t-s)}} M^*_{f^s}\phi\| \geq \|M^*_{f^s}\phi\|$. That establishes the claim.

Since $\mathcal{F}$ is hypercyclic, then $\lim_{t\to\infty} \|\varphi(t)\| = \infty$. Thus, $K := \{t \in [0, \infty) : \|\phi\| \leq \|\varphi(t)\| \leq 2\|\phi\|\}$ is a compact interval in $[0, \infty)$. Hence it follows that

$$Orb(\mathcal{F}', \phi) \cap \{h \in \mathcal{H}(G) : \|\phi\| \leq \|h\| \leq 2\|\phi\|\} = C := \{\alpha h : |\alpha| = 1, h \in \varphi(K)\}.$$

But this latter set, C is compact since it is the continuous image of the compact set $\partial\mathbb{D} \times K$ under the map $(\alpha, t) \mapsto \alpha\phi(t)$. Since compact sets in infinite dimensions have empty interior, then C cannot be dense in $\{h \in \mathcal{H}(G) : \|\phi\| \leq \|h\| \leq 2\|\phi\|\}$, thus $Orb(\mathcal{F}', \phi)$ cannot be dense there either. But this contradicts the fact that $Orb(\mathcal{F}, \phi)$ is dense. Thus it follows that (1) $\Rightarrow$ (4c) when p is irrational. Thus we have proven that (1) $\Rightarrow$ (4c) assuming that ($*$) holds. A similar argument will show (1) $\Rightarrow$ (4d) assuming $|f(z)| < 1$ for all $z \in G$ and $|g(z)| > 1$ for all $z \in G$. The theorem now follows. □

Example 3.2. *Let G be a region, $\mathcal{H}(G)$ a Hilbert space of analytic functions on G as in Definition 2.1, and $f, g \in H^\infty(G)$. Also let $\mathcal{F}$ be the semigroup generated by M^*_f and M^*_g .*

1. *If f has a zero at some point in G, then the pair (M^*_f, M^*_g) is hypercyclic if and only if $\|f\|_\infty > 1$ or $\|g\|_\infty > 1$.*
2. *If $G = \mathbb{D}$ and $f(z) = z$, then the pair (M^*_f, M^*_g) is hypercyclic if and only if $\|g\|_\infty > 1$.*
3. *If $G = \mathbb{D}$, $f(z) = e^{(z+1)}$ and $g(z) = e^{-2(z+1)}$, then the pair (M^*_f, M^*_g) is not hypercyclic.*

4. Tuples of multiplication operators on connected open sets

The case of hypercyclic $n-$tuples of adjoint multiplication operators will now follow easily from our result about hypercyclic pairs (Theorem 3.1).

Theorem 4.1. *Let $\mathcal{H}(G)$ be a Hilbert space of analytic functions as in Definition 2.1 on a region G. Let $\{f_1,,\dots,f_n\} \subseteq H^\infty(G)$. Also let $\mathcal{F}$ be the semigroup generated by the adjoints of the multiplication operators on $\mathcal{H}(G)$ with symbols $f_1,\dots,f_n$. Then the following are equivalent:*

(1) *The tuple $(M_{f_1}^*, M_{f_2}^*, \dots, M_{f_n}^*)$ is hypercyclic on $\mathcal{H}(G)$.*
(2) *The semigroup $\mathcal{F}$ generated by $(M_{f_1}^*, M_{f_2}^*, \dots, M_{f_n}^*)$ contains a hypercyclic operator.*
(3) *There exists integers $k_1, k_2, \dots, k_n \geq 0$ such that $(f_1^{k_1} f_2^{k_2} \cdots f_n^{k_n})$ is non-constant and $(f_1^{k_1} f_2^{k_2} \cdots f_n^{k_n})(G) \cap \partial\mathbb{D} \neq \emptyset$.*
(4) *There is a pair of indices i, j and integers $k_1, k_2 \geq 0$ such that $f_i^{k_1} f_j^{k_2}$ is non-constant and $(f_i^{k_1} f_j^{k_2})(G) \cap \partial\mathbb{D} \neq \emptyset$.*
(5) *There is a pair of indices i, j such that one of the following holds:*
   (a) *f_i is non-constant and $f_i(G) \cap \partial\mathbb{D} \neq \emptyset$.*
   (b) *f_j is non-constant and $f_j(G) \cap \partial\mathbb{D} \neq \emptyset$.*
   (c) *$|f_i(z)| > 1$ and $|f_j(z)| < 1$ for all $z \in G$ and there does not exist a $p > 0$ such that $|f_i(z)|^p = \frac{1}{|f_j(z)|}$ for all $z \in G$.*
   (d) *$|f_i(z)| < 1$ and $|f_j(z)| > 1$ for all $z \in G$ and there does not exist a $p > 0$ such that $|f_i(z)|^p = \frac{1}{|f_j(z)|}$ for all $z \in G$.*

Proof. It follows from Theorem 3.1 that (4) $\Leftrightarrow$ (5) holds. Also, by Theorem 2.3, (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). Hence it suffices to prove the implication (1) $\Rightarrow$ (5). So assume that (1) holds and by way of contradiction assume that (5) does not hold. Since (4) and (5) are equivalent and we are assuming that (5) is not true, then there does not exist an index i such that f_i is non-constant and $f_i(G) \cap \partial\mathbb{D} \neq \emptyset$. Thus for every pair of indices i, j we have either

(*) $\min\{\|f_i\|_{inf}, \|f_j\|_{inf}\} \geq 1$ or $\max\{\|f_i\|_\infty, \|f_j\|_\infty\} \leq 1$ or the pair (f_i, f_j) satisfies that there exits a $p > 0$ s.t. $|f_i|^p = 1/|f_j|$ and $0 < |f_j(z)| \neq 1, \forall z \in G$.

Let $A = \{i : \|f_i\|_\infty \leq 1$ and f_i is not a unimodular constant$\}$, $B = \{i : \|f_i\|_{inf} \geq 1$ and f_i is not a unimodular constant$\}$, and $C = \{i : f_i$ is a unimodular constant$\}$.

Then by the comments immediately preceding (*) we have that $A \cup B \cup C = \{1, 2, \dots, n\}$. Since we are assuming that (1) holds, then $A \neq \emptyset$ and $B \neq \emptyset$. Also since (1) holds f_i must be nonconstant for some $i \in A \cup B$, with out loss of generality assume that there is an $i \in B$ so that f_i is nonconstant (a similar argument would apply if $i \in A$). Now choose and fix an $i_1 \in A$ and a $j_1 \in B$ and choose j_1 such that f_{j_1} is nonconstant.

By (*) there is a $p > 0$ such that

$$|f_{i_1}| = \frac{1}{|f_{j_1}|^p} \text{ on } G. \tag{**}$$

Now if $k \in B$ and $k \neq j_1$, then by applying $(*)$ to f_{i_1} and f_k we see that there exists a $q > 0$ such that

$$|f_{i_1}| = \frac{1}{|f_k|^q} \text{ on } G.$$

It follows that

$$\frac{1}{|f_{j_1}|^p} = |f_{i_1}| = \frac{1}{|f_k|^q} \text{ on } G.$$

Hence $|f_k| = |f_{j_1}|^r$ where $r = p/q > 0$. Similarly, if $k \in A$, then by applying $(*)$ to f_k and f_{j_1}, we get that there exists an $p > 0$ such that $|f_k| = |f_{j_1}|^{-p}$.

It follows that for every $i \in A \cup B$, there exists a unique $p_i \in \mathbb{R} \setminus \{0\}$ such that

$$|f_i| = |f_{j_1}|^{p_i} \text{ on } G. \tag{***}$$

There are two cases to consider now, either every p_i is rational or there exists an i such that p_i is irrational.

Case I: For every $i \in A \cup B$, p_i is rational.

For simplicity let $f := f_{j_1}$. Say $p_i = a_i/b_i$ where $a_i \in \mathbb{Z} \setminus \{0\}$ and $b_i \in \mathbb{N}$. Then from (***) we have that $|f_i| = |f|^{a_i/b_i}$ on G. Hence there exists unimodular constants c_i such that

$$f_i^{b_i} = c_i f^{a_i} \text{ on } G. \tag{†}$$

Let $k_1, k_2, \ldots, k_n \geq 0$ be integers. For each $i \in A \cup B$, upon dividing k_i by b_i we see that there are integers q_i, r_i satisfying $q_i \geq 0$ and $0 \leq r_i < b_i$ and $k_i = b_i q_i + r_i$. Thus, using (†) we have

$$(f_1^{k_1} f_2^{k_2} \cdots f_n^{k_n}) = \left(\prod_{i \in C} f_i^{k_i}\right) \cdot \left(\prod_{i \in A \cup B} f_i^{b_i q_i + r_i}\right) = \alpha \cdot \left(\prod_{i \in A \cup B} f_i^{r_i}\right) \cdot \prod_{i \in A \cup B} f_i^{b_i q_i} =$$

$$= \alpha \cdot \left(\prod_{i \in A \cup B} f_i^{r_i}\right) \cdot \prod_{i \in A \cup B} (c_i f^{a_i})^{q_i} = \beta \left(\prod_{i \in A \cup B} f_i^{r_i}\right) \cdot f^m$$

from some integer $m (= \sum_{i \in A \cup B} a_i q_i)$ and for some unimodular constants α, β. Since f is invertible in $H^\infty(G)$ we see from (†) that f_i is also invertible for each $i \in A \cup B$. Let

$$\mathcal{C} = \{ \prod_{i \in A \cup B} f_i^{r_i} : 0 \leq r_i < b_i \}.$$

Then $\mathcal{C}$ consists of a finite number of invertible functions in $H^\infty(G)$.

So, let

$$\mathcal{F}' = \{ M_h^* : h = c f^m, c \in \mathbb{C}, |c| = 1, m \in \mathbb{Z} \}.$$

Since we are assuming that (1) holds, let $\phi \in \mathcal{H}(G)$ be a function such that $Orb(\mathcal{F}, \phi)$ is dense in $\mathcal{H}(G)$. It follows from the above equations that $Orb(\mathcal{F}, \phi)$ is contained in $\bigcup \{ M_g^*(Orb(\mathcal{F}', \phi)) : g \in \mathcal{C} \}$. Since $\mathcal{C}$ is finite it suffices to show that $Orb(\mathcal{F}', \phi)$ is nowhere dense in $\mathcal{H}(G)$; however this argument is identical to the one in Theorem 3.1. Thus $Orb(\mathcal{F}', \phi)$ is nowhere dense, and so it follows that $Orb(\mathcal{F}, \phi)$ is also nowhere dense, contradicting (1). Thus, (1) $\Rightarrow$ (5) when all the p_i's are rational.

Case II: There exists an $i \in A \cup B$ such that p_i is irrational.

Again for simplicity let $f := f_{j_1}$. If $i_0 \in A \cup B$ and p_{i_0} is irrational, then $|f|^{p_{i_0}} = |f_{i_0}|$ on G and $|f| > 1$ on G (because $f = f_{j_1}$ and $j_1 \in B$) hence non-vanishing on G, thus by Proposition 2.4, f has an analytic logarithm on G. Then by (***) for each $i \in A \cup B$, there exist unimodular constants c_i such that

$$f_i = c_i f^{p_i} \text{ on } G.$$

Thus for all integers $k_1, k_2, \ldots, k_n \geq 0$, there exists a $t \in \mathbb{R}$ and an $\alpha \in \partial\mathbb{D}$ such that $(f_1^{k_1} f_2^{k_2} \cdots f_n^{k_n}) = \alpha f^t$. (Since f has a logarithm, f^t is well defined.)

Hence if ϕ is a hypercyclic vector for $\mathcal{F}$ and $\mathcal{F}' = \{\alpha M_{f^t}^* : t \in \mathbb{R}, |\alpha| = 1\}$, then $Orb(\mathcal{F}, \phi) \subseteq Orb(\mathcal{F}', \phi)$. However, as in the proof of Theorem 3.1, we can show that $Orb(\mathcal{F}', \phi)$ is not dense in $\mathcal{H}(G)$, contradicting the fact that $\mathcal{F}$ is hypercyclic. It now follows that (1) implies (5) when some p_i is irrational. Hence the theorem follows. $\square$

5. Pairs of multiplication operators on disconnected open sets

The following is a basic lemma showing that when $|g| < 1$ (a similar result holds, when $|g| > 1$) which integers n, k have the property that $f^n g^k$ is non-constant and has its range intersecting the unit circle. Proposition 5.2 is a more careful look at the same question when both f and g may have ranges that hit the circle.

Lemma 5.1. *If G is an open set in $\mathbb{C}$ and $f, g \in Hol(G) \setminus \{0\}$ and $|g(z)| < 1$ for all $z \in G$, then for a pair of non-negative integers (n, k) there exists an $a, b \in G$ such that $|f(a)^n g(a)^k| < 1$ and $|f(b)^n g(b)^k| > 1$ if and only if $m < \frac{k}{n} < M$, where*

$$m = \inf_{z \in G} \frac{\ln |f(z)|}{-\ln |g(z)|} \text{ and } M = \sup_{z \in G} \frac{\ln |f(z)|}{-\ln |g(z)|}.$$

Proof. The proof of this lemma is basically contained in the proof of Theorem 3.1. □

Notice that in the above lemma, n cannot equal zero. However, for the following proposition, if $k > 0$ and $n = 0$, then interpret $\frac{k}{n}$ as infinity. However, notice that this will only occur in part (3). In parts (1) and (2) n cannot be zero.

Proposition 5.2. *Let Ω be a region in $\mathbb{C}$ and $f, g \in H^\infty(\Omega) \setminus \{0\}$, g not a unimodular constant. Define two subsets of Ω and four constants as follows:*

- $\Omega^{(1)} = \{z \in \Omega : |g(z)| > 1\}$.
- $\Omega^{(2)} = \{z \in \Omega : |g(z)| < 1\}$.
- $m^{(1)} = \inf_{z \in \Omega^{(1)}} \frac{\ln |f(z)|}{-\ln |g(z)|}$ *and* $M^{(1)} = \sup_{z \in \Omega^{(1)}} \frac{\ln |f(z)|}{-\ln |g(z)|}$
- $m^{(2)} = \inf_{z \in \Omega^{(2)}} \frac{\ln |f(z)|}{-\ln |g(z)|}$ *and* $M^{(2)} = \sup_{z \in \Omega^{(2)}} \frac{\ln |f(z)|}{-\ln |g(z)|}$

Then $\Omega^{(1)} \cup \Omega^{(2)}$ is a dense open subset of Ω and the following hold:

(1) *Suppose $\Omega^{(1)} \neq \emptyset$. Then a pair of nonnegative integers (n, k), not both zero, satisfy that there exists an $a, b \in \Omega^{(1)}$ such that $|f(a)^n g(a)^k| < 1$ and $|f(b)^n g(b)^k| > 1$ if and only if $m^{(1)} < M^{(1)}$ and $\frac{k}{n} \in (m^{(1)}, M^{(1)})$. Furthermore, if $\frac{k}{n} \notin (m^{(1)}, M^{(1)})$, then either $|f(z)^n g(z)^k| \geq 1$ for all $z \in \Omega^{(1)}$ or $|f(z)^n g(z)^k| \leq 1$ for all $z \in \Omega^{(1)}$.*

(2) *Suppose $\Omega^{(2)} \neq \emptyset$. Then a pair of nonnegative integers (n, k), not both zero, satisfy that there exists an $a, b \in \Omega^{(2)}$ such that $|f(a)^n g(a)^k| < 1$ and $|f(b)^n g(b)^k| > 1$ if and only if $m^{(2)} < M^{(2)}$ and $\frac{k}{n} \in (m^{(2)}, M^{(2)})$. Furthermore, if $\frac{k}{n} \notin (m^{(2)}, M^{(2)})$, then either $|f(z)^n g(z)^k| \geq 1$ for all $z \in \Omega^{(2)}$ or $|f(z)^n g(z)^k| \leq 1$ for all $z \in \Omega^{(2)}$.*

(3) *Suppose $\Omega^{(1)} \neq \emptyset$, $\Omega^{(2)} \neq \emptyset$, and $(m^{(1)}, M^{(1)}) \cap (m^{(2)}, M^{(2)}) = \emptyset$. Then a pair of nonnegative integers (n, k), not both zero, satisfy that there exists an $a, b \in \Omega$ such that $|f(a)^n g(a)^k| < 1$ and $|f(b)^n g(b)^k| > 1$ if and only if $\frac{k}{n} \in (-\infty, \alpha) \cup (\beta, \infty]$, where $\alpha = \min\{M^{(1)}, M^{(2)}\}$ and $\beta = \max\{m^{(1)}, m^{(2)}\}$. Furthermore, if $\frac{k}{n} \in [\alpha, \beta]$, then either $|f(z)^n g(z)^k| \geq 1$ for all $z \in \Omega$ or $|f(z)^n g(z)^k| \leq 1$ for all $z \in \Omega$.*

(4) *Suppose $\Omega^{(1)} \neq \emptyset$, $\Omega^{(2)} \neq \emptyset$, and $(m^{(1)}, M^{(1)}) \cap (m^{(2)}, M^{(2)}) \neq \emptyset$. Then for every pair of nonnegative integers (n, k), not both zero, there exists an $a, b \in \Omega$ such that $|f(a)^n g(a)^k| < 1$ and $|f(b)^n g(b)^k| > 1$.*

Proof. For a point $z \in G$, $|f(z)^n g(z)^k| < 1$ if and only if $n \ln|f(z)| + k \ln|f(z)| < 0$. Thus,

$$(*)\ |f(z)^n g(z)^k| < 1 \text{ if and only if } \ln|f(z)| < -\frac{k}{n}\ln|g(z)|.$$

It then follows that if $z \in \Omega^{(1)}$, then $-\ln|g(z)| < 0$, so $(*)$ holds if and only if $\frac{\ln|f(z)|}{-\ln|g(z)|} > \frac{k}{n}$. Similarly, if $z \in \Omega^{(2)}$, then $-\ln|g(z)| > 0$, so $(*)$ holds if and only if $\frac{\ln|f(z)|}{-\ln|g(z)|} < \frac{k}{n}$. Similar statements hold describing when $|f(z)^n g(z)^k| > 1$.

If we define $m^{(1)}, m^{(2)}, M^{(1)}, M^{(2)}$ as above, then the following statements hold:

(a) $\frac{k}{n} > m^{(1)}$ if and only if $|f(z)^n g(z)^k| > 1$ for some $z \in \Omega^{(1)}$.
(b) $\frac{k}{n} > m^{(2)}$ if and only if $|f(z)^n g(z)^k| < 1$ for some $z \in \Omega^{(2)}$.
(c) $\frac{k}{n} < M^{(1)}$ if and only if $|f(z)^n g(z)^k| < 1$ for some $z \in \Omega^{(1)}$.
(d) $\frac{k}{n} < M^{(2)}$ if and only if $|f(z)^n g(z)^k| > 1$ for some $z \in \Omega^{(2)}$.

The above results follow from these facts by considering various cases. □

Example 5.3. *Let $f(z) = e^{az+b}$ and $g(z) = e^z$, with $a, b \in \mathbb{R}$. If $\Omega = \mathbb{D}$, then*

$$m^{(1)} = \begin{cases} -\infty & b > 0 \\ -a & b = 0 \\ -(a+b) & b < 0 \end{cases}, \qquad M^{(1)} = \begin{cases} -(a+b) & b > 0 \\ -a & b = 0 \\ \infty & b < 0 \end{cases},$$

$$m^{(2)} = \begin{cases} (b-a) & b > 0 \\ -a & b = 0 \\ -\infty & b < 0 \end{cases}, \qquad M^{(2)} = \begin{cases} \infty & b > 0 \\ -a & b = 0 \\ (b-a) & b < 0 \end{cases}.$$

Thus, if $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, then there exists $a, b \in \mathbb{R}$ (with $b > 0$) such that $(m^{(1)}, M^{(1)}) = (-\infty, \alpha)$ and $(m^{(2)}, M^{(2)}) = (\beta, \infty)$. If $\alpha > \beta$, then there exists $a, b \in \mathbb{R}$ (with $b < 0$) such that $(m^{(1)}, M^{(1)}) = (\alpha, \infty)$ and $(m^{(2)}, M^{(2)}) = (-\infty, \beta)$.

Proof. For $z \in \mathbb{D}$, let $z = x + iy$. Notice that since $a, b \in \mathbb{R}$, then $Re(az+b) = ax+b$. Thus $\ln|f(z)| = (ax+b)$ and $-\ln|g(z)| = -x$. Thus,

$$\frac{\ln|f(z)|}{-\ln|g(z)|} = \frac{ax+b}{-x} = -a - \frac{b}{x}.$$

Also $\Omega^{(1)} = \{z \in \mathbb{D} : Re(z) > 0\}$ and $\Omega^{(2)} = \{z \in \mathbb{D} : Re(z) < 0\}$. So the sup/inf of the above quantities simply amount to finding the sup/inf of the real function $-a - b/x$ over the intervals $(0,1)$ and $(-1,0)$. □

Corollary 5.4. *Keeping the same notation as in Proposition 5.2, let*

$$\mathcal{P}(f,g) = \{(n,k) \in \mathbb{N} \times \mathbb{N} : f^n g^k \text{ is nonconstant on } \Omega \text{ and } (f^n g^k)(\Omega) \cap \partial\mathbb{D} \neq \emptyset\}.$$

If $\mathcal{P}(f,g) \neq \emptyset$, then one of the following holds:

(a) *There is an open interval $J = (a,b) \subseteq \mathbb{R}$ such that $(n,k) \in \mathcal{P}(f,g)$ if and only if $\frac{k}{n} \in J$, or*
(b) *There is a compact interval $K = [c,d] \subseteq \mathbb{R}$ such that $(n,k) \in \mathcal{P}(f,g)$ if and only if $\frac{k}{n} \notin K$.*

Furthermore, $\{a,b,c,d\} \subseteq \{m^{(1)}, M^{(1)}, m^{(2)}, M^{(2)}, \pm\infty\}$, $c, d \in \mathbb{R}$.

Proof. If $\mathcal{P}(f,g) = \mathbb{N} \times \mathbb{N}$, then (a) holds with $J(f,g) = (0,\infty)$. If $\Omega = \Omega^{(1)}$ or $\Omega = \Omega^{(2)}$, then by Proposition 5.2, (a) holds. The only cases when (a) is not satisfied is when condition (3) of Proposition 5.2 holds. In that case (b) holds. □

If condition (a) in Corollary 5.4 holds, then we will say that $\mathcal{P}(f,g)$ is a "**sector**" and if condition (b) in Corollary 5.4 holds, then we will say that $\mathcal{P}(f,g)$ is a "**sector complement**". Notice that if $J=(a,b)$ in Corollary 5.4, then $(n,k)\in\mathcal{P}(f,g)$ if and only if the point (n,k) lies in the sector or region strictly between the two lines $y=ax$ and $y=bx$. Similarly, if $\mathcal{P}(f,g)$ is a sector complement, then there is a sector S (the region between two lines through the origin) such that $\mathcal{P}(f,g)=(\mathbb{N}\times\mathbb{N})\setminus S$.

Theorem 5.5 (Two Components)**.** *Let G be an open set in $\mathbb{C}$ with two components, G_1 and G_2, and let $\mathcal{H}(G)$ be a Hilbert space of analytic functions as in Definition 2.1 on G. If $f,g\in H^\infty(G)$ and $\mathcal{F}=\{M^*_{f^n}M^*_{g^k}:n,k\geq 0\}$, then the following are equivalent:*

1. *The pair (M_f^*,M_g^*) is hypercyclic on $\mathcal{H}(G)$.*
2. *The semigroup $\mathcal{F}$ generated by (M_f^*,M_g^*) contains a hypercyclic operator.*
3. *There exists integers $n,k\geq 0$ such that f^ng^k is non-constant on each component of G and $(f^ng^k)(G_i)\cap\partial\mathbb{D}\neq\emptyset$ for $i\in\{1,2\}$.*

Proof. By Theorem 2.3 one easily sees that (3) implies (2) implies (1). We must show that (1) implies (3). So assume that (1) holds and by way of contradiction assume that (3) does not hold.

For $i\in\{1,2\}$, let $f_i=f|G_i$ and let $g_i=g|G_i$. Since for each $i\in\{1,2\}$, $\mathcal{F}_i:=\mathcal{F}|\mathcal{H}(G_i)$ is hypercyclic on $\mathcal{H}(G_i)$, then by Theorem 3.1,

$$(*)\forall i\in\{1,2\},\exists n,k\geq 0 \text{ such that } f_i^ng_i^k \text{ is non-constant and } (f_i^ng_i^k)(G_i)\cap\partial\mathbb{D}\neq\emptyset.$$

But the n and k may depend on i. Furthermore by Theorem 3.1, if n and k are any non-negative integers, not both zero, and $f_i^ng_i^k$ is constant on G_i for some i, then it cannot be a unimodular constant.

It follows from $(*)$ that $\mathcal{P}(f_1,g_1)\neq\emptyset$ and $\mathcal{P}(f_2,g_2)\neq\emptyset$. Since we are assuming that (3) does not hold, then

$$(**)\quad \mathcal{P}(f_1,g_1)\cap\mathcal{P}(f_2,g_2)=\emptyset.$$

Hence by Corollary 5.4 it follows that there are two cases, either $\mathcal{P}(f_1,g_1)$ and $\mathcal{P}(f_2,g_2)$ are disjoint sectors or one is a sector which is disjoint from the other one which is a sector complement.

Case 1: $\mathcal{P}(f_1,g_1)$ and $\mathcal{P}(f_2,g_2)$ are disjoint sectors.

We will use the notation from Proposition 5.2 and Corollary 5.4. For each $i\in\{1,2\}$, let $J_i=(a_i,b_i)$ with $a_i,b_i\in\{m_i^{(1)},M_i^{(1)},m_i^{(2)},M_i^{(2)},\pm\infty\}$ be as in Corollary 5.4 (also see below), so that $(n,k)\in\mathcal{P}(f_i,g_i)$ if and only if $\frac{k}{n}\in J_i$. Since $\mathcal{P}(f_1,g_1)$ and $\mathcal{P}(f_2,g_2)$ are disjoint, it follows that $J_1\cap J_2=\emptyset$.

For each $i\in\{1,2\}$, let $w_i(z)=\dfrac{\ln|f_i(z)|}{-\ln|g_i(z)|}$. Then by $(*)$ above and Proposition 5.2, w_i is nonconstant. Let $m_i^{(j)}=\inf\{w_i(z):z\in G_i^{(j)}\}$ and $M_i^{(j)}=\sup\{w_i(z):z\in G_i^{(j)}\}$. Since $J_1\cap J_2=\emptyset$ either $b_1\leq a_2$ or $b_2\leq a_1$. We will suppose that $b_1\leq a_2$, the other case is similar. Thus, $b_1\in\mathbb{R}$ and so by Proposition 5.2 we have that either $G_1=G_1^{(1)}$ ($|g_1|>1$ on G_1) or $G_1=G_1^{(2)}$ ($|g_1|<1$ on G_1). We'll suppose that $G_1=G_1^{(2)}$.

Thus $a_1=m_1^{(2)}$, $b_1=M_1^{(2)}$ and $a_2=m_2^{(j)}$ for some $j\in\{1,2\}$. Let $m\in[b_1,a_2]=[M_1^{(2)},m_2^{(j)}]$. If $\frac{k}{n}\geq m(\geq M_1^{(2)})$, then item (d) in the proof of Proposition 5.2 implies that $|f_1^ng_1^k|\leq 1$ on $G_1=G_1^{(2)}$. However, if $\frac{k}{n}<m(\leq m_2^{(j)})$, then item (a)

or (b) in the proof of Proposition 5.2 implies that $|f_2^n g_2^k| \leq 1$ on $G_2^{(1)}$ or $|f_2^n g_2^k| \geq 1$ on $G_2^{(2)}$. Since $\frac{k}{n} < m \leq a_2$, then $\frac{k}{n} \notin J_2$, so $|f_2(z)^n g_2(z)^k| \neq 1$ for any $z \in G_2$, thus either $|f_2^n g_2^k| \leq 1$ on G_2 or $|f_2^n g_2^k| \geq 1$ on G_2. Since both cases are similar, let's assume that $j = 2$ and so $|f_2^n g_2^k| \geq 1$ on G_2.

Now let $\phi \in \mathcal{H}(G)$ be a hypercyclic vector for $\mathcal{F}$ and let $\phi_i := \phi|G_i$. Now suppose that h is in the closure of the orbit of ϕ under $\mathcal{F}$. Then there exists integers n_j, k_j such that $M^*_{f^{n_j} g^{k_j}} \phi \to h$. Now either there exist infinitely many j's such that $\frac{k_j}{n_j} \geq m$ or infinitely many j's such that $\frac{k_j}{n_j} < m$. In the first case $|f_1^{n_j} g_1^{k_j}| \leq 1$ on G_1 and hence $\|M^*_{f^{n_j} g^{k_j}} | \mathcal{H}(G_1)\| \leq 1$. In the second case, $|f_2^{n_j} g_2^{k_j}| \geq 1$ on G_2 and hence $M^*_{f_2^{n_j} g_2^{k_j}} | \mathcal{H}(G_2)$ is an expansive operator (meaning $\|M^*_{f_2^{n_j} g_2^{k_j}} x\| \geq \|x\|$ for all $x \in \mathcal{H}(G_2)$). Thus it follows that either $\|h|G_1\| \leq \|\phi_1\|$ or $\|h|G_2\| \geq \|\phi_2\|$. This restriction on h contradicts the fact that $\mathcal{F}$ is hypercyclic. Thus in this case it follows that (1) implies (3).

Case 2: $\mathcal{P}(f_1, g_1)$ is a sector that is disjoint from $\mathcal{P}(f_2, g_2)$ which is a sector complement.

Since $\mathcal{P}(f_1, g_1)$ is a sector let $J_1 = (a, b)$ be as in Corollary 5.4 and let $J_2 = \mathbb{R} \setminus K = (-\infty, c) \cup (d, \infty)$ where $K = [c, d]$ is the compact interval guaranteed by Corollary 5.4. Since $\mathcal{P}(f_1, g_1) \cap \mathcal{P}(f_2, g_2) = \emptyset$, then $J_1 \cap J_2 = \emptyset$, thus $c \leq a < b \leq d$.

By Corollary 5.4, either $(a, b) = (m_1^{(1)}, M_1^{(1)})$ or $(a, b) = (m_1^{(2)}, M_1^{(2)})$. With out loss of generality we'll suppose that $(a, b) = (m_1^{(1)}, M_1^{(1)})$. Similarly without loss of generality, we'll suppose that $[c, d] = [M_2^{(2)}, m_2^{(1)}]$.

Now if $\frac{k}{n} \in [c, d] = [M_2^{(2)}, m_2^{(1)}]$, then $\frac{k}{n} \geq c = M_2^{(2)}$, so by part (d) of Proposition 5.2 it follows that $|f_2^n g_2^k| \leq 1$ for all $z \in G_2^{(2)}$. Now since $\frac{k}{n} \leq d$, then $\frac{k}{n} \notin J_2$, so the range of $f_2^n g_2^k$ cannot hit the unit circle on G_2, thus we must have $|f_2^n g_2^k| < 1$ for all $z \in G_2$. Which implies that $M^*_{f_2^n g_2^k} | \mathcal{H}(G_2)$ is a contraction, whenever $\frac{k}{n} \in [c, d]$.

Now if $\frac{k}{n} < c$, then it follows that $M^*_{f_1^n g_1^k} | \mathcal{H}(G_1)$ is a contraction and if $\frac{k}{n} > d$, then it follows that $M^*_{f_1^n g_1^k} | \mathcal{H}(G_1)$ is expansive.

Now if $\phi \in \mathcal{H}(G)$ is a hypercyclic vector for $\mathcal{F}$ and h is in the closure of the orbit of ϕ under $\mathcal{F}$. Then there exists integers n_j, k_j such that $M^*_{f^{n_j} g^{k_j}} \phi \to h$. Now either there exist infinitely many j's such that $\frac{k_j}{n_j} \in [c, d]$ or there exist infinitely many j's such that $\frac{k_j}{n_j} < c$ or there exist infinitely many j's such that $\frac{k_j}{n_j} > d$. In either of these cases, because of the contractive or expansive properties we've established, there will be restrictions on the norm of $h|G_1$ and/or the norm of $h|G_2$. Thus h cannot be arbitrary, so $\mathcal{F}$ is not hypercyclic. This contradiction implies that (1) implies (3) in case 2. It now follows that (1) always implies (3), hence the theorem follows. ☐

A similar argument as above shows that the same result holds for finitely many components. The details are left to the reader.

Theorem 5.6 (Finitely many Components)**.** *Let G be an open set in $\mathbb{C}$ with finitely many components. Suppose that $\{G_i\}_{i=1}^N$ are the components of G. Also let $\mathcal{H}(G)$ be a Hilbert space of analytic functions as in Definition 2.1 on G. If $f, g \in H^\infty(G)$ and $\mathcal{F} = \{M^*_{f^n} M^*_{g^k} : n, k \geq 0\}$, then the following are equivalent:*

(1) *The pair (M^*_f, M^*_g) is hypercyclic on $\mathcal{H}(G)$.*

(2) *The semigroup $\mathcal{F}$ generated by (M_f^*, M_g^*) contains a hypercyclic operator.*
(3) *There exists integers $n, k \geq 0$ such that $f^n g^k$ is non-constant on every component of G and $(f^n g^k)(G_i) \cap \partial\mathbb{D} \neq \emptyset$ for every $i \in \{1, \ldots, N\}$.*

We now give an example showing how one might find an f and g on a disconnected open set so that M_f^* and M_g^* are not hypercyclic, but the semigroup, $\mathcal{F}$, that they generate is hypercyclic.

Lemma 5.7. *If $0 < a < b < 1 < c < d$, then $\frac{\ln(c)}{-\ln(a)} < \frac{\ln(d)}{-\ln(b)}$. Furthermore, a pair (n, k) of positive integers satisfies $c^n a^k < 1$ and $d^n b^k > 1$ if and only if $\frac{\ln(c)}{-\ln(a)} < \frac{k}{n} < \frac{\ln(d)}{-\ln(b)}$.*

Proof. Since $\ln(x)$ is increasing and $-\ln(x)$ is decreasing it follows that

$$\frac{-\ln(b)}{-\ln(a)} < 1 < \frac{\ln(d)}{\ln(c)}.$$

From this the first inequality holds. For the second, simply take the logarithm of both inequalities and solve for $\frac{k}{n}$. $\square$

Example 5.8. *Keep the same notation as in Theorem 5.6 and assume that in addition $|f(z)| > 1$ for all $z \in G$ and $|g(z)| < 1$ for all $z \in G$. Also suppose that there exists an r, s such that $0 < r < 1 < s$ and for each $i \in \{1, \ldots, N\}$, there exists $z_i, w_i \in G_i$ such that the following hold:*

(1) *for each $i \in \{1, \ldots, N\}$, $|f(z_i)| > s$,*
(2) *for each $i \in \{1, \ldots, N\}$, $|f(w_i)| < s$,*
(3) *for each $i \in \{1, \ldots, N\}$, $|g(z_i)| > r$,*
(4) *for each $i \in \{1, \ldots, N\}$, $|g(w_i)| < r$,*

then the pair (M_f^*, M_g^*) is hypercyclic on $\mathcal{H}(G)$.

Proof. Choose $a, b, c, d \in \mathbb{R}$ such that

$$\max_i |g(w_i)| < a < r < b < \min_i |g(z_i)| < 1 < \max_i |f(w_i)| < c < s < d < \min_i |f(z_i)|.$$

Then by Lemma 5.7, there exists positive integers n, k such that $c^n a^k < 1$ and $d^n b^k > 1$. Thus for each i, $|f^n(z_i) g^k(z_i)| > d^n b^k > 1$ and $|f^n(w_i) g^k(w_i)| < c^n a^k < 1$. It follows that (3) holds from Theorem 5.6, hence the pair (M_f^*, M_g^*) is hypercyclic on $\mathcal{H}(G)$. $\square$

Next we give an example of $f, g \in H^\infty(G)$ such that the pair (M_f^*, M_g^*) is not hypercyclic on $\mathcal{H}(G)$, however, (M_f^*, M_g^*) is hypercyclic on $\mathcal{H}(G_i)$ for each i.

If $r, s \geq 0$, then let $A(r, s) = \{z \in \mathbb{C} : r < |z| < s\}$.

Example 5.9. *(a) Let G_1, G_2 be two disjoint open sets in $\mathbb{C}$. If*

$$(*) \qquad 0 < a_1 < b_1 < a_2 < b_2 < 1 < c_1 < d_1 < c_2 < d_2$$

and $f_i, g_i \in H^\infty(G_i)$ for $i \in \{1, 2\}$, and $f_i(G_i) \subseteq A(c_i, d_i)$ and $g_i(G_i) \subseteq A(a_i, b_i)$, then there is no pair of integers, $n, k \geq 0$ such that for all $i \in \{1, 2\}$, $f_i^n g_i^k$ is nonconstant on G_i and $(f_i^n g_i^k)(G_i) \cap \partial\mathbb{D} \neq \emptyset$. That is, the pair (M_f^*, M_g^*) is not hypercyclic on $\mathcal{H}(G)$, where $f|G_i = f_i$ and $g|G_i = g_i$ for $i \in \{1, 2\}$.

(b) Keeping the notation from part (a), clearly f_i and g_i may be chosen to satisfy the conditions of part (a) and yet also satisfy condition (4) of Theorem 3.1 (e.g. if f_i and g_i are all linear polynomials), then the pair (M_f^*, M_g^*) is hypercyclic on $\mathcal{H}(G_i)$ for each i, but the pair (M_f^*, M_g^*) is not hypercyclic on $\mathcal{H}(G)$.

Proof. (a) By way of contradiction, suppose that there are integer $n, k \geq 1$ such that for $i \in \{1, 2\}$, $f_i^n g_i^k$ is nonconstant on G_i and $(f_i^n g_i^k)(G_i) \cap \partial\mathbb{D} \neq \emptyset$, then by Lemma 5.7 it follows that $\frac{\ln(c_1)}{-\ln(a_1)} < \frac{k}{n} < \frac{\ln(d_1)}{-\ln(b_1)}$ and $\frac{\ln(c_2)}{-\ln(a_2)} < \frac{k}{n} < \frac{\ln(d_2)}{-\ln(b_2)}$. However, by (*) and Lemma 5.7 we have that $\frac{\ln(c_2)}{-\ln(a_2)} > \frac{\ln(d_1)}{-\ln(b_1)}$, hence we have a contradiction, and no such n, k exist. □

Remark. The previous example shows that if (A_1, A_2) and (B_1, B_2) are each hypercyclic pairs, then the pair $(A_1 \oplus B_1, A_2 \oplus B_2)$ need not be hypercyclic.

6. Some General Observations

If $\mathcal{F}$ is any collection of operators, then let $Orb(\mathcal{F}, x) = \{Tx : T \in \mathcal{F}\}$. Also let $\mathcal{HC}(\mathcal{F}) = \{x \in X : Orb(\mathcal{F}, x) \text{ is dense in } X\}$ be the set of hypercyclic vectors for the collection $\mathcal{F}$.

Theorem 6.1. *Suppose that $\mathcal{F}$ is a collection of commuting operators on a separable Banach space X. Then the following are equivalent.*

1. *$\mathcal{HC}(\mathcal{F})$ is dense in X.*
2. *$\mathcal{HC}(\mathcal{F})$ is a dense G_δ in X.*
3. *For any two nonempty open sets U, V in X, there exists a $T \in \mathcal{F}$ such that $T(U) \cap V \neq \emptyset$.*

If every operator in $\mathcal{F}$ has dense range, then the above conditions are also equivalent to:

4. *$\mathcal{HC}(\mathcal{F})$ is nonempty.*

Proof. If $\{U_n\}_n$ is a countable basis for the space X, then one easily sees that $\mathcal{HC}(\mathcal{F}) = \bigcap_n \bigcup_{T \in \mathcal{F}} T^{-1}(U_n)$. If condition (3) is satisfied, then $\bigcup_{T \in \mathcal{F}} T^{-1}(U_n)$ is a dense open set, hence by the Baire Category Theorem, $\mathcal{HC}(\mathcal{F})$ is a dense G_δ in X, so condition (2) holds. Clearly (2) implies (1). To see that (1) implies (3), assume that $\mathcal{HC}(\mathcal{F})$ is dense in X and let U, V be two nonempty open sets. Then there exists a $x \in \mathcal{HC}(\mathcal{F}) \cap U$ and a $T \in \mathcal{F}$ such that $Tx \in V$. Thus, $T(U) \cap V \neq \emptyset$. So (1) impies (3). Thus we have (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

Clearly, (1) $\Rightarrow$ (4). Now assume that (4) holds and that every operator in $\mathcal{F}$ has dense range and we will prove that (1) holds. Let $x \in X$ be such that $Orb(\mathcal{F}, x)$ is dense in X. If $T \in \mathcal{F}$, then since $\mathcal{F}$ is commutative, $Orb(\mathcal{F}, Tx) = T(Orb(\mathcal{F}, x))$ and since T has dense range and $Orb(\mathcal{F}, x)$ is dense we get that $Orb(\mathcal{F}, Tx)$ is dense in X. Thus $Tx \in \mathcal{HC}(\mathcal{F})$. Since this holds for each $T \in \mathcal{F}$, then we have $Orb(\mathcal{F}, x) \subseteq \mathcal{HC}(\mathcal{F})$. Thus $\mathcal{HC}(\mathcal{F})$ is dense and (1) holds. □

Theorem 6.2. *Let $A = \bigoplus_{i=1}^\infty A_i$ and $B = \bigoplus_{i=1}^\infty B_i$ be commuting operators on $\mathcal{H} = \bigoplus_{i=1}^\infty \mathcal{H}_i$. Suppose that for each $n \geq 1$, that the semigroup $\mathcal{F}_n$ generated by $A_1 \oplus \cdots \oplus A_n$ and $B_1 \oplus \cdots \oplus B_n$ on $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ is hypercyclic and has a dense set of hypercyclic vectors, then the semigroup $\mathcal{F}$ generated by A and B is hypercyclic on $\mathcal{H}$ and has a dense set of hypercyclic vectors.*

Proof. Let $\mathcal{F}$ be the semigroup group generated by A and B on $\mathcal{H}$. We will use Theorem 6.1 to show that $\mathcal{F}$ is hypercyclic on $\mathcal{H}$. Let U, V be two nonempty open sets. Since the set of vectors in $\mathcal{H}$ that have only finitely many nonzero coordinates is dense in $\mathcal{H}$, then there exists vectors $x, y \in \mathcal{H}$ with only finitely many nonzero coordinates such that $x \in U$ and $y \in V$. Also choose an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$ and $B(y, \epsilon) \subseteq V$. Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$. Let $n \geq 1$ be large enough such that $x_k = 0$ and $y_k = 0$ if $k \geq n$. Let $x[n] := (x_1, x_2, \ldots, x_n)$ and $y[n] = (y_1, y_2, \ldots, y_n)$.

Then by assumption the semigroup $\mathcal{F}_n$ is hypercyclic on $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ and has a dense set of hypercyclic vectors, thus there exists a $T \in \mathcal{F}_n$ such that $T(B(x[n], \epsilon)) \cap B(y[n], \epsilon) \neq \emptyset$. Let $z[n] = (z_1, z_2, \ldots, z_n) \in B(x[n], \epsilon)$ such that $Tz[n] \in B(y[n], \epsilon)$. If p and q are nonnegative integers such that $T = (A_1 \oplus \cdots \oplus A_n)^p (B_1 \oplus \cdots \oplus B_n)^q$ and we let $z = (z_1, z_2, \ldots, z_n, 0, 0, \ldots)$, then one checks that $z \in B(x, \epsilon)$ and $T'z \in B(y, \epsilon)$ where $T' = A^p B^q$. It follows that $T'(U) \cap V \neq \emptyset$. Hence by Theorem 6.1, $\mathcal{F}$ is hypercyclic on $\mathcal{H}$ and has a dense set of hypercyclic vectors. □

7. Open Sets with Infinitely Many Components

Theorem 7.1. *Let G be an open set in $\mathbb{C}$ with infinitely many components. Suppose that $\{G_i\}_{i=1}^{\infty}$ are the components of G. Also let $\mathcal{H}(G)$ be a Hilbert space of analytic functions as in Definition 2.1 on G. If $f, g \in H^{\infty}(G)$ and $\mathcal{F} = \{M_{f^n}^* M_{g^k}^* : n, k \geq 0\}$, then the following are equivalent:*

1. *The pair (M_f^*, M_g^*) is hypercyclic on $\mathcal{H}(G)$.*
2. *For every $N \geq 1$, the pair (M_f^*, M_g^*) is hypercyclic on $\mathcal{H}(\bigcup_{i=1}^N G_i)$.*
3. *For every $N \geq 1$, $\mathcal{F}|\mathcal{H}(\bigcup_{i=1}^N G_i)$ contains a hypercyclic operator.*
4. *For every $N \geq 1$, there exists integers $n, k \geq 0$ such that for every $i \in \{1, \ldots, N\}$, $f^n g^k$ is non-constant on G_i and $(f^n g^k)(G_i) \cap \partial\mathbb{D} \neq \emptyset$.*

Proof. It follows from Theorem 5.6 that (2), (3), and (4) are all equivalent. Clearly, (1) implies (2) and it follows from Theorem 6.2 that (4) implies (1). Because if $f_i = f|G_i$ for all i, then M_f on $\mathcal{H}(G)$ is unitarily equivalent to $\bigoplus_{i=1}^{\infty} M_{f_i}$ on $\bigoplus_{i=1}^{\infty} \mathcal{H}(G_i) = \mathcal{H}(G)$, similarly for M_g. Thus we may apply Theorem 6.2. □

8. A hypercyclic semigroup containing no hypercyclic operators

In this section we give an example of a hypercyclic commutative semigroup generated by a pair of pure cosubnormal operators, yet the semigroup does not contain a hypercyclic operator. The cosubnormal operators are adjoints of multiplication operators on a Hilbert space of analytic functions on an open set with infinitely many components.

As mentioned in the introduction, if A is any supercyclic operator, then one can easily see that the semigroup generated by the tuple $(A, 2I, \frac{1}{3}I, e^{i\theta}I)$ is hyperyclic if $\theta \in \mathbb{R}$ is an irrational multiple of π. This follows because $\{\frac{2^i}{3^j}e^{ik\theta} : i, j, k \geq 0\}$ is dense in $\mathbb{C}$. Now there exists a bounded open set G with infinitely many components such that if A is the adjoint of mulitplication by z on the Bergman space of G, then A is supercyclic, but no multiple of A is hypercyclic (see [11]). With that operator A, then the semigroup generated by $(A, 2I, \frac{1}{3}I, e^{i\theta}I)$ will be hypercyclic (consist entirely of cosubnormal operators and) yet contain no hypercyclic operator. However, this semigroup has four generators. We now show how to give an example of such a semigroup generated by two operators.

Theorem 8.1. *If G is a bounded open set with infinitely many components and $\mathcal{H}(G)$ is a Hilbert space of analytic functions on G as in Definition 2.1, then there exists $f, g \in H^{\infty}(G)$ such that the pair (M_f^*, M_g^*) is hypercyclic on $\mathcal{H}(G)$, but $\mathcal{F} = \{M_{f^n}^* M_{g^k}^* : n, k \geq 0\}$ contains no hypercyclic operator.*

First two simple lemmas which we leave to the reader. Recall that $A(r,s) = \{z \in \mathbb{C} : r < |z| < s\}$, $\|f\|_{inf,G} = \inf\{|f(z)| : z \in G\}$, and $\|f\|_{\infty,G} = \sup\{|f(z)| : z \in G\}$.

Lemma 8.2. *If G is any bounded open set in $\mathbb{C}$, then given any $s > r > 0$, there exists a nonconstant linear polynomial f such that $f(G) \subseteq A(r,s)$ and $\|f\|_{inf,G} = r$. There also exists a nonconstant linear polynomial g such that $g(G) \subseteq A(r,s)$ and $\|f\|_{\infty,G} = s$.*

Lemma 8.3. *Let $f(z) = az + b$ and $g(z) = cz + d$ where $ac \neq 0$. If G is an open set where f and g are non-zero, then there is no $p > 0$ such that $|f|^p = \frac{1}{|g|}$ on G.*

Proof of Theorem 8.1. Let $\{G_i\}_{i=1}^{\infty}$ be the components of G. By applying Lemma 8.2 we can choose inductively constants a_i, b_i and nonconstant linear polynomials f_i, g_i such that the following hold:

1. $\frac{1}{2} < a_{i+1} < a_i < 1$ and $2 < b_{i+1} < b_i < 3$ for all $i \geq 1$.
2. $\lim_{i\to\infty} a_i = \frac{1}{2}$ and $\lim_{i\to\infty} b_i = 2$.
3. $f_i(G_i) \subseteq A(\frac{1}{2}, a_i)$ and $g_i(G_i) \subseteq A(2, b_i)$.
4. For each i, $\|f_i\|_{inf,G_i} = \frac{1}{2}$ and $\|f_i\|_{\infty,G_i} = a_i$.
5. For each i, $\|g_i\|_{inf,G_i} = 2$ and $\|f_i\|_{\infty,G_i} = b_i$.

Now let f, g be defined on G as $f|G_i = f_i$ and $g|G_i = g_i$. Then $f, g \in H^\infty(G)$.

Let
$$w_i(z) = \frac{\ln|f_i(z)|}{-\ln|g_i(z)|}.$$
Then since f_i and g_i are nonconstant linear polynomials on G_i, then by Lemma 8.3 and the proof of Theorem 3.1, w_i is non-constant on G_i. Thus $J_i := w_i(G_i)$ is an open interval. By Lemma 5.7 we have that
$$J_i := w_i(\mathbb{D}) = \left(1, \frac{\ln(b_i)}{-\ln(a_i)}\right).$$

It follows that for each $N > 1$, $\bigcap_{i=1}^N J_i \neq \emptyset$ and if $\frac{k}{n} \in \bigcap_{i=1}^N J_i$, then $M_{f^n}^* M_{g^k}^*$ is hypercyclic on $\mathcal{H}(\bigcup_{i=1}^N G_i)$. In particular, for each $N > 1$, the pair (M_f^*, M_g^*) is hypercyclic on $\mathcal{H}(\bigcup_{i=1}^N G_i)$. It now follows from Theorem 7.1 that (M_f^*, M_g^*) is hypercyclic on $\mathcal{H}(G)$.

However since $\bigcap_{i=1}^\infty J_i = \emptyset$, then $\mathcal{F}$ contains no hypercyclic operator. Because for any nonnegative integers n, k, there exists an i such that $k/n \notin J_i$, thus by Lemma 5.1 $(f_i^n g_i^k)(G_i) \cap \partial\mathbb{D} = \emptyset$. So, $M_{f^n}^* M_{g^k}^*$ is not hypercyclic. □

9. Final Remarks & Questions

There are a lot of open questions about the hypercyclicity of pairs or tuples of operators, or equivalently of finitely generated commutative hypercyclic semigroups. Here are a few, there are many others.

Note that Kérchy [14] has some results about supercyclic semigroups in finite dimension, some results involving weighted shifts and supercyclic semigroups, and a "supercyclicity criterion" for a semigroup.

Question 9.1. *Can one characterize the finitely generated commutative hypercyclic semigroups in finite dimensions? There are non-trivial examples of such in every dimension, see Kérchy* [14].

Question 9.2. *Can one characterize the pairs (tuples) of cosubnormal (cohyponormal) operators that are hypercyclic?*

Question 9.3. *Can one characterize the pairs (tuples) of weighted shifts that are hypercyclic? Are there non-trivial examples in this case?*

Question 9.4. *Is there a "hypercyclicity criterion" for pairs or tuples of operators?*

Question 9.5. *If $\mathcal{F}$ is a finitely generated commutative hypercyclic semigroup, then must $\mathcal{F}$ contain a cyclic operator?*

Question 9.6. *If (T_1, T_2) is a hypercyclic pair, then is $(T_1 \oplus T_1, T_2 \oplus T_2)$ also a hypercyclic pair? Notice that this reduces to Herrero's question when T_2 is the identity operator.*

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