LINEAR CHAOS?

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In this article we hope to convience the reader that the dynamics of linear operators can be fantastically complex and that linear dynamics exhibits the same beauty and complexity as non-linear dynamics. It has been known for sometime that continuous linear operators on Hilbert space can actually be chaotic! In fact, we shall show that the orbits of linear operators can be as complicated as the orbits of any continuous function.

More precisely, there is a (very natural) bounded linear operator T with the property that any continuous function f on a compact metric space X, is topologically conjugate to the restriction of T to an invariant compact set.

1 Non-Linear Dynamics

A discrete dynamical system is simply a continuous function $f : X \to X$ where X is a complete separable metric space. For $x \in X$, the *orbit* of x under f is $Orb(f, x) = \{x, f(x), f^2(x), \dots\}$ where $f^n = f \circ f \circ \cdots \circ f$ is the n^{th} iterate of f obtained by composing f with itself n times. In (discrete) dynamics one is interested in understanding the behavior of orbits of a function. Roughly speaking, a function f is considered to be "chaotic" if its orbits behave in a very complicated and unpredictable way. More precisely, we have the following well known definition:

Devaney's Definition of Chaos: *Suppose that $f : X \to X$ is a continuous function on a complete separable metric space X, then f is chaotic if:*
(a) the periodic points for f are dense in X,
(b) f is transitive,
(c) f has sensitive dependence on initial conditions.

Recall that a point $x \in X$ is a *periodic point* for f if $f^n(x) = x$ for some $n \geq 1$. Also, f is *transitive* if for any two non-empty open sets U, V in X, there exists an integer $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$. It is well known that, in a complete metric space with no isolated points, being transitive is equivalent (via the Baire Category Theorem) to having a point with dense orbit, which in turn is equivalent to having a dense G_δ set of points each of which has a dense orbit. Finally, f has *sensitivie dependence on initial conditions* if there exists a $\delta > 0$ such that for any $x \in X$ and for any neighborhood U of x, there exists a $y \in U$ and an $n \geq 1$ such that $d(f^n(x), f^n(y)) > \delta$ (here d denotes the metric on X).

It was shown by Banks et. al. [1] that if f has a dense set of periodic points and is transitive, then f must have sensitive dependence on initial conditions. Hence only the first two conditions of the definition of chaos need to be verified when showing that a particular function f is chaotic. In fact, for functions on intervals in $\mathbb{R}$, it was shown by Vellekoop and Berglund [14] that transitivity implies chaos. However, in metric spaces other than $\mathbb{R}$, transitivity need not imply (a) or (c) in the definition of chaos. Although condition (c) need not be verified, sensitivity is at the heart of chaos, and for some researchers in the sciences, sensitive dependence on initial conditions is all that is required for a deterministic system to be chaotic.

The following examples are well-known, see for instance Devaney [5].

Example 1.1 (Some Chaotic Functions)

1. *$f : [0, 1] \to [0, 1]$ given by $f(x) = 4x(1 - x)$ is chaotic.*

2. *$f : [-2, 2] \to [-2, 2]$ given by $f(x) = x^2 - 2$ is chaotic.*

3. *$f : S^1 \to S^1$ given by $f(z) = z^2$ is chaotic, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.*

4. *If $f(x) = \lambda x(1 - x)$ and $\lambda > 4$, then there exists a Cantor set $\Lambda \subseteq [0, 1]$ such that $f(\Lambda) = \Lambda$ and $f : \Lambda \to \Lambda$ is chaotic.*

5. *If $c \in \mathbb{C}$ and $f_c(z) = z^2 + c$, then there exists a compact set $J_c \subseteq \mathbb{C}$ (called the Julia set for f_c) such that $f_c : J_c \to J_c$ is chaotic.*

In studying the dynamics of functions, the standard equivalence relation used to say that two functions have the "same dynamics" is *topological conjugacy.* If $f : X \to X$ and $g : Y \to Y$ are two continuous functions, then f is *topologically conjugate* to g if there exists a homeomorphism $h : X \to Y$ such that $g = h \circ f \circ h^{-1}$, that is h conjugates f to g. It is easy to see that having a dense set of periodic points, or being transitive, are properties that are preserved under topological conjugacy. Thus, it follows that being chaotic is also preserved under topological conjugacy; however, having sensitive dependence on initial conditions is not preserved, because it is a metric condition and not a topological condition. It is an easy exercise to show that the functions (1) and (2) in Example 1.1 are topologically conjugate (via a linear function).

2 Linear Dynamics

It is often said that chaos cannot appear in linear systems, and indeed it is a nice exercise using Jordan Canonical Forms to understand the behavior of all orbits of a linear operator in finite dimensions, and none are dense in the space (also see Proposition 2.1). However, we will present a result due to Rolewicz, that linear operators on an *infinite dimensional* Hilbert space can be chaotic. Furthermore, we will apply non-linear dynamics to help construct vectors whose orbits under a linear operator are "chaotic".

In what follows, $\mathcal{H}$ will always denote a separable complex Hilbert space. If T is a linear operator on $\mathcal{H}$, then $\|T\| = sup\{\|Tx\| : \|x\| \leq 1\}$; we say that

T is bounded if $\|T\| < \infty$. It's well known that T is bounded if and only if T is continuous. We will denote the set of all bounded linear operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. We start with some straightforward results that examine sensitivity and periodic points for linear operators.

Proposition 2.1 *If $T \in \mathcal{B}(\mathcal{H})$ and T is transitive, then T^* has no eigenvectors.*

Proof: Suppose that T is transitive and $T^*v = \lambda v$ where $v \neq 0$. If $x \in \mathcal{H}$ is a vector with dense orbit, then $\{T^n x : n \geq 0\}$ is dense in $\mathcal{H}$. Hence $\{\langle T^n x, v\rangle : n \geq 0\}$ is dense in $\mathbb{C}$. But $\langle T^n x, v\rangle = \langle x, (T^n)^* v\rangle = \langle x, \lambda^n v\rangle = \overline{\lambda}^n \langle x, v\rangle$. However, it is easy to see that $\{\overline{\lambda}^n \langle x, v\rangle : n \geq 0\}$ is not dense in $\mathbb{C}$, a contradiction. ■

The beauty of Proposition 2.1, is that it gives a simple answer to the question of why a linear operator in finite dimensions cannot be chaotic (or even transitive). It can also be used to show that no compact operator on a Hilbert space can be transitive.

Proposition 2.2 *If $T \in \mathcal{B}(\mathcal{H})$, then T has sensitive dependence on initial conditions if and only if $\sup \|T^n\| = \infty$ if and only if T has an unbounded orbit.*

Proof: The Principle of Uniform Boundedness says that $\sup \|T^n\| = \infty$ if and only if T has an unbounded orbit. So suppose that there is a vector $v \in \mathcal{H}$ with unbounded orbit, i.e. $\sup \|T^n v\| = \infty$. Now suppose that δ is any fixed positive number. If $x \in \mathcal{H}$ and U is a neighborhood of x, then let $y_\epsilon = x + \epsilon v$. Then for sufficiently small $\epsilon > 0$, we have $y_\epsilon \in U$ and $\|T^n x - T^n y_\epsilon\| = \epsilon \|T^n v\|$ and since $\{\|T^n v\|\}$ is an unbounded sequence, there exists a choice of n such that $\|T^n x - T^n y_\epsilon\| > \delta$. Thus T has sensitive dependence on initial conditions. Conversely, suppose T has sensitive dependence on initial conditions and that $\delta > 0$ is the number guaranteed by the sensitivity condition. Choosing $x = 0$, we have that for any $\epsilon > 0$ there exists a y and an $n \geq 1$ such that $\|y\| < \epsilon$ and $\|T^n y\| > \delta$. Thus, $\|\frac{1}{\epsilon} y\| < 1$ and $\|T^n(\frac{1}{\epsilon} y)\| > \delta/\epsilon$, thus $\|T^n\| > \delta/\epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $\sup \|T^n\| = \infty$. ■

It follows from the proposition above that a linear operator T in finite dimensions may have sensitive dependence on initial conditions, it fact this will hold whenever T has an eigenvalue λ satisfying $|\lambda| > 1$, because then $\|T^n\| \geq |\lambda|^n$. It also follows that whenever a bounded linear operator T is transitive it necessarily has sensitive dependence on initial conditions; because, if T is transitive, it will have a vector with dense orbit, and hence it has an unbounded orbit.

Proposition 2.3 *If $T \in \mathcal{B}(\mathcal{H})$, then a vector is a periodic point for T if and only if it is a finite linear combination of eigenvectors of T where the eigenvalues are n^{th} roots of unity.*

Proof: Suppose that $Tv_i = \lambda_i v_i$ for $i \in \{1, \ldots, m\}$ and for each i, there exists an $n_i \geq 1$ such that $\lambda_i^{n_i} = 1$. If $x = c_1 v_1 + \cdots + c_m v_m$ for scalars $\{c_i\}$, then x is a periodic point with period $n = n_1 n_2 \cdots n_m$. For the converse, if x is a periodic point for T, then $T^n x = x$ for some $n \geq 1$. Hence

$x \in \ker(T^n - I)$. Now factor the polynomial $(z^n - 1)$ into distinct linear terms $(z^n - 1) = (z - \lambda_1) \cdots (z - \lambda_n)$ where each λ_i is an n^{th} root of unity. Thus, $x \in \ker(T^n - I) = \ker[(T - \lambda_1) \cdots (T - \lambda_n)] = \text{span}\{\ker(T - \lambda_i) : 1 \leq i \leq n\}$. ■

A nice corollary of Proposition 2.3 is that a linear operator T on a finite dimensional Hilbert space has a dense set of periodic points if and only if there exists an $n \geq 1$ such that $T^n = I$, in which case every point is a periodic point.

The Backward Shift: The set of all sequences $x = (x_0, x_1, x_2, \dots)$ of complex numbers such that $\|x\|^2 = \sum_{n=0}^{\infty} |x_n|^2 < \infty$ is denoted by ℓ^2. There is a natural inner product on ℓ^2: if $x, y \in \ell^2$, then $\langle x, y \rangle = \sum_{n=0}^{\infty} x_n \overline{y_n}$, where x_n and y_n denote the n^{th} coordinates of x and y, respectively. With the above inner product, ℓ^2 becomes a separable infinite dimensional complex Hilbert space.

An important linear operator on ℓ^2 is the *Backward Shift* B. The Backward shift acts as follows:

$$B(x_0, x_1, x_2, \dots) = (x_1, x_2, \dots).$$

It is clear that $\|Bx\| \leq \|x\|$ for all $x \in \ell^2$. It follows that B is a continuous linear operator from ℓ^2 into ℓ^2. It is also easy to check that the orbit of any vector under B converges to zero. For if $x \in \ell^2$, then $\|B^n x\|^2 = \sum_{k=n}^{\infty} |x_k|^2 \to 0$ as $n \to \infty$. Hence the dynamics of the Backward Shift are fairly simple—all orbits converge to the fixed point 0. However, in 1969 Rolewicz [11] proved the following suprising result.

Theorem 2.4 *If B is the Backward shift on ℓ^2, then $2B$ is chaotic on ℓ^2.*

Naturally, Rolewicz [11] did not use this termoniology, and actually he only proved that twice the Backward Shift is transitive, however this is the crucial part in the defintion of chaos. In fact some authors define chaos as transitivity plus sensitive dependence on initial conditions and do not require a dense set of periodic points, see e.g. Knudsen [9] or Robinson [10]. Regardless, it is suprising—even amazing—that *linear* operators can be chaotic!

Proof: Let $T = 2B$. Notice that $T^n(x_0, x_1, x_2, \dots) = 2^n(x_n, x_{n+1}, \dots)$. In order to show that the periodic points for T are dense, let $y \in \ell^2$. If $y = (y_0, y_1, \dots)$, then define vectors x_n as:

$$x_n = (y_0, \dots, y_{n-1}, \frac{y_0}{2^n}, \dots, \frac{y_{n-1}}{2^n}, \frac{y_0}{2^{2n}}, \dots, \frac{y_{n-1}}{2^{2n}}, \frac{y_0}{2^{3n}}, \dots, \frac{y_{n-1}}{2^{3n}}, \dots).$$

It is easy to check that $x_n \in \ell^2$ and that x_n is a periodic point for T with period n. Furthermore, $x_n \to y$, hence the periodic points for T are dense.

To show that T is transitive, let U and V be two open sets in ℓ^2 and choose vectors $x \in U$ and $y \in V$. Now let

$$z_n = (x_0, x_1, \dots, x_{n-1}, \frac{y_0}{2^n}, \dots, \frac{y_{n-1}}{2^n}, 0, 0, \dots).$$

Then $z_n \to x$ and $T^n z_n = (y_0, \dots, y_{n-1}, 0, 0, \dots) \to y$. Hence for all large n, $z_n \in U$ and $T^n z_n \in V$. Thus, T is transitive.

One can also easily check that $\|T^n\| = 2^n \to \infty$, thus, by Proposition 2.2, T has sensitive dependence on initial conditions. It follows that T is chaotic on ℓ^2. ■

The proof given above will remind many of what is now called "symbolic dynamics". However, there is one important difference here, in that the vectors we construct must actually be in the Hilbert space, that is they must have a finite norm. Where as in symbolic dynamics any sequence with the appropriate symbols belongs to the "space".

The Backward Shift of Higher Multiplicity: Let's consider a natural generalization of the Backward Shift. Suppose $\mathcal{H}_n$ is a separable complex Hilbert space with dimension n $(1 \le n \le \infty)$, then $\ell^2(\mathcal{H}_n)$ will denote the set of all *sequences of vectors* $\{x_k\}_{k=0}^{\infty}$ in $\mathcal{H}_n$ satisfying $\sum_{k=0}^{\infty} \|x_k\|^2 < \infty$. If we define a norm on $\ell^2(\mathcal{H}_n)$ by $\|\{x_k\}_{k=0}^{\infty}\|^2 = \sum_{k=0}^{\infty} \|x_k\|^2$, then $\ell^2(\mathcal{H}_n)$ becomes a separable infinite dimensional Hilbert space.

The *Backward Shift of multiplicity* n is the operator B_n on $\ell^2(\mathcal{H}_n)$ defined as:

$$B_n(x_0, x_1, x_2, \dots) = (x_1, x_2, \dots).$$

Thus B_n takes a sequence of vectors and produces another sequence of vectors. If $n < \infty$, then $\mathcal{H}_n = \mathbb{C}^n$, so each of the coordinates x_k is a vector in $\mathbb{C}^n$. In particular, $\ell^2(\mathcal{H}_1) = \ell^2(\mathbb{C}) = \ell^2$, and so $B_1 = B$. If $n = \infty$, then since all separable infinite dimensional Hilbert spaces are isomorphic, one may think of $\mathcal{H}_\infty$ as being ℓ^2. It is an easy exercise to show that B_n is unitarily equivalent to $B \oplus B \oplus \cdots \oplus B$ (n times). The same proof as given in Theorem 2.4 also gives the following result.

Theorem 2.5 *If $1 \le n \le \infty$ and B_n is the Backward Shift with multiplicity n, then $2B_n$ is chaotic on $\ell^2(\mathcal{H}_n)$.*

Now that we know that there are chaotic linear operators, one may naturally ask how many linear operators are chaotic? Surprisingly, more than one might first think. In fact, it follows by work of Herrero [8] and Chan [4] that the chaotic linear operators on a Hilbert space $\mathcal{H}$ are dense in $\mathcal{B}(\mathcal{H})$ in the strong operator topology—that is the topology of pointwise convergence. Indeed, within various natural classes of linear operators there are many examples of linear operators that are chaotic; including shift operators, such as twice the Backward shift, composition operators, and adjoints of multiplication operators, see [12],[3],[7].

Now that we have seen that linear operators may have vectors with dense orbits—which in a sense is as complicated as an orbit could possibly be—we are naturally lead to ask the following question:

Question: For a linear operator, how complicated can an orbit be that is not dense?

For example, can an orbit for a linear operator be dense in the unit ball of the Hilbert space, or dense in a Cantor set, or some other fractal-like subset of the Hilbert space? The following result due to Bourdon and the author sheds some light on these types of questions, and also has other important applications to the dynamics of linear operators (see [2]).

Theorem 2.6 *If $T \in \mathcal{B}(\mathcal{H})$, $x \in \mathcal{H}$, and the orbit of x is somewhere dense in $\mathcal{H}$, then the orbit of x must be dense in $\mathcal{H}$.*

Recall that a set is somewhere dense if its closure has non-empty interior, otherwise it is called nowhere dense. Thus, for a linear operator an orbit is either nowhere dense or everywhere dense. In particular, a linear operator cannot have an orbit whose closure is the closed unit ball. On the other hand, the author has proven the following result which says that orbits of linear operators can be as complicated as the orbit of any continuous function on a compact metric space. Recall that K is an *invariant set* for T if $T(K) \subseteq K$.

Theorem 2.7 *Suppose that $1 \leq n \leq \infty$ and B_n is the Backard Shift with multiplicity n on $\ell^2(\mathcal{H}_n)$; also let $T = 2B_n$. If $f : X \to X$ is any continuous function on a closed bounded set $X \subseteq \mathcal{H}_n$, then there is an invariant closed set $K \subseteq \ell^2(\mathcal{H}_n)$ such that $T|K$ is topologically conjugate to f.*

If $1 \leq n < \infty$, then the Theorem implies that any continuous function $f : X \to X$ on a compact set $X \subseteq \mathbb{C}^n$ is topologically conjugate to the restriction of a linear operator to an invariant closed set. In particular, looking back at Example 1.1, consider the function $f(x) = 5x(1-x)$, since f is chaotic on a Cantor set $\Lambda \subseteq \mathbb{R} \subseteq \mathbb{C}$, it follows that f is topologically conjugate to a restriction of $T = 2B_1$ = twice the Backward shift (with multiplicity one). Thus, there is a Cantor set $K \subseteq \ell^2$ such that $T(K) \subseteq K$ and $T|K$ is conjugate to f. In particular, since f is transitive, it has a dense orbit, thus $T|K$ also has a dense orbit. Thus it follows that T has an orbit that is dense in a Cantor set!

Similarly, $T = 2B_1$ has orbits that are dense in compact sets homeomorphic to products of intervals and Cantor sets, Julia sets, and other fractal-like sets. In higher dimensions there are continuous functions with orbits dense in products of Julia sets, and other such "strange attractors" ([5], [10]) thus $T = 2B_n$ will have orbits dense in sets that are homeomorphic to such sets.

Lemma 2.8 *If X is a compact metric space, then X is homeomorphic to a compact subset of ℓ^2.*

Proof: If d is the metric on X and $\{x_n\}_{n=0}^{\infty}$ is a countable dense subset of X, then the map $h : X \to \ell^2$ given by $h(x) = \{d(x, x_n)/2^n\}_{n=0}^{\infty}$ is easily seen to be one-to-one and continuous. Since X is compact, h is a homeomorphism. ■

By the previous Lemma we also have the following corollary of Theorem 2.7

Corollary 2.9 *If $T = 2B_\infty$ and $f : X \to X$ is any continuous function on a compact metric space X, then there is an invariant compact set $K \subset \ell^2(\mathcal{H}_\infty)$ such that $T|K$ is topologically conjugate to f.*

This Corollary says that there is a *linear operator*, namely twice the Backward Shift of infinite multiplicity, that is *universal* in the sense that its restriction to an invariant compact set can have the "same dynamics" as any continuous function on any compact metric space!

Proof of Theorem 2.7: Suppose that $T = 2B_n$ and $f : X \to X$ is a continuous function on a closed bounded set $X \subseteq \mathcal{H}_n$. Define $h : X \to \ell^2(\mathcal{H}_n)$ by:

$$h(x) = (x, \frac{f(x)}{2}, \frac{f^2(x)}{2^2}, \frac{f^3(x)}{2^3}, \ldots).$$

Notice that $x, f(x), f^2(x), \ldots$ are all vectors in $X \subseteq \mathcal{H}_n$ and since $Orb(f,x) \subseteq X$ and X is a bounded set in $\mathcal{H}_n$ it follows easily that $\|h(x)\| < \infty$, thus h does map X into $\ell^2(\mathcal{H}_n)$. It is also clear that h is one-to-one and that $h(f(x)) = T(h(x))$ for all $x \in X$. Thus if we set $K = h(X)$, then K is invariant for T and $h \circ f \circ h^{-1} = T|K$. So, it suffices to show that $h : X \to K$ is a homeomorphism. First let's show that h is continuous. So, suppose that $x_0 \in X$ and $\epsilon > 0$. Let $d > 0$ be the diameter of X, that is $\|x - y\| \le d$ for all $x, y \in X$. Then choose an $m \ge 1$ such that $\sum_{k=m}^{\infty} \frac{d^2}{4^k} < \epsilon^2/4$. Since $f^0, f^1, f^2, \ldots, f^m$ are all continuous at x_0, there exists a $\delta > 0$ such that if $\|x - x_0\| < \delta$, then $\|f^k(x) - f^k(x_0)\| < \frac{\epsilon}{2\sqrt{(m+1)}}$ for all $k \in \{0, \ldots, m\}$. Thus if $\|x - x_0\| < \delta$, then

$$\|h(x) - h(x_0)\|^2 = \sum_{k=0}^{m} \frac{\|f^k(x) - f^k(x_0)\|^2}{4^k} + \sum_{k=m+1}^{\infty} \frac{\|f^k(x) - f^k(x_0)\|^2}{4^k} \le$$

$$\le \sum_{k=0}^{m} \frac{\epsilon^2}{4(m+1)} + \sum_{k=m+1}^{\infty} \frac{d^2}{4^k} \le \frac{\epsilon^2}{4} + \frac{\epsilon^2}{4} = \frac{\epsilon^2}{2} < \epsilon^2.$$

Thus $\|h(x) - h(x_0)\| < \epsilon$. Hence h is continuous. Also, since it is clear that $\|h(x) - h(y)\| \ge \|x - y\|$, we see that $h^{-1} : K \to X$ is also continuous, in fact Lipschitz. Thus h is a homeomorphism and the result follows. ∎

We next notice that if the continuous function f is Lipschitz on $X \subseteq \mathcal{H}_n$, that is, $\|f(x) - f(y)\| \le C\|x - y\|$, then the conjugating homeomorphism h is bi-Lipschitz (that is, h and h^{-1} are both Lipschitz).

Theorem 2.10 *Suppose $1 \le n \le \infty$ and $f : X \to X$ is a Lipschitz function on a closed bounded subset $X \subseteq \mathcal{H}_n$ with Lipschitz constant M. If $\lambda > \max\{M, 1\}$, then there is a closed set $K \subseteq \ell^2(\mathcal{H}_n)$ that is invariant for $T = \lambda B_n$ such that f is topologically conjugate to $T|K$ via a bi-Lipschitz homeomorphism.*

Proof: Define $h : X \to \ell^2(\mathcal{H}_n)$ by:

$$h(x) = (x, \frac{f(x)}{\lambda}, \frac{f^2(x)}{\lambda^2}, \frac{f^3(x)}{\lambda^3}, \dots).$$

Follow the proof of Theorem 2.7 and notice that if f is Lipschitz, then h is bi-Lipschitz, in fact $\|x-y\| \le \|h(x)-h(y)\| \le C\|x-y\|$ where $C^2 = \sum_{k=0}^{\infty} \frac{M^{2k}}{\lambda^{2k}}$. ∎

Since bi-Lipschitz homoemorphisms will preserve more of the metric structure of X than a homeomorphism would—for instance the Hausdorff dimension of X—it follows easily that there are bounded linear operators that have orbits that are dense in compact sets with any given Hausdorff dimension. Since the functions $f_c(z) = z^2 + c$ are Lipschitz on compact sets, it follows that multiples of the Backward shift will have orbits dense in compact sets that are bi-Lipschitz equivalent to Julia sets. Thus the closure of an orbit of a linear operator can have the same fine geometric structure as Julia sets and other such complicated sets. In particular, we see that the orbits of linear operators can be fantastically complicated, and that linear dynamics exhibits the same beauty and complexity as non-linear dynamics.

Here we have just touched on the dynamics of linear operators. There is an extensive literature on the subject, see for instance [13], [6] and their references. In operator theory a linear operator that is transitive is usually called *hypercyclic*, since it is a strong form of cyclicity. Two important open questions in the area are the following:

Open Question: If $T \in \mathcal{B}(\mathcal{H})$, then does T have a non-trivial invariant closed set? Here, non-trivial means not $\{0\}$ and not all of $\mathcal{H}$. This is equivalent to asking: does every bounded linear operator T have a non-zero vector whose orbit is not dense? It should be noted that the answer to this question is no for operators on certain Banach spaces, such as ℓ^1.

Open Question: If $T \in \mathcal{B}(\mathcal{H})$ and T is transitive, then is $T \oplus T$ also transitive?

Finally, the following is a question that occured to the author while writing this paper:

Question: If X is a compact metric space with no isolated points, then is there a continuous chaotic function $f : X \to X$? Or simply a transitive function?

References

[1] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey, *On Devaney's Definition of Chaos*, Amer. Math. Monthly **99**, 332–334.

[2] P.S. Bourdon and N. S. Feldman, *Somewhere dense orbits are everywhere dense*, Submitted.

[3] P. S. Bourdon and J. H. Shapiro, *Cyclic Phenomena for Composition Operators*, Memoirs of the AMS, 125, AMS, Providence, RI, 1997.

[4] K. C. Chan, *The Density of Hypercyclic Operators on a Hilbert Space*, To appear in J. Operator Theory.

[5] R.L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison Weseley, 1987.

[6] K-G. Grosse-Erdmann, *Universal families and hypercyclic operators*, Bulletin AMS, **36** no 3 (1999), 345–381.

[7] R. M. Gethner and J. H. Shapiro, *Universal vectors for operators on spaces of holomorphic functions*, Proc. AMS, **100** (1987), 281–288.

[8] D. Herrero, *Hypercyclic operators and chaos*, J. Operator Th., **28** (1992), 93–103.

[9] C. Knudsen, *Chaos without nonperiodicity*, Amer. Math. Monthly **101** (1994), no. 6, 563–565.

[10] C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, CRC Press, 2nd Edition, 1999.

[11] S. Rolewicz, *On orbits of elements*, Studia Math., **32** (1969), 17–22.

[12] H. N. Salas, *Hypercyclic weighted shifts*, Trans Amer. Math. Soc., **347** (1995), 993–1004.

[13] J.H. Shapiro, *Notes on Dynamics of Linear Operators*, Unpublished Lecture Notes, (available at www.math.msu.edu/˜shapiro).

[14] M. Vellekoop and R. Berglund, *On Intervals, Transitivity = Chaos*, Amer. Math. Monthly, **101** (1994), no. 4, 353–355.

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