Interpolating Measures for Subnormal Operators

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If $\mu \in M^+(K)$ is a positive regular Borel measure supported on a compact set K in the complex plane, then let $R^2(K,\mu)$ denote the closure of $\mathrm{Rat}(K)$, the rational functions with poles off K, in $L^2(\mu)$. If we define $S_{K,\mu} = M_z$ on $R^2(K,\mu)$, then $S_{K,\mu}$ is a typical rationally cyclic subnormal operator. When K is polynomially convex, then $R^2(K,\mu) = P^2(\mu)$, the closure of the analytic polynomials in $L^2(\mu)$, and $S_\mu := S_{K,\mu}$ will be a cyclic subnormal operator.

If $\lambda \in K$, then λ is a bounded point evaluation (b.p.e.) for $S_{K,\mu} = M_z$ on $R^2(K,\mu)$ if there is a constant $C > 0$ such that $|f(\lambda)| \leq C\|f\|_{L^2(\mu)}$ for all $f \in \mathrm{Rat}(K)$. This is equivalent to requiring that the densely defined linear operator $A : \mathrm{Rat}(K) \to \mathbb{C}$ given by $A(f) = f(\lambda)$ extends to an (onto) bounded linear operator $A : R^2(K,\mu) \to \mathbb{C}$ (the extension is also called A).

Thomson's Theorem [1] says that if $S_\mu = M_z$ on $P^2(\mu)$ is pure, then b.p.e.'s always exist for S_μ. However it is known (see [2]) that b.p.e.'s need not exist for $R^2(K,\mu)$ spaces. We are looking to generalize the idea of a b.p.e. for a $R^2(K,\mu)$ space to the notion of an interpolating measure for any subnormal operator.

For a measure $\nu \in M^+(K)$, ν will be an *interpolating measure* for $S_{K,\mu} = M_z$ on $R^2(K,\mu)$ if the densely defined map $A : \mathrm{Rat}(K) \to L^2(\nu)$ defined by $A(f) = f$ extends to be an (into and) *onto* bounded linear operator $A : R^2(K,\mu) \to L^2(\nu)$.

Question: If K is a compact set in the complex plane and μ a measure on K, then does $S_{K,\mu} = M_z$ on $R^2(K,\mu)$ have an interpolating measure?

For an arbitrary operator S on a Hilbert space $\mathcal{H}$, a measure ν is said to be an *interpolating measure* for S if there exists an (into and) *onto* bounded linear operator $A : \mathcal{H} \to L^2(\nu)$ such that $AS = N_\nu A$, where $N_\nu = M_z$ on $L^2(\nu)$.

Question: If S is a subnormal operator, then does S have an interpolating measure? If not, which subnormal operators have interpolating measures?

Theorems: (a) If $S_\mu = M_z$ on $P^2(\mu)$ is pure and G is the set of b.p.e.'s for S_μ, then a measure ν is an interpolating measure for S_μ if and only if ν is a discrete measure *carried by* G whose atoms form a $P^2(\mu)$ interpolating sequence.

(b) If $S = M_z$ on $H^2(G)$ where $G = \mathbb{D} \setminus [0,1]$, then Lebesgue measure on $[0,1]$ is an interpolating measure for S.

(c) If $S = M_{\overline{z}}$ on $L^2(\mathbb{D})^\perp$ is the dual of the Bergman operator, then for any compact set $K \subseteq \mathbb{D}$, $\nu =$ area measure on K is an interplolating measure for S.

Question: Are there bounded regions G in $\mathbb{C}$ such that $S = M_z$ on the Bergman space $L^2_a(G)$ has a continuous interpolating measure that is supported on ∂G ?

References

[1] J.E. Thomson, *Approximation in the mean by polynomials* Ann. of Math. **133** (1991), no. 3, 477–507.

[2] C. Fernstrøm, *Bounded point evaluations and approximation in L^p by analytic functions*, Spaces of analytic functions (Sem. Functional Anal. and Function Theory, Kristiansand, 1975), pp. 65–68. Lecture Notes in Math., Vol. 512, Springer, Berlin, 1976.