

# HYPERCYCLIC TUPLES OF OPERATORS & SOMEWHERE DENSE ORBITS

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ABSTRACT. In this paper we prove that there are hypercyclic  $(n + 1)$ -tuples of diagonal matrices on  $\mathbb{C}^n$  and that there are no hypercyclic  $n$ -tuples of diagonalizable matrices on  $\mathbb{C}^n$ . We use the last result to show that there are no hypercyclic subnormal tuples in infinite dimensions. We then show that on real Hilbert spaces there are tuples with somewhere dense orbits that are not dense, but we also give sufficient conditions on a tuple to insure that a somewhere dense orbit, on a real or complex space, must be dense.

## 1. INTRODUCTION

By an  $n$ -tuple of operators we mean a finite sequence of length  $n$  of *commuting* continuous linear operators on a locally convex space  $X$ . If  $T = (T_1, T_2, \dots, T_n)$  is an  $n$ -tuple of operators, then we will let  $\mathcal{F} = \mathcal{F}_T = \{T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} : k_i \geq 0\}$  be the semigroup generated by  $T$ . If  $x \in X$ , then the orbit of  $x$  under the tuple  $T$  or under  $\mathcal{F}$  is  $Orb(T, x) = Orb(\mathcal{F}, x) = \{Tx : T \in \mathcal{F}\}$ . It will also be convenient to write  $Orb(\{T_i\}_{i \in J}, x)$  for the orbit of  $x$  under the semigroup generated by the set of operators  $\{T_i\}_{i \in J}$ . The tuple  $T$  is hypercyclic if there is a vector  $x \in X$  whose orbit under  $T$  is dense in  $X$ . When  $n = 1$ , then orbits of single operators and hypercyclic operators have been widely studied. The classic example is twice the backward shift on  $\ell^2(\mathbb{N})$ , it was shown to be hypercyclic by Rolewicz [24] in 1969. In fact many natural operators are hypercyclic; they arise within the classes of weighted shifts [26], composition operators [6], co-analytic Toeplitz operators [12], and adjoints of subnormal and hyponormal operators [11].

An important result in hypercyclicity is the so called ‘‘Somewhere Dense Theorem’’ by Bourdon & Feldman [5]; it states that if  $T$  is a continuous linear operator on a locally convex space  $X$ , then an orbit for  $T$  is either dense in  $X$  or nowhere dense in  $X$ . In other words, if an orbit is somewhere dense in  $X$ , then it must actually be dense in  $X$ . Recall that a set  $E \subseteq X$  is somewhere dense in  $X$  if the closure of  $E$  has nonempty interior in  $X$ . A set  $E$  is nowhere dense in  $X$  if the closure of  $E$  has empty interior in  $X$ .

In this paper we give some examples of hypercyclic tuples of operators. Surprisingly hypercyclic  $n$ -tuples can arise in finite dimensions when  $n > 1$ , something that does not happen for single operators. For example we prove that there is an  $(n + 1)$ -tuple of diagonal matrices that is hypercyclic on  $\mathbb{C}^n$  and that no  $n$ -tuple of commuting diagonalizable matrices on  $\mathbb{C}^n$  can be hypercyclic. We also prove that there are no hypercyclic tuples of normal operators on an infinite dimensional

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Hilbert space and no hypercyclic tuples of subnormal operators having commuting normal extensions, on an infinite dimensional space.

Surprisingly, examples are given of  $n$ -tuples of operators on real Hilbert spaces that have somewhere dense orbits that are not dense! On complex Hilbert spaces the issue is not completely settled, but we present a result that says, with a little extra hypothesis, a somewhere dense orbit must be dense.

## 2. PRELIMINARY RESULTS & EXAMPLES

Throughout the paper, unless otherwise stated,  $X$  will denote a locally convex topological vector space over the complex numbers. One example where we might require  $X$  to be a separable Banach space is if the Baire Category Theorem is needed. We begin with some elementary results that are well known for single operators (when applicable to single operators). Some of the standard proofs will not be given.

**Proposition 2.1.** *If  $T = (T_1, \dots, T_n)$  is a commuting  $n$ -tuple and if the semigroup  $\mathcal{F}_T$  contains a hypercyclic operator, then  $T$  is hypercyclic.*

**Example 2.2.** *If  $B$  is the unilateral backward shift on  $\ell^2(\mathbb{N})$ , then  $T = (2I, \frac{1}{2}B)$  is hypercyclic on  $\ell^2(\mathbb{N})$ . Why? Because  $2B$  is hypercyclic on  $\ell^2(\mathbb{N})$  and  $2B \in \mathcal{F}_T$ .*

**Proposition 2.3.** *If  $\mathcal{F}$  is a set of operators on a separable Banach space  $X$ , then  $\mathcal{F}$  is hypercyclic if for any two open sets  $U, V$  there exists a  $T \in \mathcal{F}$  such that  $T(U) \cap V \neq \emptyset$ .*

*Proof.* Let  $\{V_n\}$  be a countable basis of open sets for  $X$ . By assumption, for each  $n$  the set  $\bigcup_{T \in \mathcal{F}} T^{-1}(V_n)$  is a dense open set in  $X$ . Hence by the Baire Category theorem,  $\bigcap_{n=1}^{\infty} \bigcup_{T \in \mathcal{F}} T^{-1}(V_n)$  is a dense  $G_\delta$  set in  $X$  consisting of hypercyclic vectors for  $\mathcal{F}$ .  $\square$

**Proposition 2.4.** *Suppose that  $T = (T_1, \dots, T_n)$  is a hypercyclic tuple on a separable Banach space  $X$ .*

- (1) *If  $T_k$  is invertible for each  $k$ , then the tuple  $(T_1^{-1}, \dots, T_n^{-1})$  is also hypercyclic on  $X$ .*
- (2) *If  $\mathcal{M}$  is an invariant subspace for  $T$ , then the quotient of  $T$  is hypercyclic on  $X/\mathcal{M}$ .*
- (3) *Every orbit of  $T^* = (T_1^*, \dots, T_n^*)$  is unbounded.*

**Proposition 2.5** (Hypercyclicity Criterion). *Suppose that  $(T_1, T_2)$  is a pair of operators on a separable Banach space  $Z$ . Suppose also that there exists two strictly increasing sequences of positive integers  $\{n_j\}$  and  $\{k_j\}$ , dense sets  $X$  and  $Y$  in  $Z$  and functions  $S_j : Y \rightarrow Z$  such that*

- (1) *For each  $x \in X$ ,  $T_1^{n_j} T_2^{k_j} x \rightarrow 0$  as  $j \rightarrow \infty$ .*
- (2) *For each  $y \in Y$ ,  $S_j y \rightarrow 0$  as  $j \rightarrow \infty$ .*
- (3) *For each  $y \in Y$ ,  $T_1^{n_j} T_2^{k_j} S_j y \rightarrow y$  as  $j \rightarrow \infty$ .*

*Then  $(T_1, T_2)$  is a hypercyclic pair.*

*Proof.* If  $U$  and  $V$  are two nonempty open sets in  $Z$ , then choose  $x \in X \cap U$  and  $y \in V \cap Y$  and let  $z_j = x + S_j y$ . Then as  $j \rightarrow \infty$ ,  $z_j \rightarrow x$  and  $T_1^{n_j} T_2^{k_j} z_j = T_1^{n_j} T_2^{k_j} x + T_1^{n_j} T_2^{k_j} S_j y \rightarrow y$ . Thus for large  $j$  we have  $z_j \in U$  and  $T_1^{n_j} T_2^{k_j} z_j \in V$ . Thus Proposition 2.3 implies that the pair  $(T_1, T_2)$  is hypercyclic.  $\square$

*Remark.* If  $T_2$  is the identity, then the conditions in Proposition 2.5 reduce to the well known “Hypercyclicity Criterion ” for a single operator.

**Corollary 2.6.** *If  $(T_1, T_2)$  satisfies the hypercyclicity criterion, then*

$$(T_1 \oplus T_1, T_2 \oplus T_2)$$

*also satisfies the hypercyclicity criterion, hence is a hypercyclic pair.*

**Example 2.7.** *Let  $T = (T_1, T_2, T_3) = (2I_1, \frac{1}{3}I_1, e^{i\theta}I_1)$ , where  $I_1$  is the identity operator on  $\mathbb{C}$  and  $\theta$  is an irrational multiple of  $\pi$ . Then  $T$  is hypercyclic on  $\mathbb{C}$ , but  $T$  does not satisfy the hypercyclicity criterion.*

*Proof.* It follows from Corollary 4.2 and the fact that  $\{e^{in\theta} : n \in \mathbb{N}\}$  is dense in  $\{z : |z| = 1\}$  that  $T$  is hypercyclic on  $\mathbb{C}$ . However,  $T$  does not satisfy the hypercyclicity criterion since  $(T_1 \oplus T_1, T_2 \oplus T_2, T_3 \oplus T_3) = (2I_2, \frac{1}{3}I_2, e^{i\theta}I_2)$  is not hypercyclic on  $\mathbb{C}^2$ , where  $I_2$  is the identity operator on  $\mathbb{C}^2$ . □

We saw above that the pair  $(2I, \frac{1}{2}B)$  is hypercyclic on  $\ell^2(\mathbb{N})$ , it is also easy to check that this pair satisfies the hypercyclicity criterion with respect to any two sequences  $\{n_j\}$  and  $\{k_j\}$  where  $n_j - k_j \rightarrow \infty$ .

**Example 2.8.** *Let  $T_1 = 2I$  and let  $T_2 = \frac{1}{2}B$  where  $B$  is the backward shift on  $\ell^2(\mathbb{N})$ . Also let  $S$  be the forward shift on  $\ell^2(\mathbb{N})$ . Then  $T = (T_1, T_2)$  satisfies the hypercyclicity criterion on  $\ell^2(\mathbb{N})$ . Furthermore,*

- (1)  $T_1^n T_2^k = 2^{n-k} B^k$ .
- (2) *If  $X$  is the set of vectors with finite support, then for each  $x \in X$ ,  $T_1^n T_2^k x \rightarrow 0$  as  $k \rightarrow \infty$ .*
- (3)  $S_{n,k} := 2^{k-n} S^k$  is a right inverse for  $T_1^n T_2^k$  on all of  $\ell^2(\mathbb{N})$ .
- (4) *If  $x$  is a non-zero vector in  $\ell^2(\mathbb{N})$ , then  $S_{n,k} x \rightarrow 0$  if and only if  $(n-k) \rightarrow \infty$ .*

*Proof.* Let  $\{n_j\}$  and  $\{k_j\}$  be any two (strictly increasing) sequences of positive integers such that  $n_j - k_j \rightarrow \infty$ . Let  $X$  be the vectors with finite support and let  $Y = \ell^2(\mathbb{N})$ . Notice that  $T_1^{n_j} T_2^{k_j} x = 0$  for all large  $j$  when  $x \in X$  and since  $n_j - k_j \rightarrow \infty$ , then  $S_j y := S_{n_j, k_j} y \rightarrow 0$  as  $j \rightarrow \infty$ . □

The next result is proven in Feldman [10] and generalizes the example that  $(2I, \frac{1}{2}B)$  is hypercyclic on  $\ell^2(\mathbb{N})$ ; it shows that “most” pairs of coanalytic Toeplitz operators are hypercyclic, if not, the symbols are closely related.

**Theorem 2.9.** *Let  $f, g \in H^\infty(\mathbb{D}) \setminus \{0\}$  satisfy  $|f(z)| > 1$  for all  $z \in \mathbb{D}$  and  $|g(z)| < 1$  for all  $z \in \mathbb{D}$ , and let  $M_f$  and  $M_g$  be the corresponding multiplication operators on  $H^2(\mathbb{D})$ . Then neither  $M_f^*$  nor  $M_g^*$  is hypercyclic on  $H^2(\mathbb{D})$ , but the following are equivalent:*

- (1) *The pair  $T = (M_f^*, M_g^*)$  is hypercyclic on  $H^2(\mathbb{D})$ .*
- (2) *The semigroup  $\mathcal{F}_T$  contains a hypercyclic operator.*
- (3) *There exists  $n, k \in \mathbb{N}$  so that  $f^n g^k$  is nonconstant and  $(f^n g^k)(\mathbb{D}) \cap \partial\mathbb{D} \neq \emptyset$ .*
- (4) *There does not exist a  $p > 0$  and a  $\theta \in \mathbb{R}$  such that  $g(z) = \frac{e^{i\theta}}{f(z)^p}$  for all  $z \in \mathbb{D}$ .*

The next example shows that the study of hypercyclic tuples of operators includes the study of supercyclic operators. Recall that an operator  $A$  on a space  $X$  is supercyclic if there is a vector  $x \in X$  such that  $\{\alpha A^n x : \alpha \in \mathbb{C} \text{ and } n \geq 0\}$  is dense in  $X$ .

**Example 2.10.** *If  $A$  is an operator and  $T = (2I, \frac{1}{3}I, e^{i\theta}I, A)$  where  $\theta$  is an irrational multiple of  $\pi$  and  $I$  is the identity operator, then  $T$  is a hypercyclic tuple if and only if  $A$  is a supercyclic operator. Furthermore, the hypercyclic vectors for  $T$  are the same as the supercyclic vectors for  $A$ .*

*Proof.* The semigroup generated by  $T$  is  $\mathcal{F}_T = \left\{ \frac{2^{n_1} e^{in_2\theta}}{3^{n_3}} A^{n_4} : n_1, n_2, n_3, n_4 \geq 0 \right\}$ .

Since  $\left\{ \frac{2^{n_1} e^{in_2\theta}}{3^{n_3}} : n_1, n_2, n_3 \geq 0 \right\}$  is dense in  $\mathbb{C}$  (see Corollary 4.2) it follows that the hypercyclic vectors for  $T$  are precisely the same as the supercyclic vectors for  $A$ .  $\square$

Notice that by using Proposition 3.1 instead of Corollary 4.2, one can find two complex numbers  $a$  and  $b$  so that the set of hypercyclic vectors of the 3-tuple  $T = (aI, bI, A)$  is the same as the set of supercyclic vectors for  $A$ .

**Example 2.11.** *If  $A$  is a supercyclic operator so that no multiple of  $A$  is hypercyclic, then the tuple  $T = (2I, \frac{1}{3}I, e^{i\theta}I, A)$  where  $\theta$  is an irrational multiple of  $\pi$  has the property that the semigroup  $\mathcal{F}_T$  generated by  $T$  is a hypercyclic semigroup, but it does not contain any hypercyclic operators.*

**Example 2.12.** *If  $A$  and  $B$  are hypercyclic operators, and we let  $T_1 = A \oplus I$  and  $T_2 = I \oplus B$ , then  $T = (T_1, T_2)$  is a hypercyclic pair, but neither  $T_1$  nor  $T_2$  is cyclic.*

*Proof.* Let  $x$  and  $y$  be hypercyclic vectors for  $A$  and  $B$  respectively. Notice that  $T_1^n T_2^k(x \oplus y) = A^n x \oplus B^k y$ . It follows easily that  $x \oplus y$  is a hypercyclic vector for the pair  $(T_1, T_2)$ . Since  $A$  and  $B$  are both hypercyclic, then they act on infinite dimensional spaces and the identity operator  $I$  on an infinite dimensional space is not cyclic, thus  $T_1$  and  $T_2$  are not cyclic either.  $\square$

We see in the previous example that  $T_1$  and  $T_2$  need not be cyclic for  $T$  to be hypercyclic. However, it is possible that in the above example  $T_1 T_2 (= A \oplus B)$  is cyclic. We will see in Example 2.16 that given a  $p > 0$ , there are hypercyclic pairs  $(T_1, T_2)$  where  $T_1^n T_2^k, 0 \leq n, k \leq p$  are not cyclic.

The following proposition is a slightly more general result than the previous example.

**Proposition 2.13.** *Let  $A$  and  $B$  be hypercyclic operators and let  $C$  be an operator with dense range that commutes with  $B$ . If we define*

$$T_1 = A \oplus C \text{ and } T_2 = I \oplus B$$

*then  $(T_1, T_2)$  is a hypercyclic pair.*

*Proof.* Let  $x$  and  $y$  be hypercyclic vectors for  $A$  and  $B$  respectively. We claim that  $x \oplus y$  is a hypercyclic vector for the pair  $(T_1, T_2)$ . To prove this let  $U \times V$  be a basic (product) open set. Notice that  $T_1^n T_2^k = A^n \oplus C^n B^k$ . Now since  $x$  is a hypercyclic vector for  $A$ , then we can choose an  $n$  such that  $A^n x \in U$ . Next since  $C$  has dense range, then so does  $C^n$ , thus the inverse image  $(C^n)^{-1}(V)$  is a non-empty open set. Since  $y$  is a hypercyclic vector for  $B$ , then there is a  $k$  such that  $B^k y \in (C^n)^{-1}(V)$ . Thus,  $C^n B^k y \in V$ . Thus,  $T_1^n T_2^k(x \oplus y) \in U \times V$ . So,  $(T_1, T_2)$  is a hypercyclic pair with hypercyclic vector  $x \oplus y$ .  $\square$

If  $T$  is a hypercyclic operator, then  $T^*$  cannot have any eigenvalues. However, this is not the case for tuples of operators as the next example shows.

**Definition 2.14.** A hypercyclic tuple  $T = (T_1, \dots, T_n)$  is said to be **minimal** if no proper subset of  $\{T_1, \dots, T_n\}$  forms a hypercyclic tuple.

**Example 2.15.** If  $A$  is a hypercyclic operator and  $B$  is an operator that satisfies the hypercyclicity criterion and  $\{\lambda_n\}_{n=1}^\infty$  is a bounded set of nonzero complex numbers, then define:

$$T_1 = A \oplus \lambda_1 I \oplus \lambda_2 I \oplus \dots \oplus \lambda_n I \oplus \dots \text{ and } T_2 = I \oplus B \oplus B \oplus B \oplus \dots$$

where  $I$  is the identity operator. Then  $(T_1, T_2)$  is a minimal hypercyclic pair and  $\sigma_p(T_1^*) = \{\overline{\lambda_n}\}_{n=1}^\infty$ .

*Proof.* Let  $C = (\lambda_1 I \oplus \lambda_2 I \oplus \dots \oplus \lambda_n I \oplus \dots)$  and apply Proposition 2.13 with  $T_1 = A \oplus C$  and  $T_2 = I \oplus B^{(\infty)}$  where  $B^{(\infty)} = (B \oplus B \oplus B \oplus \dots)$ . Notice that  $B^{(\infty)}$  is hypercyclic because  $B$  satisfies the hypercyclicity criterion. The pair  $(T_1, T_2)$  is minimal because neither operator  $T_1$  nor  $T_2$  is hypercyclic (because each of their adjoints have eigenvectors). □

A natural question is that if  $T = (T_1, \dots, T_n)$  is a hypercyclic tuple of operators, then must the semigroup  $\mathcal{F}_T$  contain a cyclic operator? The following example shows that the first finitely many operators in the semigroup need not be cyclic.

**Example 2.16.** (a) If  $A$  is an operator such that both  $A$  and  $A^*$  are hypercyclic and  $A$  has a real matrix representation, and we let  $T_1 = A \oplus I$  and  $T_2 = I \oplus A^*$ , then  $(T_1, T_2)$  is a hypercyclic pair, but  $T_1 T_2$  is not a cyclic operator.

(b) If

$$T_1 = A \oplus I \oplus A^2 \oplus I \oplus \dots \oplus A^p \oplus I$$

and

$$T_2 = I \oplus A^* \oplus I \oplus A^{*2} \oplus \dots \oplus I \oplus A^{*p}$$

then  $(T_1, T_2)$  is a hypercyclic pair, but the set  $\{T_1^n T_2^k : 0 \leq n, k \leq p\}$  does not contain any cyclic operators.

*Proof.* (a) It is a well known unpublished result of J.A. Deddens (see [16, Proposition 4.2]) that if  $A$  has a real matrix representation with respect to some orthonormal basis, then  $A \oplus A^*$  cannot be cyclic. Thus,  $T_1 T_2 = A \oplus A^*$  is not cyclic.

(b) If  $0 \leq n, k \leq p$ , then since  $A^k$  is a summand of  $T_1$  and  $A^{*n}$  is a summand of  $T_2$ , then  $A^{kn}$  will be a summand of  $T_1^n$  and  $A^{*nk}$  will be a summand of  $T_2^k$ , thus  $A^{kn} \oplus A^{*nk}$  will be a summand of  $T_1^n T_2^k$  and since  $A^{kn}$  has a real matrix representation, then  $A^{kn} \oplus A^{*nk}$  cannot be cyclic, hence  $T_1^n T_2^k$  cannot be cyclic either. □

**Question 2.17.** If  $T$  is a hypercyclic tuple, then is there a cyclic operator  $A$  in the semigroup generated by  $T$ ? Is there a cyclic operator  $A$  that commutes with  $T$ ?

### 3. HYPERCYCLIC NORMAL TUPLES

In this section we show that there are hypercyclic tuples of diagonal matrices on  $\mathbb{R}^n$  and  $\mathbb{C}^n$ . This was done independently by Kérchy [18]. We also use this result to show that there are no hypercyclic tuples of normal operators on an infinite dimensional Hilbert space, and even no hypercyclic tuples of subnormal operators having commuting normal extensions. It is not currently known if there are hypercyclic tuples of hyponormal operators in infinite dimensions.

**Proposition 3.1.** *There exist two complex numbers  $a, b$  such that  $\{a^n b^k : n, k \in \mathbb{N}\}$  is dense in  $\mathbb{C}$ .*

*Proof.* Let  $f : [0, 1] \rightarrow [0, 1]$  be defined by  $f(x) = \text{frac}(10x)$  where  $\text{frac}(x)$  denotes the fractional part of the real number  $x$ . Also let  $F = f \times f$ . Thus  $F$  maps  $[0, 1]^2 \rightarrow [0, 1]^2$  as follows:

$$F(x_1, x_2) = (f(x_1), f(x_2)).$$

It is easy to see that  $f$  is a “mixing” function, meaning for any two open sets  $U$  and  $V$  in  $[0, 1]$ , there exists an integer  $N$  such that  $f^{[n]}(U) \cap V \neq \emptyset$  for all  $n \geq N$ , where  $f^{[n]}$  denotes the  $n^{\text{th}}$  iterate of  $f$ . In fact given any open set  $U$ , there is an integer  $N$  such that  $f^{[n]}(U) = [0, 1]$  for all  $n \geq N$ . Since  $f$  is mixing, it follows easily that  $F$  is topologically transitive.

Let  $x = (x_1, x_2) \in [0, 1]^2$  be a point that has dense orbit under  $F$ . Also choose any positive number  $\alpha$  and a positive integer  $q$  and a  $\theta \in (0, 1]$ .

Let

$$(1) \quad a = e^{q\alpha x_1 + 2\pi i(q\theta x_1 + x_2)} \text{ and } b = e^{-\alpha - 2\pi i\theta}.$$

*Claim:*  $\{a^n b^k : n, k \in \mathbb{N}\}$  is dense in  $\mathbb{C}$ .

Since  $x$  has dense orbit under  $F$ , then

$$\left\{ \begin{bmatrix} \text{frac}(10^n x_1) \\ \text{frac}(10^n x_2) \end{bmatrix} : n \in \mathbb{N} \right\} \text{ is dense in } [0, 1]^2.$$

Thus,

$$\left\{ \begin{bmatrix} 10^n x_1 - k \\ 10^n x_2 + m \end{bmatrix} : n, k \in \mathbb{N}, m \in \mathbb{Z} \right\} \text{ is dense in } \mathbb{R}^2.$$

Notice that  $k$  above is only required to be in  $\mathbb{N}$  since the integer part of  $10^n x_1$  can be arbitrarily large and positive. Now multiplying the above set by the invertible diagonal matrix  $A = \begin{bmatrix} q\alpha(1 + \frac{2\pi i\theta}{\alpha}) & \\ & 1 \end{bmatrix}$  gives that

$$\left\{ \begin{bmatrix} 10^n q\alpha(1 + \frac{2\pi i\theta}{\alpha})x_1 - qk\alpha(1 + \frac{2\pi i\theta}{\alpha}) \\ 10^n x_2 + m \end{bmatrix} : n, k \in \mathbb{N}, m \in \mathbb{Z} \right\} \text{ is dense in } L \times \mathbb{R}$$

where  $L = \{t(1 + \frac{2\pi i\theta}{\alpha}) : t \in \mathbb{R}\}$ . Thus, a small argument shows that

$$\left\{ \left[ 10^n q\alpha \left(1 + \frac{2\pi i\theta}{\alpha}\right) x_1 - qk\alpha \left(1 + \frac{2\pi i\theta}{\alpha}\right) \right] + 2\pi i [10^n x_2 + m] : n, k \in \mathbb{N}, m \in \mathbb{Z} \right\}$$

is dense in  $\mathbb{C}$ . Simplifying this gives,

$$\left\{ 10^n q\alpha x_1 + 10^n q 2\pi i\theta x_1 - qk\alpha - qk 2\pi i\theta + 2\pi i 10^n x_2 + 2\pi i m : n, k \in \mathbb{N}, m \in \mathbb{Z} \right\}$$

is dense in  $\mathbb{C}$ . By taking the exponential of the previous set we get:

$$\left\{ e^{10^n [q\alpha x_1 + q 2\pi i\theta x_1 + 2\pi i x_2]} \cdot e^{qk[-\alpha - 2\pi i\theta]} \cdot e^{2\pi i m} : n, k \in \mathbb{N}, m \in \mathbb{Z} \right\} \text{ is dense in } \mathbb{C}.$$

Or simplifying gives

$$\left\{ (e^{q\alpha x_1 + 2\pi i(q\theta x_1 + x_2)})^{10^n} (e^{-\alpha - 2\pi i\theta})^{qk} : n, k \in \mathbb{N} \right\} \text{ is dense in } \mathbb{C}.$$

Thus we have established our claim that  $\{a^n b^k : n, k \in \mathbb{N}\}$  is dense in  $\mathbb{C}$ .  $\square$

**Corollary 3.2.** *If  $b \in \mathbb{D} \setminus \{0\}$ , then there is a dense set  $\Delta_b \subseteq \{z \in \mathbb{C} : |z| > 1\}$  such that for any  $a \in \Delta_b$ , we have  $\{a^n b^k : n, k \in \mathbb{N}\}$  is dense in  $\mathbb{C}$ .*

*Proof.* Keeping the notation from the previous proof. Let  $\Delta = \{(x_1, x_2) : (x_1, x_2) \text{ has dense orbit under } F_2\}$ . Then  $\Delta$  is a dense  $G_\delta$  set in  $[0, 1]^2$ . Suppose we are given  $b \in \mathbb{D} \setminus \{0\}$ . Then define  $\alpha = -\ln |b|$  and  $\theta = \frac{-1}{2\pi} \arg(b)$  where  $\arg(b)$  denotes the value of the argument of  $b$  that lies in  $(0, 2\pi]$ . Now let

$$\Delta_b = \{e^{q\alpha x_1 + 2\pi i(q\theta x_1 + x_2)} : (x_1, x_2) \in \Delta, q \in \mathbb{N}\}.$$

Then by checking equation (1) in Proposition 3.1 we see that  $\Delta_b$  has the required properties. □

We also have the following result equivalent to the above result.

**Corollary 3.3.** *If  $a \in \mathbb{C}$  and  $|a| > 1$ , then there is a dense set  $\Delta_a \subseteq \mathbb{D}$  such that for any  $b \in \Delta_a$ , we have  $\{a^n b^k : n, k \in \mathbb{N}\}$  is dense in  $\mathbb{C}$ .*

*Proof.* Given  $a$  with  $|a| > 1$ , let  $b = 1/a$  and then by the previous Corollary we get a set  $\Delta_b$  such that if  $c \in \Delta_b$ , then  $\{c^n b^k : n, k \in \mathbb{N}\} = \{c^n (1/a)^k : n, k \in \mathbb{N}\}$  is dense in  $\mathbb{C}$ . Then it follows that  $\{(1/c)^n a^k : n, k \in \mathbb{N}\}$  is dense in  $\mathbb{C}$ . Thus, let  $\Delta_a = \{1/c : c \in \Delta_b\}$ . □

The next theorem is a natural generalization of Proposition 3.1. Kerchy [18] has independently shown the existence of supercyclic tuples of diagonal matrices on  $\mathbb{C}^n$ . His techniques can also be used to construct hypercyclic tuples on  $\mathbb{C}^n$ . The methods used here are different.

**Theorem 3.4.** *For each  $n \geq 1$ , there exists a hypercyclic  $(n + 1)$ -tuple of diagonal matrices on  $\mathbb{C}^n$ .*

*Proof.* Fix an integer  $p \geq 1$  and we will construct a hypercyclic  $(p + 1)$ -tuple of diagonal matrices on  $\mathbb{C}^p$ . As in Proposition 3.1, let  $f : [0, 1] \rightarrow [0, 1]$  be defined by  $f(x) = \text{frac}(10x)$  and for an integer  $n$  let  $F_n = f \times f \times f \times \cdots \times f$  ( $n$  times). Thus  $F_n$  maps  $[0, 1]^n \rightarrow [0, 1]^n$  as follows:

$$F_n(x_1, x_2, \dots, x_n) = (f(x_1), f(x_2), \dots, f(x_n)).$$

As in Proposition 3.1, since  $f$  is mixing, it follows that  $F_n$  is topologically transitive.

Let  $x = \{x_j\}_{j=1}^{2p} \in [0, 1]^{2p}$  be a point that has dense orbit under  $F_{2p}$ . Also choose finite sets  $\{\alpha_j\}_{j=1}^p$  of positive real numbers,  $\{q_j\}_{j=1}^p$  of positive integers, and  $\{\theta_j\}_{j=1}^p \subseteq (0, 1]$ .

For  $1 \leq j \leq p$ , let

$$(2) \quad a_j = \exp[q_j \alpha_j x_{2j-1} + 2\pi i(q_j \theta_j x_{2j-1} + x_{2j})] \text{ and let } b_j = \exp(-\alpha_j - 2\pi i \theta_j)$$

where  $\exp(z) = e^z$  is the exponential function.

*Claim:* For each  $p \geq 1$ ,  $\left\{ \begin{bmatrix} a_1^n b_1^{k_1} \\ a_2^n b_2^{k_2} \\ \vdots \\ a_p^n b_p^{k_p} \end{bmatrix} : n, k_i \in \mathbb{N} \right\}$  is dense in  $\mathbb{C}^p$ .



is dense in  $\mathbb{C}^p$ .

By taking the exponential of each coordinate in the previous set we get:

$$\left\{ \begin{bmatrix} \exp [10^n (q_1 \alpha_1 x_1 + q_1 2\pi i \theta_1 x_1 + 2\pi i x_2)] \exp [-q_1 k_1 (\alpha_1 + 2\pi i \theta_1)] \exp [2\pi i m_1] \\ \exp [10^n (q_2 \alpha_2 x_3 + q_2 2\pi i \theta_2 x_3 + 2\pi i x_4)] \exp [-q_2 k_2 (\alpha_2 + 2\pi i \theta_2)] \exp [2\pi i m_2] \\ \vdots \\ \exp [10^n (q_p \alpha_p x_{2p-1} + q_p 2\pi i \theta_p x_{2p-1} + 2\pi i x_{2p})] \exp [-q_p k_p (\alpha_p + 2\pi i \theta_p)] \exp [2\pi i m_p] \end{bmatrix} : n, k_i \in \mathbb{N}, m_i \in \mathbb{Z} \right\}$$

is dense in  $\mathbb{C}^p$ . Or simplifying gives

$$\left\{ \begin{bmatrix} (\exp [q_1 \alpha_1 x_1 + 2\pi i (q_1 \theta_1 x_1 + x_2)])^{10^n} (\exp [-(\alpha_1 + 2\pi i \theta_1)])^{q_1 k_1} \\ (\exp [q_2 \alpha_2 x_3 + 2\pi i (q_2 \theta_2 x_3 + x_4)])^{10^n} (\exp [-(\alpha_2 + 2\pi i \theta_2)])^{q_2 k_2} \\ \vdots \\ (\exp [q_p \alpha_p x_{2p-1} + 2\pi i (q_p \theta_p x_{2p-1} + x_{2p})])^{10^n} (\exp [-(\alpha_p + 2\pi i \theta_p)])^{q_p k_p} \end{bmatrix} : n, k_i \in \mathbb{N} \right\}$$

is dense in  $\mathbb{C}^p$ . Thus using our definitions of  $a_i, b_i$  we see that,

$$\left\{ \begin{bmatrix} a_1^n b_1^{k_1} \\ a_2^n b_2^{k_2} \\ \vdots \\ a_p^n b_p^{k_p} \end{bmatrix} : n, k_i \in \mathbb{N} \right\}$$

is dense in  $\mathbb{C}^p$ .

So if we let

$$(3) \quad A = \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_p \end{bmatrix}, B_1 = \begin{bmatrix} b_1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1 & & & \\ & b_2 & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{bmatrix}, \dots, B_p = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & & b_p \end{bmatrix}, \text{ and } v = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

then  $(A, B_1, B_2, \dots, B_p)$  is a hypercyclic  $(p + 1)$ -tuple of diagonal matrices on  $\mathbb{C}^p$  with  $v$  as the hypercyclic vector. In fact, the hypercyclic vectors for the tuple are those vectors where each coordinate is non-zero.  $\square$

**Corollary 3.5.** *If  $b = \{b_j\}_{j=1}^p \subseteq \mathbb{D} \setminus \{0\}$ , then there is a dense set  $\Delta_b \subseteq \{z \in \mathbb{C}^p : |z_j| > 1 \text{ for all } j\}$  such that for any  $a \in \Delta_b$ , we have the tuple  $(A, B_1, B_2, \dots, B_p)$  is hypercyclic on  $\mathbb{C}^p$  where the matrices  $A$  and  $B_j$  are given above in equation (3).*

*Proof.* Keeping the notation from the previous proof. Let  $\Delta = \{(x_1, \dots, x_{2p}) : (x_1, \dots, x_{2p}) \text{ has dense orbit under } F_{2p}\}$ . Then  $\Delta$  is a dense  $G_\delta$  set in  $[0, 1]^{2p}$ . Suppose we are given  $b = \{b_j\}_{j=1}^p \subseteq \mathbb{D} \setminus \{0\}$ . Then define  $\alpha_j = -\ln |b_j|$  and  $\theta_j = \frac{-1}{2\pi} \arg(b_j)$  where  $\arg(b_j)$  denotes the value of the argument of  $b_j$  that lies in

$(0, 2\pi]$ . Now let

$$\Delta_b = \left\{ \left[ \begin{array}{c} \exp [q_1 \alpha_1 x_1 + 2\pi i (q_1 \theta_1 x_1 + x_2)] \\ \exp [q_2 \alpha_2 x_3 + 2\pi i (q_2 \theta_2 x_3 + x_4)] \\ \vdots \\ \exp [q_p \alpha_p x_{2p-1} + 2\pi i (q_p \theta_p x_{2p-1} + x_{2p})] \end{array} \right] : (x_1, \dots, x_{2p}) \in \Delta, q_j \in \mathbb{N} \right\}.$$

Then by checking equation (2) in Theorem 3.4 we see that  $\Delta_b$  has the required properties. Thus using our definitions of  $a_i, b_i$  we have that,

$$\left\{ \left[ \begin{array}{c} a_1^n b_1^{k_1} \\ a_2^n b_2^{k_2} \\ \vdots \\ a_p^n b_p^{k_p} \end{array} \right] : n, k_j \in \mathbb{N} \right\} \text{ is dense in } \mathbb{C}^p.$$

Hence the diagonal tuple  $(A, B_1, \dots, B_p)$  is hypercyclic.  $\square$

We now show that the previous result is the best possible in the sense that we cannot reduce the size of the tuple.

**Theorem 3.6.** *There does not exist a hypercyclic  $n$ -tuple of diagonalizable matrices on  $\mathbb{C}^n$ .*

*Proof.* To keep the notation simple we will prove this for  $n = 3$ ; this case contains all the main ideas. By way of contradiction, assume that there exists a hypercyclic 3-tuple of diagonalizable matrices on  $\mathbb{C}^3$ . Then the tuple is simultaneously diagonalizable. Hence we may assume that there is a hypercyclic 3-tuple of diagonal matrices on  $\mathbb{C}^3$ , call them  $(A, B, C)$ . Suppose that

$$A = \begin{bmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{bmatrix}, B = \begin{bmatrix} b_1 & & \\ & b_2 & \\ & & b_3 \end{bmatrix}, C = \begin{bmatrix} c_1 & & \\ & c_2 & \\ & & c_3 \end{bmatrix}.$$

Suppose that  $v = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$  is the corresponding hypercyclic vector. Then

$$\left\{ \left[ \begin{array}{c} a_1^n b_1^k c_1^l \alpha \\ a_2^n b_2^k c_2^l \beta \\ a_3^n b_3^k c_3^l \gamma \end{array} \right] : n, k, l \geq 0 \right\}$$

is dense in  $\mathbb{C}^3$ . Clearly we must have  $\alpha \neq 0, \beta \neq 0, \gamma \neq 0$ . Hence by applying the invertible matrix  $\begin{bmatrix} \alpha^{-1} & & \\ & \beta^{-1} & \\ & & \gamma^{-1} \end{bmatrix}$  to the above dense set we also have that

$$\left\{ \left[ \begin{array}{c} a_1^n b_1^k c_1^l \\ a_2^n b_2^k c_2^l \\ a_3^n b_3^k c_3^l \end{array} \right] : n, k, l \geq 0 \right\}$$

is dense in  $\mathbb{C}^3$ .

Now apply the function  $\log |z|$  to each coordinate and we get that

$$(4) \quad \left\{ \begin{bmatrix} n \log |a_1| + k \log |b_1| + l \log |c_1| \\ n \log |a_2| + k \log |b_2| + l \log |c_2| \\ n \log |a_3| + k \log |b_3| + l \log |c_3| \end{bmatrix} : n, k, l \geq 0 \right\} \text{ is dense in } \mathbb{R}^3.$$

Let

$$S := \left\{ \begin{bmatrix} n \\ k \\ l \end{bmatrix} : n, k, l \geq 0 \right\} \text{ and } T := \begin{bmatrix} \log |a_1| & \log |b_1| & \log |c_1| \\ \log |a_2| & \log |b_2| & \log |c_2| \\ \log |a_3| & \log |b_3| & \log |c_3| \end{bmatrix}$$

then  $T$  is a  $3 \times 3$  real matrix and Equation (4) implies that  $T(S)$  is a dense set in  $\mathbb{R}^3$ . Thus  $T$  has dense range, hence  $T$  is onto, thus  $T$  is invertible. However, since  $T$  is invertible and  $T(S)$  is dense in  $\mathbb{R}^3$ , then  $S = T^{-1}(T(S))$  is also dense in  $\mathbb{R}^3$ . However clearly  $S$  is not dense in  $\mathbb{R}^3$ , thus we have a contradiction. So, there are no hypercyclic diagonalizable 3-tuples on  $\mathbb{C}^3$ . And in general there are no hypercyclic diagonalizable  $n$ -tuples on  $\mathbb{C}^n$ .  $\square$

**Corollary 3.7.** *There do not exist positive real numbers  $a_i, b_i, c_i, 1 \leq i \leq 3$ , such that*

$$\left\{ \begin{bmatrix} a_1^n b_1^k c_1^l \\ a_2^n b_2^k c_2^l \\ a_3^n b_3^k c_3^l \end{bmatrix} : n, k, l \in \mathbb{R}^+ \right\} \text{ is dense in } (\mathbb{R}^+)^3.$$

*Proof.* Following the proof of Theorem 3.6, take the logarithm of each coordinate, get the matrix  $T$  and the set  $S$ . In this case, the set  $S$  is not a discrete lattice, but  $S$  is essentially the first octant in  $\mathbb{R}^3$ . However the important point is that  $S$  is not dense in  $\mathbb{R}^3$ , so  $T(S)$  cannot be dense either.  $\square$

**Corollary 3.8.** *If  $\mathcal{H}$  is an infinite dimensional Hilbert space and  $n \geq 1$ , then there does not exist a hypercyclic normal  $n$ -tuple on  $\mathcal{H}$ .*

*Proof.* By way of contradiction, suppose that  $A = (A_1, A_2, \dots, A_n)$  is a hypercyclic normal  $n$ -tuple on  $\mathcal{H}$ . Then the tuple  $A$  is cyclic. Thus  $A$  is unitarily equivalent to a tuple of multiplication operators. More precisely, there is a finite positive Borel measure  $\mu$  on a compact set in  $\mathbb{C}^n$  so that  $A$  is unitarily equivalent to the  $n$ -tuple,  $M = (M_{z_1}, M_{z_2}, \dots, M_{z_n})$ , of multiplication operators on  $L^2(\mu)$  by the coordinate functions. Let  $\{f_j\}_{j=1}^\infty$  be a countable set of functions that are continuous on the support of  $\mu$  and dense in  $L^2(\mu)$ . Suppose that  $\phi \in L^2(\mu)$  is a hypercyclic vector for the tuple  $M$ . Then for each  $j$  there is a sequence of multi-indices  $n_{k,j}$  such that  $M^{n_{k,j}} \phi \rightarrow f_j$  in  $L^2(\mu)$  as  $k \rightarrow \infty$  and by passing to a subsequence we may also assume that  $\mu$ -almost everywhere convergence holds. Let  $\Delta_j$  be a set of full  $\mu$  measure so that  $(M^{n_{k,j}} \phi)(z) \rightarrow f_j(z)$  as  $k \rightarrow \infty$  for every  $z \in \Delta_j$ . Let  $\Delta = \bigcap_{j=1}^\infty \Delta_j$ , then  $\Delta$  also has full measure. Since  $\mathcal{H}$  and hence  $L^2(\mu)$  is infinite dimensional, then  $\Delta$  must be an infinite set. Let  $w_1, \dots, w_n$  be  $n$  distinct points in  $\Delta$ . Then

$$(M^{n_{k,j}} \phi)(w_i) \rightarrow f_j(w_i) \text{ as } k \rightarrow \infty$$

for each  $j$  and each  $i$ .

Since  $\{f_j\}$  is dense in  $L^2(\mu)$ , then the set

$$\left\{ \begin{bmatrix} f_j(w_1) \\ \vdots \\ f_j(w_n) \end{bmatrix} : j \geq 1 \right\}$$

is dense in  $\mathbb{C}^n$  and from above we have that for each  $j \geq 1$ ,

$$\begin{bmatrix} (M^{n_{k,j}\phi})(w_1) \\ \vdots \\ (M^{n_{k,j}\phi})(w_n) \end{bmatrix} \rightarrow \begin{bmatrix} f_j(w_1) \\ \vdots \\ f_j(w_n) \end{bmatrix} \text{ as } k \rightarrow \infty.$$

Now define  $n$  diagonal matrices  $B_1, \dots, B_n$  on  $\mathbb{C}^n$  as:

$$B_i = \begin{bmatrix} w_1(i) & & & \\ & w_2(i) & & \\ & & \ddots & \\ & & & w_n(i) \end{bmatrix}$$

where  $w_j(i)$  denotes the  $i^{\text{th}}$  coordinate of the vector  $w_j$ .

Also define a vector  $v = (\phi(w_1), \dots, \phi(w_n))$ , then the tuple  $B = (B_1, \dots, B_n)$  is a hypercyclic  $n$ -tuple of diagonal matrices on  $\mathbb{C}^n$  with hypercyclic vector  $v$ . However this contradicts Theorem 3.6. Thus there is no hypercyclic  $n$ -tuple of normal operators on an infinite dimensional Hilbert space.  $\square$

The definition of a “subnormal tuple” is a commuting tuple of subnormal operators that have commuting normal extensions [7]. It’s known that there are commuting tuples of subnormal operators that do not have commuting normal extensions (see [20] or [8, p.79]). Thus one should be careful to distinguish between “subnormal tuple” and “a commuting tuple of subnormal operators”. The next corollary applies to subnormal tuples.

**Corollary 3.9.** *On an infinite dimensional Hilbert space, there is no hypercyclic  $n$ -tuple of subnormal operators with commuting normal extensions.*

*Proof.* The proof is exactly the same as in the normal case except one uses the fact that a commuting cyclic tuple of subnormal operators that has a commuting tuple of normal extensions is unitarily equivalent to a tuple of multiplication operators by the coordinate functions on  $P^2(\mu)$ , where  $P^2(\mu)$  denotes the closure of the analytic polynomials in  $L^2(\mu)$  and  $\mu$  is a compactly support regular Borel measure in  $\mathbb{C}^n$  ([7]). Now proceed as above with  $\{f_j\}$  being the set of all polynomials with rational coefficients.  $\square$

Bourdon [4] gave a nice proof that a hyponormal operator on a Hilbert space cannot be supercyclic. The following question if answered affirmatively would give a natural extension of Bourdon’s supercyclicity result (recall Example 2.10).

**Question 3.10.** *Is there a hypercyclic tuple of commuting hyponormal operators on an infinite dimensional Hilbert space?*

## 4. REAL HILBERT SPACES: SOMEWHERE DENSE ORBITS THAT ARE NOT DENSE!

An operator  $T$  on a space  $X$  is said to be *multi-hypercyclic* if there is a finite set of vectors  $\{x_1, \dots, x_k\} \subseteq X$  such that  $\bigcup_{k=1}^n \text{Orb}(T, x_i)$  is dense in  $X$ . Herrero [15] conjectured that a multi-hypercyclic operator would be hypercyclic. His conjecture was established independently by Costakis [9] and Peris [22]. Later Bourdon and Feldman [5] proved that if a bounded linear operator on a (real or complex) locally convex space has a somewhere dense orbit, then that orbit must actually be dense. It was well known that this fact can be used to give a simple proof of Herrero's conjecture.

The same definition of multi-hypercyclic applies to a tuple  $T$ . Namely that a tuple  $T$  is multi-hypercyclic if there are a finite number of orbits under  $T$  whose union is dense. We will say that  $T$  is  $n$ -hypercyclic if there are  $n$  orbits for  $T$  whose union is dense in  $X$ .

In this section we will show that on Hilbert spaces over the field of real numbers, there are tuples of operators that have a somewhere dense orbit that is not dense! Furthermore these tuples of operators are multi-hypercyclic but not hypercyclic! Furthermore, examples are given of tuples  $T$  that are hypercyclic, but  $T^n$  is not hypercyclic!

**Theorem 4.1.** *If  $a, b > 1$  and  $\frac{\ln(a)}{\ln(b)}$  is irrational, then  $\left\{ \frac{a^n}{b^k} : n, k \in \mathbb{N} \right\}$  is dense in  $\mathbb{R}^+$ .*

*Proof.* The one-dimensional version of Kronecker's theorem states: If  $\theta$  is a positive irrational number, then  $\{n\theta - k : n, k \in \mathbb{N}\}$  is dense in  $\mathbb{R}$ , see [14, Theorem 438, p. 375]. Applying Kronecker's Theorem with  $\theta = \ln(a)/\ln(b)$  gives that

$$\left\{ n \frac{\ln(a)}{\ln(b)} - k : n, k \in \mathbb{N} \right\}$$

is dense in  $\mathbb{R}$ . Multiplying by  $\ln(b)$  gives that

$$\{n \ln(a) - k \ln(b) : n, k \in \mathbb{N}\}$$

is dense in  $\mathbb{R}$ . Simplifying gives that

$$\left\{ \ln \left( \frac{a^n}{b^k} \right) : n, k \in \mathbb{N} \right\}$$

is dense in  $\mathbb{R}$ . Thus by taking the exponential of the above set we see that

$$\left\{ \frac{a^n}{b^k} : n, k \in \mathbb{N} \right\}$$

is dense in  $\mathbb{R}^+$ . □

**Corollary 4.2.** *If  $a, b > 1$  are relatively prime integers, then  $\left\{ \frac{a^n}{b^k} : n, k \in \mathbb{N} \right\}$  is dense in  $\mathbb{R}^+$ .*

*Proof.* We simply need to show that  $\ln(a)/\ln(b)$  is irrational. By way of contradiction, assume that  $\ln(a)/\ln(b)$  is rational. Then there are integers  $p, q \in \mathbb{N}$  such that  $\ln(a)/\ln(b) = p/q$ . So,  $q \ln(a) = p \ln(b)$  or equivalently,  $a^q = b^p$ . If  $a$  and  $b$  are relatively prime, then the Fundamental Theorem of arithmetic implies that  $a$  and  $b$  cannot be powers of each other. Thus we have a contradiction. So  $\ln(a)/\ln(b)$  is irrational. Thus Theorem 4.1 applies. □

**Example 4.3** (Multi-Hypercyclic Tuples of Matrices on  $\mathbb{R}^n$ ).

- (1) Let  $I$  be the identity operator on the real Hilbert space  $\mathbb{R}$  and let  $v_1 = 1$  and  $v_2 = -1$ . If we let  $T = (T_1, T_2)$  where  $T_1 = 2I$  and  $T_2 = \frac{1}{3}I$ , then by Corollary 4.2  $clOrb(T, 1) = [0, \infty)$  and  $clOrb(T, -1) = (-\infty, 0]$ . Hence  $T$  has somewhere dense orbits that are not dense.

Furthermore  $Orb(T, 1) \cup Orb(T, -1)$  is dense in  $\mathbb{R}$ , but  $T$  is not hypercyclic. Thus  $T$  is multi-hypercyclic, but not hypercyclic.

- (2) Let

$$T_1 = \begin{bmatrix} 2 & \\ & 1 \end{bmatrix}, T_2 = \begin{bmatrix} 1/3 & \\ & 1 \end{bmatrix}, T_3 = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}, T_4 = \begin{bmatrix} 1 & \\ & 1/3 \end{bmatrix},$$

$T = (T_1, T_2, T_3, T_4)$ , and  $v_{i,j} = \begin{bmatrix} (-1)^i \\ (-1)^j \end{bmatrix}$  where  $i, j \in \{0, 1\}$ . Then the vectors  $v_{0,0}, v_{1,0}, v_{1,1}, v_{0,1}$  lie in the first, second, third, and fourth quadrants of  $\mathbb{R}^2$  respectively. Furthermore,  $Orb(T, v_{i,j})$  is dense in the quadrant that contains  $v_{i,j}$ . Thus we see that  $T$  has somewhere dense orbits that are not dense in  $\mathbb{R}^2$  and that  $T$  is multi-hypercyclic but not hypercyclic. In fact,  $T$  is 4-hypercyclic, but not 3-hypercyclic (since each of the 4 quadrants are invariant sets for  $T$  with disjoint interiors).

- (3) Generalizing the previous example, it is easy to construct a  $2n$ -tuple on  $\mathbb{R}^n$  that has somewhere dense orbits that are not dense and is  $2^n$ -hypercyclic.

To see this simply follow the example above: For each  $1 \leq i \leq n$ , let  $A_i$  be the  $n \times n$  diagonal matrix with ones on the main diagonal except in the  $(i, i)$  position which is 2. Also let  $B_i$  be the  $n \times n$  diagonal matrix with ones on the main diagonal except in the  $(i, i)$  position which is  $1/3$ . Then  $T = (A_1, \dots, A_n, B_1, \dots, B_n)$  has the required properties.

**Theorem 4.4.** *There is an  $(n + 1)$ -tuple of diagonal matrices on  $\mathbb{R}^n$  that has an orbit dense in  $(\mathbb{R}^+)^n$ . However, there is no  $n$ -tuple of diagonalizable matrices on  $\mathbb{R}^n$  or  $\mathbb{C}^n$  that has a somewhere dense orbit.*

*Proof.* By taking the absolute values of each entry in the diagonal matrices constructed in Theorem 3.4 we see that there is an  $(n + 1)$ -tuple of diagonal matrices on  $\mathbb{R}^n$  that has an orbit that is dense in  $(\mathbb{R}^+)^n$ . To see that there is no  $n$ -tuple with a somewhere dense orbit on  $\mathbb{R}^n$  or  $\mathbb{C}^n$ , follow the reasoning in the proof of Theorem 3.6.  $\square$

The example and theorem above are related to work of Bermudez, Bonilla, and Peris [2], where they show that if an operator  $T$  on a Banach space  $X$  is  $\mathbb{R}$ -supercyclic, then it must be  $\mathbb{R}^+$ -supercyclic. Also see Corollary 5.9 where another proof of this fact is given. The above examples can be modified to give tuples of matrices that are  $\mathbb{R}$ -supercyclic but not  $\mathbb{R}^+$ -supercyclic. In fact, the pair in (1) from Example 4.3 is such an example.

**Example 4.5** (A Multi-Hypercyclic Triple in Infinite Dimensions).

Let  $\ell_{\mathbb{R}}^2(\mathbb{N})$  denote the real Hilbert space of all real sequences that are square summable. Also let  $B$  denote the backward shift on  $\ell_{\mathbb{R}}^2(\mathbb{N})$ . Let  $\mathcal{H} = \mathbb{R} \oplus \ell_{\mathbb{R}}^2(\mathbb{N})$  and define operators  $T_1 = 2I_{\mathbb{R}} \oplus I_{\ell^2}$ ,  $T_2 = \frac{1}{3}I_{\mathbb{R}} \oplus I_{\ell^2}$ ,  $T_3 = I_{\mathbb{R}} \oplus 2B$ . Then  $T = (T_1, T_2, T_3)$  has a somewhere dense orbit that is not dense and is 2-hypercyclic, but not hypercyclic.

More precisely, if  $x$  is a hypercyclic vector for  $2B$  on  $\ell_{\mathbb{R}}^2(\mathbb{N})$  and we let  $v_1 = (1, x)$  and  $v_2 = (-1, x)$ , then  $\text{Orb}(T, v_1) \cup \text{Orb}(T, v_2)$  is dense in  $\mathcal{H}$ , yet neither orbit is dense in  $\mathcal{H}$ .

**Example 4.6** (A Multi-Hypercyclic Pair in Infinite Dimensions).

On  $\mathcal{H} = \mathbb{R} \oplus \ell_{\mathbb{R}}^2(\mathbb{N})$  define operators  $T_1 = 2I_{\mathbb{R}} \oplus I_{\ell^2}$ ,  $T_2 = \frac{1}{3}I_{\mathbb{R}} \oplus 2B$ . Then  $T = (T_1, T_2)$  is 2-hypercyclic, but not hypercyclic.

*Proof.* We simply use the fact that  $2B$  is a mixing operator and show that the pair  $(T_1, T_2)$  is transitive on each of the two closed invariant subsets:  $\mathbb{R}^+ \oplus \ell_{\mathbb{R}}^2(\mathbb{N})$  and  $\mathbb{R}^- \oplus \ell_{\mathbb{R}}^2(\mathbb{N})$ , where  $\mathbb{R}^+ = [0, \infty)$  and  $\mathbb{R}^- = (-\infty, 0]$ . So, let  $U_1, V_1$  be two non-empty open sets in  $\mathbb{R}^+$  and let  $U_2, V_2$  be two non-empty open sets in  $\ell_{\mathbb{R}}^2(\mathbb{N})$ . We must find integers  $n, k \geq 0$  so that  $T_1^n T_2^k(U_1 \oplus U_2) \cap (V_1 \oplus V_2) \neq \emptyset$ . Notice that  $T_1^n T_2^k = \frac{2^n}{3^k} I_{\mathbb{R}} \oplus (2B)^k$ . Since  $2B$  is a mixing operator, there exists a  $K \geq 0$  such that  $(2B)^k(U_2) \cap V_2 \neq \emptyset$  for all  $k \geq K$ . Since  $U_1$  and  $U_2$  are both open sets of non-negative real numbers and since  $\{2^n/3^k : n, k \geq 0\}$  is dense in  $\mathbb{R}^+$ , then there exists a  $k \geq K$  and an  $n \geq 0$  such that  $\frac{2^n}{3^k} U_1 \cap V_1 \neq \emptyset$ . With this choice of  $n$  and  $k$  we have that  $T_1^n T_2^k(U_1 \oplus U_2) \cap (V_1 \oplus V_2) \neq \emptyset$ .  $\square$

Ansari [1] gave a truly original proof to show that if  $T$  is a hypercyclic operator, then  $T^n$  is also hypercyclic for every positive integer  $n$ . This also follows from the results of Costakis [9], Peris [22] and Bourdon & Feldman [5], since  $T^n$  is multi-hypercyclic whenever  $T$  is hypercyclic. It is therefore natural to ask if a similar result holds for tuples of operators.

If  $T = (T_1, T_2)$  is a commuting pair of operators and  $n = (n_1, n_2)$  is a pair of non-negative integers (a multi-index), then we define  $T^n$  to be the pair  $(T_1^{n_1}, T_2^{n_2})$ . In view of Ansari's Theorem it is natural to ask if  $T$  is a hypercyclic pair and  $n$  is a multi-index, then must  $T^n$  be hypercyclic? The following example shows that it need not be true. However, see Corollary 5.8 for a positive result along these lines.

**Example 4.7** (Powers of a Hypercyclic Pair).

Slightly different than the previous example, define  $T_1 = -2I_{\mathbb{R}} \oplus I_{\ell^2}$ ,  $T_2 = \frac{1}{3}I_{\mathbb{R}} \oplus 2B$  on  $\mathcal{H} = \mathbb{R} \oplus \ell_{\mathbb{R}}^2(\mathbb{N})$ . Then  $T = (T_1, T_2)$  is hypercyclic on  $\mathcal{H}$ . However, if  $n = (2, 1)$ , then  $T^n = (T_1^2, T_2)$  is not hypercyclic on  $\mathcal{H}$ .

*Proof.* This follows since the first coordinates of any term in a given orbit must all have the same sign, and hence cannot be dense in  $\mathbb{R}$ .  $\square$

## 5. WHEN SOMEWHERE DENSE ORBITS ARE EVERYWHERE DENSE

In this section we will give some sufficient conditions for a somewhere dense orbit to be dense. These apply to both real and complex spaces. In view of the results in the previous section, some extra conditions are necessary. As above,  $T = (T_1, T_2, \dots, T_n)$  will denote a (commuting)  $n$ -tuple of operators and  $\mathcal{F} = \mathcal{F}_T$  will be the semigroup they generate. Also, in this section,  $X$  will denote a real or complex locally convex space unless otherwise stated.

Define  $W$  as the following set of polynomials in  $n$ -variables,

$$W = \{p : p(z_1, z_2, \dots, z_n) = \lambda z_1^{k_1} z_2^{k_2} \dots z_n^{k_n} + \mu \text{ where } \lambda, \mu \in \mathbb{C}, k_i \geq 0\}.$$

Notice that  $\{p(T) : p \in W\} = \{\lambda A + \mu I : A \in \mathcal{F}, \lambda, \mu \in \mathbb{C}\}$ .

**Lemma 5.1.** *Let  $T = (T_1, T_2, \dots, T_n)$  be an  $n$ -tuple of commuting operators on  $X$ . If  $x \in X$  and  $U$  is the interior of the closure of  $\text{Orb}(T, x)$  and  $T_i$  does not have dense range for some  $i \in \{1, \dots, n\}$ , then  $U$  is also equal to the interior of the closure of  $\text{Orb}(\{T_j\}_{j \neq i}, x)$ .*

*Proof.* Suppose that  $T_i$  does not have dense range for some  $i \in \{1, \dots, n\}$ . Let  $R(T_i)$  denote the closure of the range of  $T_i$ . Since  $T_i$  does not have dense range, then  $R(T_i)$  is a proper closed subspace of  $X$ , hence it is nowhere dense in  $X$ . If  $k_j \geq 0$  for  $j \neq i$  and  $k_i \geq 1$ , then  $T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \in R(T_i)$ . Thus we have that  $\text{Orb}(T, x) \subseteq \text{Orb}(\{T_j\}_{j \neq i}, x) \cup R(T_i)$ . So,  $U \subseteq \text{clOrb}(T, x) \subseteq \text{clOrb}(\{T_j\}_{j \neq i}, x) \cup R(T_i)$ . Thus,  $U \setminus R(T_i) \subseteq \text{clOrb}(\{T_j\}_{j \neq i}, x)$ . Thus,  $U \subseteq \text{cl}(U \setminus R(T_i)) \subseteq \text{clOrb}(\{T_j\}_{j \neq i}, x)$ . So we have that  $U \subseteq \text{int}[\text{clOrb}(\{T_j\}_{j \neq i}, x)] \subseteq \text{int}[\text{clOrb}(T, x)] = U$ , so  $U = \text{int}[\text{clOrb}(\{T_j\}_{j \neq i}, x)]$ .  $\square$

**Lemma 5.2.** *If  $T = (T_1, T_2, \dots, T_n)$  is an  $n$ -tuple of commuting operators on  $X$  and if there exists an  $x \in X$  such that  $\text{Orb}(T, x)$  is somewhere dense in  $X$  and  $x \in \text{int}[\text{clOrb}(T, x)]$ , then  $\{p(T)x : p \in W\}$  is dense in  $X$ .*

*Proof.* Let  $U$  be the interior of the closure of  $\text{Orb}(T, x)$ . Then  $U - \{x\} := \{y - x : y \in U\} \subseteq \text{cl}\{(A - I)x : A \in \mathcal{F}_T\} = \text{cl}\{p(T)x : p = (z_1^{k_1} \dots z_n^{k_n} - 1), k_i \geq 0\}$ . Since  $U - \{x\}$  is an open set containing zero, then the set of all multiples of  $U - \{x\}$  is dense in  $X$ . Now since  $\{p(T)x : p \in W\}$  is invariant under scalar multiplication and  $U - \{x\} \subseteq \{p(T)x : p \in W\}$ , then  $\{p(T)x : p \in W\}$  is dense in  $X$ .  $\square$

**Lemma 5.3.** *Let  $T = (T_1, T_2, \dots, T_n)$  be an  $n$ -tuple of commuting operators on  $X$  such that  $T_k$  has dense range for each  $1 \leq k \leq n$ . Suppose that  $x \in X$  and  $U$  is the interior of the closure of  $\text{Orb}(T, x)$ . Suppose also that for each  $i$  and for each  $k \geq 0$ ,  $\text{Orb}(\{T_j\}_{j \neq i}, T_i^k x)$  is nowhere dense. If  $A = T_1^{p_1} T_2^{p_2} \dots T_n^{p_n} \in \mathcal{F}_T$ , then  $U$  is equal to the interior of the closure of  $\text{Orb}(T, Ax) = \{T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x : k_i \geq p_i\}$ .*

*Proof.* If  $\text{Orb}(T, x)$  is dense in  $X$ , then  $U = X$  and we must show that  $\text{Orb}(T, Ax)$  is also dense in  $X$ . However this follows easily since  $\text{Orb}(T, Ax) = A(\text{Orb}(T, x))$  and  $A$  has dense range. So, we may assume that  $\text{Orb}(T, x)$  is not dense in  $X$  and that  $U$  is nonempty. Notice that  $\text{Orb}(T, x) =$

$$\begin{aligned} &= \{T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x : k_i \geq p_i\} \cup \{T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x : k_i \geq 0 \text{ \& \ } \exists i \text{ s.t. } k_i < p_i\} = \\ &= \text{Orb}(T, Ax) \cup \{T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x : k_i \geq 0 \text{ \& \ } \exists i \text{ s.t. } k_i < p_i\}. \end{aligned}$$

*Claim:*  $E := \{T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x : k_i \geq 0 \text{ \& \ } \exists i \text{ s.t. } k_i < p_i\}$  is nowhere dense.

To verify the Claim, notice that  $E = \bigcup_{i=1}^n \bigcup_{k=0}^{p_i-1} (\text{Orb}(\{T_j\}_{j \neq i}, T_i^k(x)))$ . By assumption,  $\text{Orb}(\{T_j\}_{j \neq i}, T_i^k(x))$  is nowhere dense for each  $i$  and for each  $k \geq 0$ , thus  $E$  is a finite union of nowhere dense sets, hence  $E$  itself is also nowhere dense in  $X$ . It now follows easily that  $U \subseteq \text{clOrb}(T, Ax)$ . Thus,  $U \subseteq \text{int}[\text{clOrb}(T, Ax)] \subseteq \text{int}[\text{clOrb}(T, x)] = U$ . So,  $U$  is equal to the interior of the closure of  $\text{Orb}(T, Ax)$ .  $\square$

The next lemma is similar to one from Bourdon and Feldman [5]. If  $T = (T_1, T_2, \dots, T_n)$  is a tuple of operators on  $X$  and  $E \subseteq X$ , then we say that  $E$  is invariant for  $T$  if  $T_i(E) \subseteq E$  for each  $1 \leq i \leq n$ .

**Lemma 5.4.** *Let  $T = (T_1, T_2, \dots, T_n)$  be an  $n$ -tuple of commuting operators on  $X$  such that  $T_k$  has dense range for each  $1 \leq k \leq n$ . If  $x \in X$  and  $U$  is the interior of the closure of  $\text{Orb}(T, x)$  and if  $\text{Orb}(\{T_j\}_{j \neq i}, T_i^k(x))$  is nowhere dense for each  $i \in \{1, \dots, n\}$  and for each  $k \geq 0$ , then  $X \setminus U$  is invariant under  $T$ .*

*Proof.* Let  $F$  denote the closure of  $\text{Orb}(T, x)$  and  $U$  the interior of  $F$ . Observe that  $F$  is invariant under  $T$ .

If  $U$  is empty or  $U = X$ , then there is nothing to prove. Suppose that  $U$  is nonempty and  $U \neq X$ . Choose an  $A \in \mathcal{F}_T$  such that  $Ax$  belongs to  $U$  and set  $x_0 = Ax$ . Thus  $x_0 \in U$ . By Lemma 5.3,  $U$  is contained in the closure of  $\text{Orb}(T, x_0) = \text{Orb}(T, Ax)$ . Since  $x_0 \in U$ , it follows that  $x_0$  is a limit point of  $\text{Orb}(T, x_0)$  and that  $U$  is equal to the interior of the closure of  $\text{Orb}(T, x_0)$ . Since  $x_0 \in U$ , Lemma 5.2, implies that  $\{p(T)x_0 : p \in W\}$  is dense in  $X$ .

Suppose, in order to obtain a contradiction, that there exists an  $i \in \{1, \dots, n\}$  such that  $T_i$  maps some point  $y \in X \setminus U$  into  $U$ . Without loss of generality, we may assume  $y \in X \setminus F$ . Because,  $X \setminus F$  is open and  $\{p(T)x_0 : p \in W\}$  is dense in  $X$ , we may find a polynomial  $p \in W$  so that  $p(T)x_0$  is close enough to  $y$  to ensure (1)  $p(T)x_0 \in X \setminus F$  and (2)  $T_i(p(T)x_0) \in U$ . Because  $U$  is contained in the closed  $T$ -invariant set  $F$ , it follows that the closure of the orbit under  $T$  of  $T_i(p(T)x_0)$  belongs to  $F$ ; that is  $\text{Orb}(T, T_i p(T)x_0) \subseteq F$ . However,  $\text{Orb}(T, T_i p(T)x_0) = p(T)\text{Orb}(T, T_i x_0)$ . Now by Lemma 5.3,  $\text{intclOrb}(T, T_i x_0) = \text{intclOrb}(T, x_0)$  and  $x_0$  is a limit point of  $\text{intclOrb}(T, x_0)$ , hence also a limit point of  $\text{intclOrb}(T, T_i x_0)$ . This together with the continuity of  $p(T)$  yields  $p(T)x_0 \in F$ . Thus  $p(T)x_0 \in F$  and  $p(T)x_0 \in X \setminus F$ , a contradiction. Thus  $X \setminus U$  is invariant under  $T$ .  $\square$

Observe that the preceding lemma shows that if  $\text{Orb}(T, x)$  is somewhere dense, and if  $\text{Orb}(\{T_j\}_{j \neq i}, T_i^k(x))$  is nowhere dense for each  $1 \leq i \leq n$  and for each  $k \geq 0$ , then every element of  $\text{Orb}(T, x)$ , including  $x$  itself, belongs to the interior of the closure of  $\text{Orb}(T, x)$ .

**Theorem 5.5** (Somewhere Dense Theorem). *Suppose that  $T = (T_1, T_2, \dots, T_n)$  is an  $n$ -tuple of commuting operators on a locally convex space  $X$  over  $\mathbb{F}$  ( $= \mathbb{R}$  or  $\mathbb{C}$ ). If  $x \in X$  and  $\text{Orb}(T, x)$  is somewhere dense in  $X$ , then  $\text{Orb}(T, x)$  is dense in  $X$  provided one of the following holds:*

*If  $\mathbb{F} = \mathbb{C}$ :*

(a) *The set  $K := \mathbb{C} \setminus \bigcup \{\sigma_p(A^*) : A \in \mathcal{F}_T\}$  is nonempty and has a connected component that is unbounded;*

*or*

(b) *There is a cyclic operator  $B$  that commutes with  $T$  and satisfies  $\sigma_p(B^*)$  has no interior in  $\mathbb{C}$  and  $\mathbb{C} \setminus \sigma_p(B^*)$  is connected.*

*If  $\mathbb{F} = \mathbb{R}$ :*

(a')  *$\sigma_p(A^*) \subseteq [-1, 1]$  for each  $A \in \mathcal{F}_T$ ;*

*or*

(b') *There is a cyclic operator  $B$  that commutes with  $T$  and satisfies  $p(B)$  has dense range for every non-zero real polynomial  $p$ .*

*Proof.* We shall assume that  $\mathbb{F} = \mathbb{C}$  and that either (a) or (b) holds. The real versions are very similar. We shall proceed by induction on  $n$ . For  $n = 1$  the result follows from the Bourdon-Feldman Somewhere Dense Theorem (see [5]). So, now assume that  $n \geq 2$ . Then our induction hypothesis is that for any  $(n - 1)$ -tuple  $T'$  of commuting operators that satisfies either (a) or (b) above (or (a') or (b')) in the case where  $\mathbb{F} = \mathbb{R}$ ) that  $\text{Orb}(T', x)$  is either dense in  $X$  or nowhere dense in  $X$ , for every  $x \in X$ .

Now let  $T = (T_1, T_2, \dots, T_n)$  be a commuting  $n$ -tuple of operators on  $X$ . Let  $x \in X$  and let  $F$  denote the closure of  $\text{Orb}(T, x)$  and let  $U$  denote the interior of  $F$ . Suppose that  $U$  is nonempty and, by way of contradiction, that  $F$  is a proper subset of  $X$ .

**Claim 1:** For every  $i \in \{1, \dots, n\}$ ,  $T_i$  has dense range.

To prove the claim, suppose there exists an  $i$  such that  $T_i$  does not have dense range. Then by Lemma 5.1, we have that  $U$  is the interior of the closure of  $\text{Orb}(\{T_j\}_{j \neq i}, x)$ . But this contradicts our inductive assumption because  $\{T_j\}_{j \neq i}$  is a set of  $(n-1)$  commuting operators that satisfies either (a) or (b) and has an orbit that is somewhere dense in  $X$ , but not dense in  $X$ . So the claim holds.

**Claim 2:**  $(X \setminus U)$  is invariant under  $T$ ; that is  $T_i(X \setminus U) \subseteq (X \setminus U)$  for all  $i \in \{1, \dots, n\}$ .

Simply notice that for each  $i \in \{1, \dots, n\}$  and for each  $k \geq 0$ ,  $\text{Orb}(\{T_j\}_{j \neq i}, T_i^k(x)) \subseteq F$ , so  $\text{Orb}(\{T_j\}_{j \neq i}, T_i^k(x))$  is not dense in  $X$ . Thus by our inductive hypothesis,  $\text{Orb}(\{T_j\}_{j \neq i}, T_i^k(x))$  is nowhere dense in  $X$ . So by Claim 1 and Lemma 5.4 we get that  $X \setminus U$  is invariant under  $T$ .

**Claim 3:**  $Ax \in U$ , for every  $A \in \mathcal{F}_T$ .

Suppose not, then  $Ax \in (X \setminus U)$ . However, by Claim 2,  $X \setminus U$  is invariant under  $T$ , thus by Lemma 5.3 we get  $U \subseteq \text{clOrb}(T, Ax) \subseteq (X \setminus U)$ , clearly a contradiction (since  $U \neq \emptyset$ ). So, we must have that  $Ax \in U$ .

**Claim 4:** If  $B$  is an operator on  $X$  that has dense range and commutes with  $T_i$  for each  $i$ , and  $A \in \mathcal{F}_T$ , then  $B(Ax) \notin \partial U$ .

First notice that by Lemma 5.3,  $U$  is the interior of the closure of  $\text{Orb}(T, Ax)$ . Since, by Claim 3,  $F = \text{cl}U$ , we also have that  $F = \text{clOrb}(T, Ax)$ .

Now, by way of contradiction, suppose that  $B(Ax) \in \partial U$ . Because  $U$  is the interior of the closed set  $F$ ,  $X \setminus U$  is the closure of  $X \setminus F$ . Thus by Claim 3, Lemma 5.2 and the fact that  $A$  has dense range (Claim 1), we may choose a collection  $\mathcal{Q} \subseteq W$  of polynomials such that  $\{q(T)Ax : q \in \mathcal{Q}\}$  is a dense subset of  $X \setminus U$ . Since  $B$  has dense range,  $B$  must map the dense set  $D := U \cup \{q(T)Ax : q \in \mathcal{Q}\}$  to a dense set; however, we will show that  $B(D) \subseteq X \setminus U$ .

We'll first show that  $B(U) \subseteq (X \setminus U)$ . Since  $B(Ax) \in \partial U$ , then  $B(Ax) \in X \setminus U$  and by Claim 2,  $X \setminus U$  is invariant under  $T$ , thus  $\text{Orb}(T, B(Ax)) \subseteq (X \setminus U)$ , so  $\text{clOrb}(T, B(Ax)) \subseteq (X \setminus U)$ . Since  $U \subseteq \text{clOrb}(T, Ax)$ , we have  $B(U) \subseteq B(\text{clOrb}(T, Ax)) \subseteq \text{cl}B(\text{Orb}(T, Ax)) = \text{clOrb}(T, B(Ax)) \subseteq (X \setminus U)$ . Thus we have  $B(U) \subseteq (X \setminus U)$ .

Now let  $q \in \mathcal{Q}$ . We claim that  $Bq(T)Ax \in (X \setminus U)$ . To see this notice that since  $q(T)Ax \in (X \setminus U)$  and since  $X \setminus U$  is invariant under  $T$  (Claim 2), then  $\text{Orb}(T, q(T)Ax) \subseteq (X \setminus U)$ . Using this we get that  $Bq(T)Ax = q(T)BAx \in q(T)(\partial U) \subseteq q(T)F = q(T)\text{clOrb}(T, Ax) \subseteq \text{cl}q(T)\text{Orb}(T, Ax) = \text{clOrb}(T, q(T)Ax) \subseteq (X \setminus U)$ .

Thus we have that  $B(U) \subseteq (X \setminus U)$  and  $B(q(T)Ax) \in (X \setminus U)$  for every  $q \in \mathcal{Q}$ , thus  $B(D) \subseteq X \setminus U$ . However, this contradicts the fact that  $B$  has dense range. Thus we must have that  $BAx \notin \partial U$  and Claim 4 follows.

**Claim 5:** Suppose that  $\mathcal{C}$  is a set of operators on  $X$  such that each operator in  $\mathcal{C}$  has dense range and commutes with  $T$  and such that  $\mathcal{C}v$  is a connected set in  $X$  for each  $v \in X$ . If there exists a

$C_0 \in \mathcal{C}$  and an  $A \in \mathcal{F}_T$  such that  $C_0(Ax) \in U$ , then  $C(Ax) \in U$  for all  $C \in \mathcal{C}$ .

Notice that  $\mathcal{C}Ax := \{C(Ax) : C \in \mathcal{C}\}$  is a connected set of vectors that intersects  $U$ , and by Claim 4,  $\mathcal{C}Ax$  does not intersect  $\partial U$ . Thus by the connectivity of  $\mathcal{C}Ax$ , we must have that  $\mathcal{C}Ax \subseteq U$ .

**Claim 6:** If  $\mathbb{F} = \mathbb{C}$ , then  $U$  is invariant under multiplication by non-zero scalars. If  $\mathbb{F} = \mathbb{R}$ , then  $U$  is invariant under multiplication by positive scalars.

Let  $\mathcal{L}$  denote the component of  $\mathbb{F} \setminus \{0\}$  that contains 1. We will prove that  $U$  is invariant under multiplication by scalars in  $\mathcal{L}$ . To see this simply note that if  $\mathcal{C} = \{\lambda I : \lambda \in \mathcal{L}\}$ , then each operator in  $\mathcal{C}$  has dense range, commutes with  $T$ , and for each  $v \in X$ ,  $\mathcal{C}v$  is a connected set in  $X$ . Also since  $I \in \mathcal{C}$ , for every  $A \in \mathcal{F}_T$ ,  $I(Ax) \in U$  (by Claim 3), thus by Claim 5,  $C(Ax) \in U$  for all  $C \in \mathcal{C}$ . Since  $\{Ax : A \in \mathcal{F}_T\}$  is dense in  $U$ , we must have that  $C(U) \subseteq F$  for all  $C \in \mathcal{C}$ . However, since  $C$  is invertible,  $C(U)$  is an open set contained in  $F$  and since  $U$  is the interior of  $F$ , it follows that  $C(U) \subseteq U$ . Also since  $\mathcal{C}$  is closed under the operation of taking inverses, then in fact we have that  $C(U) = U$  for all  $C \in \mathcal{C}$ . Or, in other words,  $\lambda U = U$  for all  $\lambda \in \mathcal{L}$ .

Now suppose that condition (a) holds, as stated in the Theorem. Namely, that the set  $K := \mathbb{C} \setminus \bigcup \{\sigma_p(A^*) : A \in \mathcal{F}_T\}$  is nonempty and has an unbounded connected component, call this unbounded component  $\mathcal{E}$ . With this, we must reach a contradiction. Fix an operator  $A \in \mathcal{F}_T$  and let's apply Claim 5 to the following collection  $\mathcal{C}_A = \{(A - \lambda I) : \lambda \in \mathcal{E}\}$ . By the definition of  $K$ , each member of  $\mathcal{C}_A$  has dense range and clearly commutes with  $T$ . Also, since  $\mathcal{E}$  is connected, then  $\mathcal{C}_A v$  is connected in  $X$  for each  $v \in X$ . We claim that there exists a  $\lambda_0 \in \mathcal{E}$  such that  $(A - \lambda_0 I)x \in U$ . Since  $x \in U$  and  $U$  is open, then  $(\frac{1}{\lambda} Ax - x) \in U$  for all sufficiently large  $\lambda$ . Since  $\mathcal{E}$  is unbounded, there exists a nonzero  $\lambda_0 \in \mathcal{E}$  such that  $(\frac{1}{\lambda_0} Ax - x) \in U$ . Now by Claim 6,  $U$  is invariant under multiplication by  $\lambda_0$ , thus  $(A - \lambda_0 I)x = \lambda_0(\frac{1}{\lambda_0} Ax - x) \in U$ . Thus,  $\mathcal{C}_A$  satisfies the hypothesis of Claim 5, thus  $\mathcal{C}_A x \subseteq U$ . Thus, since  $A \in \mathcal{F}_T$  was arbitrary, we have  $(A - \lambda I)x \in U$  for all  $\lambda \in \mathcal{E}$  and for all  $A \in \mathcal{F}_T$ .

Now fix a number  $\lambda_1 \in \mathcal{E}$ . Then  $(A - \lambda_1 I)x \in U \subseteq F$  for all  $A \in \mathcal{F}_T$ . Since  $\{Ax : A \in \mathcal{F}_T\}$  is dense in  $F$ , by taking limits of  $(A - \lambda_1 I)x = (Ax - \lambda_1 x)$  we see that  $F - \lambda_1 x = \{y - \lambda_1 x : y \in F\} \subseteq F$ . However,  $\lambda_1 x \in U$  by Claim 6, thus  $0 \in U - \lambda_1 x \subseteq (F - \lambda_1 x) \subseteq F$ . Thus, 0 is in the interior of  $F$ , since  $0 \in U - \lambda_1 x \subseteq F$  and  $U - \lambda_1 x$  is an open set. Since  $F$  is invariant under multiplication by positive scalars (Claim 6) it follows that  $F = X$ . But this contradicts our initial assumption that  $F \neq X$ . Thus we have the desired contradiction, and we can now conclude that  $F = X$ . By induction, the proof is now complete in this case where (a) holds.

For the case where  $\mathbb{F} = \mathbb{R}$  and (a') holds we simply let  $\mathcal{E} = (1, \infty)$  and proceed as above. We must have  $\mathcal{E}$  being an interval of positive numbers, since Claim 6 only gives that  $U$  is invariant under multiplication by positive scalars.

Now suppose that (b) holds. Namely that there is a cyclic operator  $B$  that commutes with  $T$  and satisfies  $\sigma_p(B^*)$  has no interior in  $\mathbb{C}$  and  $\mathbb{C} \setminus \sigma_p(B^*)$  is connected. Then, as before we will apply Claim 5 to a certain collection of operators.

Let  $\mathcal{P}$  be the set of all analytic polynomials  $p$  such that  $p(z)$  is nonzero on  $\sigma_p(B^*)$ . Also let  $\mathcal{C} = \{p(B) : p \in \mathcal{P}\}$  and  $\mathcal{C}_n = \{p(B) : p \in \mathcal{P} \text{ and the degree of } p \text{ is } n\}$ . So  $\mathcal{C} = \bigcup_n \mathcal{C}_n$ . Clearly  $T$  commutes with  $\mathcal{C}$  and by definition of  $\mathcal{P}$  each operator in  $\mathcal{C}$  has dense range.

Also, for every  $v \in X$ , the set  $\mathcal{C}_n v$  is connected in  $X$ , because it is the image of the connected set  $(\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \sigma_p(B^*))^n$  under the continuous map  $(c, \lambda_1, \dots, \lambda_n) \mapsto p(B)v = c(B - \lambda_1 I) \cdots (B - \lambda_n I)v$ .

Notice that since  $\sigma_p(B^*)$  is a bounded set, then  $p_{\epsilon,n}(z) = \epsilon z^n + 1 \in \mathcal{C}_n$  for all small  $\epsilon$ . Also, if  $A \in \mathcal{F}_T$ , then since  $U$  is an open set and  $Ax \in U$ , then  $p_{\epsilon,n}(B)Ax = (\epsilon B^n + I)Ax = \epsilon B^n Ax + Ax \in U$  for all small  $\epsilon > 0$ . Thus according to Claim 5,  $p(B)Ax \in U$  for all  $p \in \mathcal{C}_n$ . Hence  $p(B)Ax \in U$  for all  $p \in \mathcal{C}$ . By taking limits, we see that  $p(B)F \subseteq U$  for every  $p \in \mathcal{P}$ . However, since  $\sigma_p(B^*)$  has empty interior, it follows that  $\mathcal{P}$  is dense in the set of all analytic polynomials (in the topology of uniform convergence on compact sets), thus  $p(B)F \subseteq F$  for all polynomials  $p$ .

Now since  $\sigma_p(B^*)$  has empty interior, it follows that  $B$  has a dense set of cyclic vectors (to see this simply note that if  $v$  is cyclic for  $B$ , then  $\{p(B)v : p \in \mathcal{P}\}$  is a dense set that also consists of cyclic vectors for  $B$ ). So, let  $y \in U$  be a cyclic vector for  $B$ , then  $X = cl\{p(B)y : p \text{ is a polynomial}\} \subseteq F$ . So,  $F = X$ . But this contradicts our assumption that  $F \neq X$ . With this contradiction, we have that  $F = X$ . By induction, the proof is now complete in this case where (b) holds.

For the case where  $\mathbb{F} = \mathbb{R}$  and (b') holds we simply let  $\mathcal{P}$  be the set of all analytic polynomials with real coefficients and let  $\mathcal{P}_n^\pm$  be the set of all polynomials of degree  $n$  with real coefficients and having a positive (respectively, negative) leading coefficient. Now let  $\mathcal{C}_n^\pm = \{p(B) : p \in \mathcal{P}_n^\pm\}$ , then one can check that  $\mathcal{C}_n^+ v$  and  $\mathcal{C}_n^- v$  are both convex sets (hence connected) and the operators in  $\mathcal{C}_n^\pm$  all have dense range (by assumption), thus we may proceed as above using  $p_{\pm\epsilon,n}(z) = \pm\epsilon z^n + 1 \in \mathcal{C}_n^\pm$  for  $\epsilon > 0$ .

This completes the proof.  $\square$

**Corollary 5.6.** *Suppose that  $T = (T_1, \dots, T_n)$  is a tuple of operators on a real or complex locally convex space  $X$  and  $\mathcal{F}$  is the semigroup they generate. If  $A^*$  has no eigenvalues for every  $A \in \mathcal{F}$ , then any orbit of  $T$  that is somewhere dense in  $X$  will be dense in  $X$ .*

**Corollary 5.7.** *Suppose that  $T = (T_1, \dots, T_n)$  is a tuple of operators on a complex locally convex space  $X$  and  $\mathcal{F}$  is the semigroup they generate. If  $A^*$  has a countable number of eigenvalues for every  $A \in \mathcal{F}$ , then any orbit of  $T$  that is somewhere dense in  $X$  will be dense in  $X$ .*

*In particular, if  $T$  is an  $n$ -tuple of matrices on  $\mathbb{C}^k$ , then every somewhere dense orbit of  $T$  will be everywhere dense in  $\mathbb{C}^k$ .*

The fact that multi-hypercyclic operators are in fact hypercyclic was proven independently by Costakis [9] and Peris [22]. Ansari [1] gave a truly original proof of the fact that if  $T$  is a hypercyclic operator, then  $T^n$  is also hypercyclic for every  $n \geq 1$ .

**Corollary 5.8.** *Suppose that  $T = (T_1, T_2, \dots, T_n)$  is an  $n$ -tuple of commuting operators on a locally convex space  $X$  and  $T$  satisfies the hypothesis from Theorem 5.5, then the following hold:*

- (1) If  $T$  is multi-hypercyclic, then  $T$  is actually hypercyclic.
- (2) If  $T$  is hypercyclic and  $k = (k_1, k_2, \dots, k_n)$  is a multi-index with  $k_i \geq 1$  for all  $i$ , then  $T^k = (T_1^{k_1}, T_2^{k_2}, \dots, T_n^{k_n})$  is also hypercyclic with the same set of hypercyclic vectors as  $T$ .

*Proof.* (1) If a finite set has dense orbit under  $T$ , then some element of that set will have a somewhere dense orbit, hence by Theorem 5.5 its orbit must be dense. Hence  $T$  is hypercyclic.

(2) Suppose that  $k = (k_1, k_2, \dots, k_n)$  is given and  $k_i \geq 1$  for all  $i$ . Let  $p = (p_1, p_2, \dots, p_n)$  be any multi-index, then by applying the division algorithm we see that there exists integers  $q_i \geq 0$  and  $0 \leq r_i < k_i$  such that  $p_i = k_i q_i + r_i$ . Thus,  $T^p x = T_1^{p_1} \dots T_n^{p_n} x = (T_1^{k_1})^{q_1} \dots (T_n^{k_n})^{q_n} T_1^{r_1} \dots T_n^{r_n} x$ . Thus, we see that

$$\text{Orb}(T, x) = \bigcup_{0 \leq r_i < k_i} \text{Orb}(T^k, T_1^{r_1} \dots T_n^{r_n} x).$$

Now suppose that  $x$  is a hypercyclic vector for  $T$ . Then the left hand side above is dense, hence the right hand side above is also dense. Thus for some choice of  $r_i$  we must have that  $\text{Orb}(T^k, T_1^{r_1} \dots T_n^{r_n} x)$  is somewhere dense in  $X$ . It then follows from Theorem 5.5 that  $\text{Orb}(T^k, T_1^{r_1} \dots T_n^{r_n} x)$  is dense in  $X$ , hence  $T^k$  is hypercyclic.

Continuing, let's also show that  $x$  is a hypercyclic vector for  $T^k$ . Since we have that  $\text{Orb}(T^k, T_1^{r_1} \dots T_n^{r_n} x)$  is dense in  $X$ , and since  $\text{Orb}(T^k, T_1^{r_1} \dots T_n^{r_n} x) = T_1^{r_1} \dots T_n^{r_n} \text{Orb}(T^k, x)$ , it follows that  $T_1^{r_1} \dots T_n^{r_n}$  has dense range, which implies that  $T_i$  has dense range whenever  $r_i \neq 0$ . Now define  $s_i = k_i - r_i$  when  $r_i \neq 0$  and define  $s_i = 0$  when  $r_i = 0$ . Then  $T_1^{s_1} \dots T_n^{s_n}$  will have dense range and so  $T_1^{s_1} \dots T_n^{s_n} \text{Orb}(T^k, T_1^{r_1} \dots T_n^{r_n} x)$  is dense in  $X$ . But this last set is a subset of  $\text{Orb}(T^k, x)$ . Thus,  $x$  is hypercyclic for  $T^k$  as well. So  $T$  and  $T^k$  have the same set of hypercyclic vectors.  $\square$

An operator  $T$  on a space  $X$  is said to be  $F$ -supercyclic, where  $F \subseteq \mathbb{C}$ , if there is a vector  $x \in X$  such that  $F \cdot \text{Orb}(T, x) = \{\alpha T^n x : \alpha \in F, n \geq 0\}$  is dense in  $X$ . In [2], Bermdez, Bonilla, and Peris proved that if an operator  $T$  is  $\mathbb{R}$ -supercyclic, then in fact  $T$  is  $\mathbb{R}^+$ -supercyclic. We give another proof of this fact, because it follows easily from the results above and because it gives a simple example of how tuples of operators can be used to solve problems for a single operator.

**Corollary 5.9.** *If  $T$  is an operator on a complex locally convex space and  $T$  is  $\mathbb{R}$ -supercyclic, then  $T$  is actually  $\mathbb{R}^+$ -supercyclic.*

*Proof.* Consider the tuple of operators  $S = (S_1, S_2, T)$  where  $S_1 = -2I$  and  $S_2 = \frac{1}{3}I$ . Since  $S_1$  and  $S_2$  are multiples of the identity operators, then  $S$  is a commuting tuple. By Corollary 4.2,  $\{2^n/3^k\}$  is dense in  $\mathbb{R}^+$ , from which one can show that  $\{(-2)^n/3^k\}$  is dense in  $\mathbb{R}$ . This together with the fact that  $T$  is  $\mathbb{R}$ -supercyclic imply that  $S$  is a hypercyclic tuple. Also since  $T$  is  $\mathbb{R}$ -supercyclic, then  $T$  is cyclic and  $T^*$  has at most one eigenvalue. Thus with  $B = T$ , we see that  $S$  satisfies condition (b) of Theorem 5.5. Hence by Corollary 5.8, we have that  $(S_1^2, S_2, T)$  is also hypercyclic, which implies that  $T$  is  $\mathbb{R}^+$ -supercyclic.  $\square$

Similarly one can show that if  $F$  is the closure of  $\{\alpha_1^{n_1} \alpha_2^{n_2} \dots \alpha_k^{n_k} : n_i \geq 0\}$ , where  $\alpha_i \in \mathbb{C}$ , and  $T$  is an operator that is  $F$ -supercyclic, and  $p = (p_1, \dots, p_k) \in \mathbb{N}^k$ , then  $T$  is also  $F^p$ -supercyclic where  $F^p$  is the closure of  $\{\alpha_1^{p_1 n_1} \alpha_2^{p_2 n_2} \dots \alpha_k^{p_k n_k} : n_i \geq 0\}$ .

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## 6. QUESTIONS

- (1) Is there an  $n$ -tuple on a complex Hilbert space that has a somewhere dense orbit that is not dense?
- (2) If  $T$  is a hypercyclic tuple, then must the semigroup generated by  $T$  contain a cyclic operator?
- (3) If  $T$  is a hypercyclic tuple, then is there a cyclic operator  $B$  that commutes with  $T$ ?
- (4) Is there a hypercyclic  $n$ -tuple of hyponormal operators on an infinite dimensional Hilbert space?
- (5) Is there a hypercyclic (nondiagonalizable)  $n$ -tuple on  $\mathbb{C}^n$ ?
- (6) Are there non-diagonalizable  $n$ -tuples on  $\mathbb{R}^k$  that have somewhere dense orbits?
- (7) If an orbit of a tuple  $T$  is somewhere dense, but not dense in a real locally convex space  $X$ , then is the closure of the orbit invariant under multiplication by positive scalars?

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