SUBNORMAL AND HYPONORMAL GENERATORS OF C^*-ALGEBRAS

NATHAN S. FELDMAN AND PAUL MCGUIRE

Abstract. For a Hilbert space operator A, that is not a normal operator, we give some necessary conditions on the thickness of the spectrum of A for $C^*(A)$ to have a subnormal generator. When A is irreducible and essentially normal, whether or not $C^*(A)$ has a subnormal generator depends only on the spectral picture of A. We show that under certain conditions on the essential spectrum and Fredholm index function of A that $C^*(A)$ has a subnormal or a hyponormal generator.

1. Introduction

If A is a bounded linear operator on a separable complex Hilbert space, then let $C^*(A)$ be the C^*-algebra of operators generated by A and the identity operator. If $S \in C^*(A)$, then S is a generator for $C^*(A)$ if $C^*(S) = C^*(A)$. We are interested in determining which operators A have the property that $C^*(A)$ has a subnormal or hyponormal generator. Subnormal and hyponormal generators of von Neumann algebras have been studied by Wogen [21] and Behncke [1]. In 1984 Putnam [18] showed that certain hyponormal operators have C^*-algebras generated by a unilateral shift, and raised a related question. This question and more was answered by Conway and McGuire [5] where they characterized the operators whose C^*-algebra is generated by a unilateral shift. In 1988, McGuire [12] extended that result to operators whose C^*-algebras are generated by subnormal operators whose essential spectrum is a finite union of *disjoint* Jordan curves. In particular, McGuire [12] proved the following fundamental result:

> For an irreducible essentially normal operator A whose essential spectrum is a *finite union of disjoint Jordan curves*, $C^*(A)$ has a subnormal generator if and only if $\operatorname{ind}(A - \lambda I) \neq 0$ for some $\lambda \in \mathbb{C} \setminus \sigma_e(A)$.

In this paper we mainly consider the problem of determining which irreducible essentially normal operators A have a subnormal (or hyponormal) generator for their C^*-algebra. It is not hard to see that the answer depends only on the spectral picture of A, that is the essential spectrum of A and the values of the Fredholm index function, $\operatorname{ind}(A - \lambda I)$, off the essential spectrum. As we shall see many such operators do have subnormal generators and yet many will not. There does not appear to be a simple answer at this point—Theorem 2.10 does give a necessary

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and sufficient condition, but it is difficult to verify in practice—however, this simply makes the problem all the more interesting.

After some preliminaries, we begin by showing that if A is any operator whose spectrum is "small" in the sense that $\sigma(A)$ is polynomially convex and has no interior, then $C^*(A)$ has a subnormal generator if and only if A is normal. However, we also give an example of an operator A whose spectrum has area zero, yet $C^*(A)$ has a subnormal generator. Another example is given of an irreducible hyponormal operator T with rank one self-commutator such that $C^*(T)$ does not have a subnormal generator. Yet another example is that of an irreducible essentially normal operator A with the same spectral picture as a *pure* subnormal operator, yet $C^*(A)$ does not have a subnormal generator!

There are two major cases to consider: One where the Fredholm index function is identically zero and the other where the index is nonzero at some point. The following result is one of the principal results.

> Theorem 5.6: If $A \in \mathcal{B}(\mathcal{H})$ is an irreducible essentially normal operator and $\text{ind}(A - \lambda I) = 0$ for all $\lambda \notin \sigma_e(A)$, then $C^*(A)$ has a hyponormal generator if and only if $\sigma_e(A)$ has no isolated points.

In the above theorem, if $C^*(A)$ has a subnormal generator, then $\sigma_e(A)$ must be connected and a reasonably thick set (see Theorem 5.1).

Another principal result is the following:

> Theorem 7.8: For an irreducible essentially normal operator A whose essential spectrum is a Swiss-cheese type set, $C^*(A)$ has a subnormal generator if and only if $C^*(A)$ has a hyponormal generator if and only if the Fredholm index function $\text{ind}(A - \lambda I)$ is bounded above or bounded below on $\mathbb{C} \setminus \sigma_e(A)$.

An important condition that implies that $C^*(A)$ has a subnormal generator is the following theorem.

> Theorem 6.6: Suppose A is an irreducible essentially normal operator such that one of the following hold:
> (1) $M := \sup\{ind(A - \lambda I) : \lambda \in \mathbb{C} \setminus \sigma_e(A)\} < \infty$ and $int(K_M)$ is connected and dense in K_M where $K_M := \sigma_e(A) \cup \{\lambda \in \mathbb{C} \setminus \sigma_e(A) : \text{ind}(A - \lambda I) < M\}$.
> or
> (2) $m := \inf\{ind(A - \lambda I) : \lambda \in \mathbb{C} \setminus \sigma_e(A)\} > -\infty$ and $int(K_m)$ is connected and dense in K_m where $K_m := \sigma_e(A) \cup \{\lambda \in \mathbb{C} \setminus \sigma_e(A) : \text{ind}(A - \lambda I) > m\}$.
> Then $C^*(A)$ has a subnormal generator.

Another principal result is Theorem 6.9 which gives conditions under which the above theorem becomes not only sufficient, but also necessary for $C^*(A)$ to have a subnormal generator. A surprising and interesting example of Theorem 6.9 comes by considering irreducible essentially normal operators whose essential spectra are "checkerboards". By a checkerboard we mean a set of the form $(X \times [0,1]) \cup ([0,1] \times Y)$ where X, Y are closed subsets of $[0,1]$ and $\{0,1\} \subseteq X \cap Y$. Theorem 6.9 provides interesting results even when X and Y are finite sets, but also applies to some infinite checkerboard sets (that is when X and Y are infinite sets.)

Several examples are given in Section 7 and we close with some open questions.

2. Preliminaries

In what follows $\mathcal{H}$ will denote a separable infinite dimensional complex Hilbert space, $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$, $\mathcal{B}_0(\mathcal{H})$ is the ideal of all compact operators on $\mathcal{H}$ and $\mathcal{B}/\mathcal{B}_0$ the Calkin algebra. An operator $S \in \mathcal{B}(\mathcal{H})$ is subnormal if it has a normal extension and an operator T is hyponomral if its self-commutator, $[T^*, T] = T^*T - TT^*$, is a non-negative operator. For a subnormal operator S, the normal spectrum of S, denoted by $\sigma_n(S)$, is defined to be the spectrum of the minimal normal extension of S. For more on subnormal and hyponormal operators see Conway [3]. An operator is essentially normal if its self-commutator is compact. The kernel of an operator is $ker(A) = \{x : Ax = 0\}$. An operator A is Fredholm if it has closed range, $\dim ker(A) < \infty$, and $\dim ker(A^*) < \infty$. When A is Fredholm, then its (Fredholm) index is defined as $\text{ind}(A) = \dim ker(A) - \dim ker(A^*)$. The essential spectrum of A is $\sigma_e(A) = \{\lambda \in \mathbb{C} : (A - \lambda I) \text{ is not Fredholm}\}$. The (Fredholm) index function for A is the integer valued continuous function $\lambda \mapsto \text{ind}(A - \lambda I)$ defined on $\mathbb{C} \setminus \sigma_e(A)$.

The term *spectral picture* of an operator A generally refers to the essential spectrum of A, $\sigma_e(A)$, and the values of its index function on the components of $\mathbb{C} \setminus \sigma_e(A)$, and also perhaps to the spectrum of A and some other subsets of the spectrum. *In this paper*, the term **spectral picture** of an operator will mean the essential spectrum and the index function of that operator.

With this terminology, the well known Brown-Douglas-Fillmore Theorem [2] takes the following form:

Theorem 2.1 (BDF Theorem)**.** *Two essentially normal operators A and B are unitarily equivalent modulo the compacts if and only if A and B have the same spectral picture.*

That is, there exists a unitary operator U and a compact operator C such that $U^*AU = B + C$ if and only if $\sigma_e(A) = \sigma_e(B)$ and $ind(A - \lambda I) = ind(B - \lambda I)$ for all $\lambda \in \mathbb{C} \setminus K$ where $K = \sigma_e(A) = \sigma_e(B)$.

Some Topological Tools: For a set $E \subseteq \mathbb{C}$, $int(E)$ and $\text{cl}(E)$ will denote the interior and closure of E, respectively. For a Jordan curve γ in $\mathbb{C}$, $inside(\gamma)$ will denote the bounded component of $\mathbb{C} \setminus \gamma$ and $outside(\gamma)$ will denote the unbounded component of $\mathbb{C} \setminus \gamma$. For a compact set $K \subseteq \mathbb{C}$, the outer boundary of K will be the boundary of the unbounded component of $\mathbb{C} \setminus K$.

Jordan Regions & Winding Numbers: If $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable continuous closed curve in the complex plane and λ is a point not on the curve, then $n(\gamma, \lambda) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-\lambda} dz$ is the winding number of γ about λ. The winding number is well known to be a homotopy invariant. If γ is only a continuous curve, then one can approximate it by rectifiable curves and use the homotopy invariance to define the winding number of γ. Alternatively, if $\gamma : [a, b] \to \mathbb{C}$ is a continuous closed curve and, say, 0 is a point not on γ, then let $\theta : [a, b] \to \mathbb{R}$ be a continuous branch of the argument of γ. So, $\theta(t)$ is a continuous function and $\gamma(t) = |\gamma(t)| e^{i\theta(t)}$ for $t \in [a, b]$. Then $n(\gamma, 0) := \frac{1}{2\pi}(\theta(b) - \theta(a))$. If Γ is a finite system of closed curves $\gamma_1, \ldots, \gamma_n$, then $n(\Gamma, \lambda) := \sum_{k=1}^n n(\gamma_k, \lambda)$ for $\lambda \notin \bigcup_k \gamma_k$. The inside and outside of a system Γ of closed curves are defined by $\text{inside}(\Gamma) = \{\lambda \in \mathbb{C} : n(\Gamma, \lambda) = 1\}$ and $\text{outside}(\Gamma) = \{\lambda \in \mathbb{C} : n(\Gamma, \lambda) = 0\}$.

A Jordan region is a region bounded by a finite number of disjoint rectifiable Jordan curves. A Jordan region G is positively oriented if each Jordan curve in the boundary of G is oriented such that inside$(\partial G) = G$ and outside$(\partial G) = \mathbb{C} \setminus clG$.

Theorem 2.2. [16, page 60 - Brouwer's Theorem] *A compact set K in $\mathbb{C}$ is homeomorphic to a compact set in $\mathbb{C}$ with 2-dimensional Lebesgue measure zero if and only if K has empty interior.*

A compact set L is said to have *positive area density at each point* if whenever U is an open set satisfying $U \cap L \neq \emptyset$, then $m_2(U \cap L) > 0$, where m_2 denotes 2-dimensional Lebesgue measure on $\mathbb{C}$. A set is called *perfect* if it is closed and has no isolated points.

Theorem 2.3. *If $K \subseteq \mathbb{C}$ is a compact set, then K is homeomorphic to a compact set $L \subseteq \mathbb{C}$ that has positive area density at each point if and only if K is a perfect set.*

The previous "folklore" result will be used, a brief sketch of the proof will be provided, since a reference could not be found. The proof was provided to the authors by Stanley C. Williams. The proof uses the fact that there are Cantor sets in the plane with positive area density at each point and that any two Cantor sets in the plane are homeomorphic via a homeomorphism of the plane (see [15, Corollary 5]).

Sketch of Stanley C. Williams' Proof . Let $D = \{d_i\}_{i=1}^{\infty}$ be a countable dense subset of K. Let $\{\epsilon_i\}_{i=1}^{\infty}$ be a sequence of strictly positive real numbers such that $\sum_{i=1}^{\infty} \epsilon_i < \infty$. Let R_1 be an open rectangle centered at d_1 with diameter $< \epsilon_1$. Since K is perfect, there is a Cantor set C_1 such that $d_1 \in C_1 \subseteq R_1 \cap K$. Let $A_1 \subseteq R_1$ be a Cantor set that has positive area density at each point and let h_1 be a homeomorphism on the plane that satisfies $h_1(C_1) = A_1$ and $h(z) = z$ if $z \notin R_1$. If $h_1(D) \subseteq A_1$, then stop. Otherwise, let i_2 be the smallest element in $\{i : h_1(d_i) \notin A_1\}$. Since $h_1(d_{i_2}) \notin A_1$, then there exists a rectangle R_2 with diameter $< \epsilon_2$ such that $h_1(d_{i_2}) \in R_2$ and $R_2 \cap A_1 = \emptyset$. Since $h_1(K)$ is perfect there is a Cantor set C_2 such that $h_1(d_2) \in C_2 \subseteq R_2 \cap h_1(K)$. Let $A_2 \subseteq R_2$ be a Cantor set that has positive area density at each point and let h_2 be a homeomorphism of the plane such that $h_2(C_2) = A_2$ and $h_2(z) = z$ if $z \notin R_2$. Let $H_2 = h_2 \circ h_1$. Now either $H_2(D) \subseteq A_1 \cup A_2$ in which case we stop and H_2 is the homeomorphism we are looking for, or else there is a smallest index i_3 such that $H_2(d_{i_3}) \notin A_1 \cup A_2$. Proceeding as above we continue to construct finite sequences of homeomorphisms $\{h_i\}_{i=1}^{n}$, of open rectangles $\{R_i\}_{i=1}^{n}$, of Cantor sets $\{C_i\}_{i=1}^{n}$ and of Cantor sets $\{A_i\}_{i=1}^{n}$ having positive area density at each point and an increasing sequence of natural numbers $\{i_k\}$ satisfying:

(1) $i_1 = 1$.
(2) i_k is the smallest index i such that $h_{k-1} \circ h_{k-2} \circ \cdots \circ h_1(d_i) \notin \bigcup_{j=1}^{k-1} A_j$.
(3) $h_{k-1} \circ h_{k-2} \circ \cdots \circ h_1(d_{i_k}) \in C_k \subseteq R_k \cap h_{k-1} \circ h_{k-2} \circ \cdots \circ h_1(K)$
(4) R_k is disjoint from $\bigcup_{j=1}^{k-1} A_j$ and has diameter $< \epsilon_k$.
(5) $A_k \subseteq R_k$
(6) $h_k(C_k) = A_k$ and $h_k(z) = z$ if $z \notin R_k$.

Either this process terminates or continues. For each k let $H_k = h_{k-1} \circ h_{k-2} \circ \cdots \circ h_1$. If $H_k(D) \subseteq \bigcup_{j=1}^{k-1} A_j$, then the process terminates and H_k is the required homeomorphism. Otherwise let $H = \lim_k H_k$, since $\sum_k \epsilon_k < \infty$, the limit exists and H is the required homeomorphism. □

The following is a simple result that will be used many times.

Lemma 2.4. *If $K \subseteq \mathbb{C}$, γ is a Jordan curve contained in K, $h : K \to \mathbb{C}$ is a one-to-one continuous function and $K \setminus \gamma$ is a connected set, then either $h(K \setminus \gamma) \subseteq inside(h(\gamma))$ or $h(K \setminus \gamma) \subseteq outside(h(\gamma))$.*

For a proof of the following result see Newman [13, p. 173, Corollary 3].

Theorem 2.5. *If γ_1 and γ_2 are two Jordan curves and $h : \gamma_1 \to \gamma_2$ is a homeomorphism, then h extends to a homeomorphism $\hat{h} : cl[inside(\gamma_1)] \to cl[inside(\gamma_2)]$.*

The spectral picture of $f(A)$: For a compact set K in the complex plane, $C(K)$ will denote the set of all continuous complex valued functions on K. If A is an essentially normal operator and $f \in C(\sigma_e(A))$, and $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}/\mathcal{B}_0$ is the natural projection into the Calkin algebra, then $\pi(A)$ is a normal element of the C^*-algebra $\mathcal{B}/\mathcal{B}_0$, thus $f(\pi(A))$ is a well-defined element of $\mathcal{B}/\mathcal{B}_0$. Since any two operators in $\pi^{-1}(f(\pi(A))$ differ by a compact operator, they must have the same spectral picture, hence we may define the spectral picture of $f(A)$ to be the spectral picture of any operator in $\pi^{-1}(f(\pi(A)))$. In general we will use $f(A)$ to denote any operator in $\pi^{-1}(f(\pi(A))$ and anything done with $f(A)$ will be invariant under compact perturbations.

Theorem 2.6 (Functions of Spectral Pictures)**.** *If A is an essentially normal operator and $f \in C(\sigma_e(A))$, then the following hold:*

(1) $\sigma_e(f(A)) = f(\sigma_e(A))$.

(2) *Let $\{G_n\}_{n=1}^{\infty}$ be the bounded components of $\mathbb{C} \setminus \sigma_e(A)$ and let $\hat{f}$ denote any continuous extension of f to $\sigma(A)$. For each $n \geq 1$, let $a_n \in G_n$. If $\lambda \in \mathbb{C} \setminus \sigma_e(f(A))$, then there exists an integer $0 < N < \infty$ and a compact set $K \subseteq \bigcup_{n=1}^{N} int(clG_n)$ such that if $\{\Omega_n\}_{n=1}^{N}$ is any finite collection of positively oriented Jordan regions satisfying $K \subseteq \bigcup_{n=1}^{N} \Omega_n \subseteq \bigcup_{n=1}^{N} cl\Omega_n \subseteq \bigcup_{n=1}^{N} int(clG_n)$, then*

$$ind(f(A) - \lambda I) = \sum_{n=1}^{N} n(\hat{f}(\partial\Omega_n), \lambda) ind(A - a_n I).$$

(3) *Keeping the notation from (2), if each component G_n is bounded by a (positively oriented) Jordan curve, then for each $\lambda \in \mathbb{C} \setminus \sigma_e(f(A))$, then there exists an integer $0 < N < \infty$ such that*

$$ind(f(A) - \lambda I) = \sum_{n=1}^{N} n(f(\partial G_n), \lambda) ind(A - a_n I).$$

In fact, $n(f(\partial G_n), \lambda) = 0$ for all $n > N$, so

$$ind(f(A) - \lambda I) = \sum_{n=1}^{\infty} n(f(\partial G_n), \lambda) ind(A - a_n I).$$

Proof. Part (1) follows simply from the fact that $\pi(A)$ is a normal element of the Calkin algebra, $\sigma_e(A) = \sigma(\pi(A))$, and the functional calculus for normal elements of C^*-algebras.

Parts (2) and (3) were proven in Feldman and McGuire [7] when A is an essentially normal subnormal operator. If A is an arbitrary essentially normal operator, then there are essentially normal subnormal operators S and T such that A has the same spectral picture as $S \oplus T^*$, hence the result follows from [7]. □

See Proposition 3.6 of Feldman & McGuire [7] for a proof of the following proposition.

Proposition 2.7. *Let S be an essentially normal subnormal operator.*

(a) If $\sigma(S) \neq \sigma_e(S)$, $ind(S - \lambda I) \neq 0$ for some $\lambda \in \mathbb{C} \setminus \sigma_e(S)$, $f \in C(\sigma_n(S))$, and f is one-to-one on $\sigma_e(S)$, then $ind(f(S) - \lambda I) \neq 0$ for some $\lambda \in \mathbb{C} \setminus \sigma_e(f(S))$.

(b) If $\sigma(S) = \sigma_e(S)$ and $f \in C(\sigma(S))$, and f is one-to-one on $\sigma(S)$, then $ind(f(S) - \lambda I) = 0$ for all $\lambda \notin \sigma_e(f(S))$.

Lemma 2.8. *If $a \in \mathcal{C}$ is a normal element in a C^*-algebra $\mathcal{C}$, then $b \in \mathcal{C}$ is a generator of $C^*(a)$ if and only if $b = h(a)$ for some one-to-one continuous function $h : \sigma(a) \to \mathbb{C}$.*

Proposition 2.9 (Generators have homeomorphic spectral pictures)**.** *If A and B are irreducible essentially normal operators, then*

(a) $C^*(A) = \pi^{-1}(C^*(\pi(A))) = \{f(A) + K : f \in C(\sigma_e(A)), K \in \mathcal{B}_0\}$.

(b) B is unitarily equivalent to a generator of $C^*(A)$ if and only if there is a homeomorphism $h : \sigma_e(A) \to \sigma_e(B)$ such that $h(A)$ and B have the same spectral picture.

Proof. (a) Since $\pi : C^*(A) \to \mathbb{C}^*(\pi(A))$ is onto and since $C^*(A)$ contains all the compact operators it follows that $\pi^{-1}(C^*(\pi(A))) \subseteq C^*(A)$. However, $\pi^{-1}(C^*(\pi(A)))$ is a C^*-algebra that contains A, hence we must have $C^*(A) \subseteq \pi^{-1}(C^*(\pi(A)))$. Thus, $C^*(A) = \pi^{-1}(C^*(\pi(A)))$.

(b) If B is a generator of $C^*(A)$, then $C^*(A) = C^*(B)$. Hence also, $C^*(\pi(A)) = C^*(\pi(B))$ and as $\pi(A)$ and $\pi(B)$ are normal elements of the Calkin algebra, then by Lemma 2.8 there is a homeomorphism $h : \sigma(\pi(A)) \to \sigma(\pi(B))$ such that $h(\pi(A)) = \pi(B)$. It follows that $h(\sigma_e(A)) = \sigma_e(B)$ and that $h(A)$ is a compact perturbation of B. Hence $h(A)$ and B have the same spectral picture.

Conversely, if there exists a homeomorphism h such that $h(A)$ and B have the same spectral picture, then by the BDF Theorem 2.1 there is a unitary operator U and a compact operator K such that $U^*BU = h(A) + K$. It then follows by part (a) that $C^*(\pi(U^*BU)) = C^*(h(\pi(A))) = C^*(\pi(A))$ where this last equality holds by Lemma 2.8. Now since U^*BU and A are both irreducible essentially normal operators, then by part (a) $C^*(U^*BU) = \pi^{-1}(C^*(\pi(U^*BU))) = \pi^{-1}(C^*(\pi(A))) = C^*(A)$. Hence B is unitarily equivalent to a generator of $C^*(A)$. □

Gleason parts: If K is a compact set in $\mathbb{C}$, then $P(K)$ will denote the uniform closure of the (analytic) polynomials in $C(K)$ and $R(K)$ will denote the uniform closure in $C(K)$ of the rational functions with poles off K. Two points $a, b \in K$ belong to the same Gleason part of $R(K)$ if a Harnack type inequality holds, that is, if there exists a constant $c > 0$ such that $\frac{1}{c} Ref(a) \leq Ref(b) \leq cRef(a)$ for all

$f \in R(K)$ with $Ref > 0$ (where Ref denotes the real part of f). A Gleason part of $R(K)$ is said to be non-trivial if it contains more than one point. It is known that if $a, b \in K$ belong to the same Gleason part, then they have mutually absolutely continuous representing measures supported on ∂K. However, if a, b belong to distinct Gleason parts, then every representing measure for a is mutually singular with respect to every representing measure for b. This last condition is useful in showing that two points belong to the same Gleason part.

A crucial tool in this paper is the ability to be able to construct irreducible essentially normal subnormal operators with prescribed spectral pictures (subject to certain natural necessary conditions). This was established by the authors in [6], and in this paper is neatly tucked away in the proof of the following theorem.

Theorem 2.10 (Subnormal generators)**.** *If A is an irreducible essentially normal operator, then $C^*(A)$ has a subnormal generator if and only if*

(a) there is a compact set $K_e \subseteq \mathbb{C}$ and a homeomorphism $h : \sigma_e(A) \to K_e$ such that $ind(h(A) - \lambda I) \leq 0$ for all $\lambda \in \mathbb{C} \setminus K_e$ and

(b) if $K := K_e \cup \{\lambda : ind(h(A) - \lambda I) < 0\}$, then $R(K)$ has only one non-trivial Gleason part which is dense in K.

Proof. If $C^*(A) = C^*(S)$ for some subnormal operator S, then S must be irreducible and essentially normal. Then by applying Proposition 2.9 to $\pi(A)$ and $\pi(S)$ we see that there is a homeomorphism $h : \sigma_e(A) \to \sigma_e(S)$ such that $h(A)$ has the same spectral picture as S. Let $K_e = \sigma_e(S)$. Since S is subnormal, then $\text{ind}(h(A) - \lambda I) = \text{ind}(S - \lambda I) \leq 0$ for all $\lambda \notin K_e$. Also, if $K = K_e \cup \{\lambda : \text{ind}(h(A) - \lambda I) < 0\}$, then since S is pure, $K = \sigma(S)$ and it is well known (see [14]) that since S is irreducible that $R(\sigma(S))$ has only one non-trivial Gleason part which is dense in $\sigma(S)$. Hence the same holds for K.

Conversely, if conditions (a) and (b) hold, then by [6] there exists an irreducible essentially normal subnormal operator S such that $\sigma(S) = K$, $\sigma_e(S) = K_e$ and $\text{ind}(S - \lambda I) = \text{ind}(h(A) - \lambda I)$ for all $\lambda \notin K_e$. Thus, $h(A)$ and S have the same spectral picture. It now follows from Proposition 2.9 that $C^*(A)$ has a generator that is unitarily equivalent to S, hence subnormal. □

Prescribed spectral pictures: As mentioned above, being able to prescribe the spectral picture of an irreducible subnormal operator is crucial, see Theorem 2.10 and the comments before it. But here we want to show that there exist irreducible essentially normal operators with any prescribed spectral picture (essential spectrum and Fredholm index function). We will also show that in a certain case (Theorem 2.12) we can also prescribe the spectrum as well.

Theorem 2.11. *If K is any compact set in $\mathbb{C}$, $\{G_n\}_{n=1}^{\infty}$ are the bounded components of $\mathbb{C} \setminus K$ and $\{a_n\}_{n=1}^{\infty}$ is any sequence of integers, then there exists an irreducible essentially normal operator A such that $\sigma_e(A) = K$ and $ind(A - \lambda I) = a_n$ for $\lambda \in G_n$.*

Proof. Let T be a direct sum of a normal operator, some Bergman operators and their adjoints so that T is essentially normal and has the prescribed essential spectrum and index function. Then there is a compact operator K such that $T + K$ is irreducible (see [10, p.119]). It follows that $A = T + K$ is the required operator. □

We also need the following result which is surely known, however, since the authors could not find a reference for it we shall supply a proof.

Theorem 2.12. *If K is a compact perfect set in $\mathbb{C}$, then there exists an irreducible essentially normal operator A such that $\sigma(A) = \sigma_e(A) = K$.*

Proof. Let D be a diagonal normal operator with distinct eigenvalues that are dense in K, then apply Lemma 2.13 below to find an irreducible operator A that is similar to D and also a compact perturbation of D. □

Lemma 2.13. *If D is a diagonal normal operator on ℓ^2 with distinct eigenvalues, then there is an irreducible operator A that is similar to D and is also equal to a compact perturbation of D.*

Proof. The operator A will equal WDW^{-1} where $W : \ell^2 \to \ell^2$ is given by $W = I + K$ and K is a compact operator to be determined. Notice then that $A = WDW^{-1} = D + K'$ for some other compact operator K'. We must choose K such that W is invertible and A is irreducible. Let $K : \ell^2 \to \ell^2$ be a strictly upper-triangular matrix with all entries above the main diagonal being strictly positive and such that the sum of all its entries is finite and small enough so that $\|K\| < 1$. If this is true, then W will be invertible. We will now show that A is irreducible. Suppose that A is reducible, then A has a pair of orthogonal invariant subspaces, $\mathcal{M}'$ and $\mathcal{N}'$, that span ℓ^2. If we set $\mathcal{M} = W^{-1}(\mathcal{M}')$ and $\mathcal{N} = W^{-1}(\mathcal{N}')$, then it follows that $\mathcal{M}$ and $\mathcal{N}$ are complementary invariant subspaces for D (but not necessarily orthogonal), that is $\mathcal{M} + \mathcal{N} = \ell^2$, $\mathcal{M} \cap \mathcal{N} = (0)$, and $\mathcal{M}$ and $\mathcal{N}$ are invariant for D. Let $\{e_n\}_{n=0}^\infty$ be the eigenvectors of D and define $x_n = W(e_n)$ for every $n \geq 0$. Then $\{x_n\}_{n=0}^\infty$ are eigenvectors for A. By definition, the vectors $\{x_n\}_{n=0}^\infty$ are the columns of W. By construction, no two columns of W are orthogonal. Thus either $\mathcal{M}' \cap \{x_n\}_{n=0}^\infty = \emptyset$ or $\mathcal{N}' \cap \{x_n\}_{n=0}^\infty = \emptyset$. Thus either $\mathcal{M} \cap \{e_n\}_{n=0}^\infty = \emptyset$ or $\mathcal{N} \cap \{e_n\}_{n=0}^\infty = \emptyset$. Let's suppose that $\mathcal{M} \cap \{e_n\}_{n=0}^\infty = \emptyset$. Let $S = D|\mathcal{M}$. Then S is a subnormal operator.

If we let $P : \ell^2 \to \mathcal{M}$ be a continuous surjective idempotent such that $PD = DP$, then $\{P(e_n)\}_{n=0}^\infty$ has dense linear span in $\mathcal{M}$ and $P(e_n)$ is an eigenvector for S for every $n \geq 0$. It follows that S is a diagonalizable subnormal operator. Thus S is a normal operator. Which implies that $\mathcal{M}$ is a reducing subspace for D. Since D is diagonalizable, then every reducing subspace of D contains an eigenvector for D. Since the eigenvalues of D are distinct, the only eigenvectors for D are multiples of the e_n's. Thus $\mathcal{M} \cap \{e_n\}_{n=0}^\infty \neq \emptyset$. A contradiction, thus A is irreducible. □

Hyponormal Operators: For a hyponormal operator T with trace class self-commutator, there is a function $g \in L^1(\sigma(T), dA)$ called the *principal function* of T and is uniquely determined by the fact that

$$tr[p(T^*), q(T)] = \frac{1}{\pi} \int_{\sigma(T)} \overline{p'(z)} q'(z) g(z) dA(z)$$

for all analytic polynomials p, q. It is known that the spectrum of T is the essential support of g, that $g(z) = -\text{ind}(T - zI)$ for all $z \notin \sigma_e(T)$ and that $0 \leq g(z) \leq rank[T^*, T]$ for all z.

The following result was first proved in Pincus [17] and the reader is referred to Martin & Putinar [11, p. 261].

Theorem 2.14 (Hyponormals with prescribed spectra)**.** *If g is a measurable function with compact support on $\mathbb{C}$ and $0 \leq g(z) \leq 1$ for all z, then there exists an irreducible hyponormal operator T with rank one self-commutator whose principal function is g.*

If $rank[T^*, T] = 1$ and $\lambda \in \mathbb{C}$, then Carey and Pincus have shown that $\lambda \in \sigma_e(T)$ if and only if for every $r > 0$, g is not almost everywhere equal to 0 or 1 on $\{z : |z - \lambda| < r\}$ (see Martin & Putinar [11, p. 263]). Using this and the previous theorem, one may construct irreducible hyponormal operators with rank one self-commutators having prescribed spectral pictures. For these and other basic properties of g see Martin & Putinar [11].

3. Arbitrary Operators with thin spectra

In this section we consider the general question of when an operator has a subnormal generator for its C^*-algebra without additional assumptions—such as irreducibility or essential normality—imposed on the operator.

Our first result, definitely has the flavor of a subnormal operator type of result, it says that if the spectrum of an operator is "thin", then its C^*-algebra cannot have a subnormal generator except in the trivial case.

A compact set $K \subseteq \mathbb{C}$ is said to be polynomially convex if $\mathbb{C} \backslash K$ is connected. Requiring a compact set to be polynomially convex and have no interior, is a measure of "smallness" or "thinness" of the set. By Lavrentiev's Theorem (see Conway [3, page 232]) it is equivalent to requiring that $P(K) = C(K)$, that is the (analytic) polynomials are uniformly dense in the continuous functions on K.

Theorem 3.1. *If $A \in \mathcal{B}(\mathcal{H})$ and $\sigma(A)$ is a polynomially convex set with no interior, then $C^*(A)$ has a subnormal generator if and only if A is a normal operator.*

Proof. Suppose that S is a subnormal operator and $C^*(S) = C^*(A)$. Let $\mathcal{I}$ denote the commutator ideal of $C^*(S) = C^*(A)$. It follows by a result of Bunce (see Conway [3, page 89]) that $C(\sigma_{ap}(S)) \cong C^*(S)/\mathcal{I} = C^*(A)/\mathcal{I}$. Since the equivalence class of A, $[A]$, in $C^*(S)/\mathcal{I}$ generates $C^*(S)/\mathcal{I}$, it follows from Lemma 2.8 that $[A]$ has the form $[A] = \phi([S])$ where ϕ is a one-to-one continuous function on $\sigma_{ap}(S)$ and $[S]$ is the equivalence class of S. It then follows that $\phi(\sigma_{ap}(S)) = \phi(\sigma([S])) = \sigma([A]) \subseteq \sigma(A)$. Since a compact subset of a polynomially convex set with no interior, is also polynomially convex with no interior, it follows that $\phi(\sigma_{ap}(S))$ is polynomially convex and has no interior. Since polynomial convexity and having interior are both invariant under homeomorphisms, it follows that $\sigma_{ap}(S)$ is polynomially convex and has no interior. Now for any subnormal operator S, $\sigma(S)$ is contained in the polynomially convex hull of $\sigma_{ap}(S)$. Since, $\sigma_{ap}(S)$ is already polynomially convex, it follows that $\sigma(S) = \sigma_{ap}(S)$. Hence $\sigma(S)$ is polynomially convex and has no interior. So, $P(\sigma(S)) = C(\sigma(S))$, in particular, S must be normal. Thus, $C^*(S)$ is abelian. Since $A, A^* \in C^*(S)$, A must commute with A^*, so A is also normal. □

Corollary 3.2. *If A is compact, quasinilpotent, has a countable spectrum, or has spectrum contained in a line segment, then $C^*(A)$ has a subnormal generator if and only if A is a normal operator.*

The following proposition follows from the fact that $*$-homomorphisms preserve subnormality (see Conway [3, p. 35]).

Proposition 3.3. *If $C^*(A)$ has a subnormal generator and $\phi : C^*(A) \to C^*(B)$ is an onto $*$-homomorphism, then $C^*(B)$ also has a subnormal generator. In particular, if $C^*(A \oplus B)$ has a subnormal generator, then $C^*(A)$ and $C^*(B)$ both have subnormal generators.*

Corollary 3.4. *(a) If A, B are any two operators, the spectrum of B is polynomially convex with no interior, and B is not normal, then $C^*(A \oplus B)$ does not have a subnormal generator.*

(b) If A is any operator and B is a non-zero compact operator that is not normal, then $C^*(A \oplus B)$ does not have a subnormal generator.

Corollary 3.5 (A hyponormal operator with no subnormal generator)**.** *There exists an irreducible hyponormal operator A with rank one self-commutator such that $C^*(A)$ does not have a subnormal generator.*

Proof. There is an irreducible hyponormal operator A with rank one self-commutator such that $\sigma(A)$ is a Cantor set with positive area density at each of its points (simply let K be such a Cantor set, g equal to the characteristic function of K and apply Theorem 2.14). However, a Cantor set is polynomially convex with no interior, hence Theorem 3.1 applies. □

In addition to polynomially convex with no interior, there are other measures of smallness, such as the rational functions being dense in the continuous functions, or even more restrictive, having area zero. We now give an example showing that one cannot improve Theorem 3.1 to either of these measures of smallness.

Example 3.6. *There is an irreducible essentially normal operator A acting on an infinite dimensional Hilbert space whose spectrum has area zero and yet $C^*(A)$ has a subnormal generator.*

Proof. Let K be a Swiss-cheese set with only one non-trivial Gleason part which is dense in K (see Theorem 7.5) and let S be an irreducible essentially normal (rationally cyclic) subnormal operator whose spectrum equals K. By Brouwer's Theorem (see Theorem 2.2) there is a compact set L with area zero that is homeomorphic to K. Let $\phi : K \to L$ be a homeomorphism. Also, by Theorem 2.12, there is an irreducible essentially normal operator A with $\sigma(A) = \sigma_e(A) = L$ and thus has $\text{ind}(A - \lambda I) = 0$ for all $\lambda \notin L$. By Proposition 2.7, $\text{ind}(\phi(S) - \lambda I) = 0$ for all $\lambda \notin \sigma_e(\phi(S)) = \phi(\sigma_e(S)) = L$, thus $\phi(S)$ and A have the same spectral picture, so it follows from Proposition 2.9(b) that $C^*(A)$ has a generator unitarily equivalent to S. □

An operator A is pure if it has no reducing subspace on which it is normal.

Proposition 3.7. *If A is a pure operator, then every generator of $C^*(A)$ is also pure.*

Proof. Simply note that A is pure if and only if the commutator ideal, $\mathcal{I}$, of $C^*(A)$ acts non-degenerately on the underlying Hilbert space $\mathcal{H}$; meaning that $\{Tx : x \in \mathcal{H}, T \in \mathcal{I}\}$ is dense in $\mathcal{H}$. Since this condition is independent of the generator, if one generator is pure, then all the other generators must also be pure. □

4. Pure Essentially Normal Operators

Question 4.1. *Which essentially normal operators $A \in \mathcal{B}(\mathcal{H})$ have the property that $C^*(A)$ has a subnormal generator?*

We will see that even this question is difficult to answer and indeed the answer depends on more than just the spectral picture of the operator A (see Example 4.7). We begin by considering those operators whose index function vanishes identically.

Proposition 4.2. *If A is a pure essentially normal operator, $ind(A - \lambda I) = 0$ for all $\lambda \notin \sigma_e(A)$ and if $C^*(A)$ has a subnormal generator, then $\sigma_e(A)$ is homeomorphic to the spectrum of a pure subnormal operator. In fact, $\sigma_e(A)$ is homeomorphic to the spectrum of any subnormal generator of $C^*(A)$.*

Proof. Since $C^*(A)$ has a subnormal generator, there is a subnormal operator S such that $C^*(S) = C^*(A)$. Since A is pure, it follows from Proposition 3.7 that S is also pure. Since A is a generator of $C^*(S)$, A has the form $A = \phi(S) + C$ where $\phi \in C(\sigma_n(S))$, $\phi|\sigma_e(S)$ is one-to-one, and C is in the commutator ideal $\mathcal{I} \subseteq \mathcal{B}_0$ of $C^*(S)$. It follows that $\phi(\sigma_e(S)) = \sigma_e(\phi(S)) = \sigma_e(\phi(S) + C) = \sigma_e(A)$. Thus, $\sigma_e(A)$ is homeomorphic to $\sigma_e(S)$. We now show that in fact $\sigma_e(S) = \sigma(S)$.

If $\sigma(S) \neq \sigma_e(S)$, then since S is pure, $\mathrm{ind}(S - \lambda I) \neq 0$ for all $\lambda \in \sigma(S) \setminus \sigma_e(S)$. So by Proposition 2.7, $\mathrm{ind}(A - \lambda I) = \mathrm{ind}(\phi(S) - \lambda I) \neq 0$ for some $\lambda \in \mathbb{C}$. A contradiction, since we are assuming that $\mathrm{ind}(A - \lambda I) = 0$. Thus $\sigma(S) = \sigma_e(S)$ and so $\sigma_e(A)$ is homeomorphic to $\sigma(S)$. □

The above Theorem naturally suggests the following question.

Question 4.3. *(a) Which compact sets L have the property that L is homeomorphic to the spectrum of a pure subnormal operator?*

(b) Which compact sets L have the property that L is homeomorphic to a compact set K satisfying $R(K) \neq C(K)$?

Here is a necessary condition for question (a) above, the authors do not know if it is also sufficient.

Proposition 4.4. *If L is a compact set homeomorphic to the spectrum of a pure subnormal operator, then for any open disk Δ*

$$(*) \qquad \Delta \cap L \neq \emptyset \implies P(cl\Delta \cap L) \neq C(cl\Delta \cap L).$$

Proof. Suppose that S is a pure subnormal operator and L is homeomorphic to $\sigma(S)$. By the Clancey-Putnam condition (see Conway [3, p. 180]), for any open disk Δ satisfying $\Delta \cap \sigma(S) \neq \emptyset$, we must have $R(cl\Delta \cap \sigma(S)) \neq C(cl\Delta \cap \sigma(S))$. Thus $\sigma(S)$ has property $(*)$. Since $(*)$ is a topological invariant, that is invariant under homeomorphisms, and L is homeomorphic to $\sigma(S)$, then L must also satisfy condition $(*)$. □

Corollary 4.5. *If $A \in \mathcal{B}(\mathcal{H})$ is a pure essentially normal operator and ind$(A - \lambda I) = 0$ for all $\lambda \notin \sigma_e(A)$, and $C^*(A)$ has a subnormal generator, then $\sigma_e(A)$ has the following property: for any open disk Δ*

$$(*) \qquad \Delta \cap \sigma_e(A) \neq \emptyset \implies P(cl\Delta \cap \sigma_e(A)) \neq C(cl\Delta \cap \sigma_e(A)).$$

Proof. If $C^*(A)$ has a subnormal generator, say S, then by Proposition 4.2, $\sigma_e(A)$ is homeomorphic to $\sigma(S)$. Since A is pure, then by Proposition 3.7, S is also pure. Now apply Proposition 4.4. □

With regard to question 4.3(b), it is known that if K is any Jordan curve, then $R(K) = C(K)$. Hence the unit circle is not homemomorphic to a compact set K satisfying $R(K) \neq C(K)$.

Corollary 4.6. *If A is an essentially normal operator that is not normal and $\sigma_e(A)$ is a Jordan curve and ind$(A - \lambda I) = 0$ for all $\lambda \notin \sigma_e(A)$, then $C^*(A)$ does not have a subnormal generator.*

Example 4.7. *(a) If S is the unilateral shift and $A = S \oplus S^*$, then $C^*(A)$ does not have a subnormal generator, because $\sigma_e(A) = \{z : |z| = 1\}$ and ind$(A - \lambda I) = 0$ for all $\lambda \notin \sigma_e(A)$.*

(b) If S is the unilateral shift and $T = S \oplus S \oplus S^*$, then S and T have the same spectral picture, $C^*(S)$ clearly has a subnormal generator, yet $C^*(T)$ does not have a subnormal generator (use part (a) and Proposition 3.3).

It follows from the previous example that two essentially normal operators may have the same spectral picture, and yet one has a subnormal generator for its C^*-algebra, but the other does not.

5. Irreducible Essentially Normal Operators with Zero Index

Unlike in Example 4.7, if A is *irreducible* and *essentially normal*, then it follows from Proposition 2.9 or Theorem 2.10 that whether or not $C^*(A)$ has a subnormal or hyponormal generator depends only on the spectral picture of A. It is equivalent to asking if the spectral picture of A is homeomorphic to the spectral picture of an irreducible subnormal (or hyponormal) operator.

There are basically two cases to understand: when ind$(A - \lambda I) = 0$ for all $\lambda \notin \sigma_e(A)$ and when ind$(A - \lambda I) \neq 0$ for some $\lambda \notin \sigma_e(A)$. In this section we consider the first case.

Theorem 5.1 (Zero index, subnormal generator)**.** *If $A \in \mathcal{B}(\mathcal{H})$ is an irreducible essentially normal operator and ind$(A - \lambda I) = 0$ for all $\lambda \notin \sigma_e(A)$, then $C^*(A)$ has a subnormal generator if and only if $\sigma_e(A)$ is homeomorphic to a compact set K such that $R(K)$ has exactly one nontrivial Gleason part which is dense in K.*

Proof. Suppose $A \in \mathcal{B}(\mathcal{H})$ is an irreducible essentially normal operator and ind$(A - \lambda I) = 0$ for all $\lambda \notin \sigma_e(A)$. First suppose that $C^*(A)$ has a subnormal generator, say S. Then by Proposition 4.2, the essential spectrum of A is homeomorphic to the spectrum of S. Since A is irreducible, so is S. Furthermore, Olin and Thomson [14] have shown that the spectrum of an irreducible subnormal operator must satisfy that $R(\sigma(S))$ has exactly one non-trivial Gleason part which must be dense in $\sigma(S)$. Thus, we may choose K to be $\sigma(S)$.

Conversely, suppose that $\sigma_e(A)$ is homeomorphic to a compact set K such that $R(K)$ has exactly one non-trivial Gleason part which is dense in K. Then by [6], there exists an irreducible essentially normal subnormal operator S such that $\sigma(S) = \sigma_e(S) = K$. Thus, $\operatorname{ind}(S - \lambda I) = 0$ for all $\lambda \notin \sigma_e(S)$. Let $\phi : \sigma_e(A) \to K$ be a homeomorphism. Then we have $\sigma_e(\phi(A)) = \phi(\sigma_e(A)) = K$. Since $\operatorname{ind}(A-\lambda I) = 0$ for all $\lambda \notin \sigma_e(A)$, then by Theorem 2.6, $\operatorname{ind}(\phi(A) - \lambda I) = 0$ for all $\lambda \notin \sigma_e(\phi(A)) = K$. Thus, $\phi(A)$ and S have the same spectral picture, so by Proposition 2.9, $C^*(A)$ has a generator unitarily equivalent to S. □

Question 5.2. *(a) Which compact sets L have the property that L is homeomorphic to the spectrum of an irreducible subnormal operator? This is equivalent to asking: (b) which compact sets L have the property that L is homeomorphic to a compact set K such that R(K) has exactly one nontrivial Gleason part which is dense in K.*

The deficiency of Theorem 5.1 lies in the inability to give a practical answer to Question 5.2(b). Proposition 5.3 and Corollary 5.4 address this issue in a modest way and are applied in Example 5.5.

Proposition 5.3. *If L is a compact set homeomorphic to the spectrum of an irreducible subnormal operator, then L is connected and for any open disk* Δ

$$(*) \qquad \Delta \cap L \neq \emptyset \implies P(cl\Delta \cap L) \neq C(cl\Delta \cap L).$$

Proof. The spectrum of an irreducible subnormal operator is connected and hence L must be also. For condition (*) simply apply Proposition 4.4. □

Corollary 5.4. *If* $A \in \mathcal{B}(\mathcal{H})$ *is an irreducible essentially normal operator and* $ind(A - \lambda I) = 0$ *for all* $\lambda \notin \sigma_e(A)$*, and* $C^*(A)$ *has a subnormal generator, then* $\sigma_e(A)$ *is a connected set with the following property: for any open disk* Δ

$$(*) \qquad \Delta \cap \sigma_e(A) \neq \emptyset \implies P(cl\Delta \cap \sigma_e(A)) \neq C(cl\Delta \cap \sigma_e(A)).$$

Proof. If $C^*(A)$ has a subnormal generator, say S, then by Theorem 5.1 $\sigma_e(A)$ is homeomorphic to $\sigma(S)$. Since A is irreducible, then S is also irreducible. Thus we may apply Proposition 5.3. □

Example 5.5. *Suppose that A is irreducible, essentially normal, and* $ind(A-\lambda I) = 0$ *for all* $\lambda \notin \sigma_e(A)$*.*

(a) If $\sigma_e(A)$ *is contained in a Jordan arc or a smooth curve, then* $C^*(A)$ *does not have a subnormal generator.*

(b) If $\sigma_e(A) = \{z : |z| \leq 1\} \cup \{z : |z - 2| \leq 1\}$ *consists of two externally tangent closed disks, then* $C^*(A)$ *does not have a subnormal generator, even though* $\sigma_e(A)$ *is connected and satisfies property* (*) *in Corollary 5.4.*

(c) If $\sigma_e(A)$ *is a string of beads, then* $C^*(A)$ *does have a subnormal generator—even though for a string of beads set K,* $R(K)$ *may have one or two nontrival Gleason parts.*

Proof. (a) Here one can check that property ($*$) of Corollary 5.4 is not satisfied.

(b) In this case, any compact set K homeomorphic to $\sigma_e(A)$ will have the form of two Jordan curves that touch at a point, each in the unbounded component of the other, together with their insides. For such a set K, $\mathbb{C} \setminus K$ is connected, thus the Gleason parts of $R(K)$ are precisely the components of the interior of K. Thus $R(K)$ has two non-trivial Gleason parts. So by Theorem 5.1, $C^*(A)$ does not have a subnormal generator.

(c) A string of beads is a compact set of the form $K = \text{cl}\mathbb{D} \setminus \bigcup_{k=1}^{\infty} \Delta_k$ where $\{\Delta_k\}$ is a sequence of open disks centered on the interval $[-1, 1]$, with pairwise disjoint closures, $cl\Delta_k \subseteq \mathbb{D}$ for all k, and $F := [-1, 1] \setminus \bigcup_{k=1}^{\infty} \Delta_k$ has no interior (considering F as a subset in the real line). Notice that F is a Cantor set in $[-1, 1]$. Since for any two Cantor sets in $[-1, 1]$ there is a homemorphism of $[-1, 1]$ that maps one Cantor set to the other, it follows rather easily, using Theorem 2.5, that any two string of beads sets are also homeomorphic. Also, whenever F has positive one-dimensional Lebesgue measure, then $R(K)$ has only one non-trivial Gleason part which is dense in K. Thus, Theorem 5.1 applies. □

The next result should be compared to Theorem 5.1 to see the difference between having a subnormal generator and a hyponormal generator. Recall a perfect set is a closed set with no isolated points.

Theorem 5.6 (Zero index, hyponormal generator)**.** *If $A \in \mathcal{B}(\mathcal{H})$ is an irreducible essentially normal operator and ind$(A - \lambda I) = 0$ for all $\lambda \notin \sigma_e(A)$, then $C^*(A)$ has a hyponormal generator if and only if $\sigma_e(A)$ is a perfect set.*

Remark. Notice that $\sigma_e(A)$ need not be connected to have a hyponormal generator, unlike in the subnormal case. Recall that there exist irreducible essentially normal operators with any spectral picture (see Theorem 2.11).

Proof of Theorem 5.6. If $C^*(A)$ has a hyponormal generator, say T, then by Proposition 2.9, there is a homeomorphism $h : \sigma_e(A) \to \sigma_e(T)$ such that $h(A)$ and T have the same spectral picture. Since $\text{ind}(A - \lambda I) = 0$ for all $\lambda \in \mathbb{C} \setminus \sigma_e(A)$, then by the Brown-Douglas-Fillmore Theory (Theorem 2.1) A is unitarily equivalent modulo the compacts to a normal operator. Since h is continuous, it follows that $h(A)$ is also unitarily equivalent modulo the compacts to a normal operator. Thus the index function of $h(A)$ is identically zero. Since $h(A)$ and T have the same spectral picture, it follows that the index function of T is identically zero. Since A is irreducible, T is also irreducible, hence pure. Thus, $\sigma_e(T) = \sigma(T)$. So, $\sigma_e(A)$ is homeomorphic (via h) to $\sigma(T)$, the spectrum of an irreducible hyponormal operator. It is known (see [11, p. 132]) that the spectrum of a pure hyponormal operator has positive area density at each of its points. Thus $\sigma(T)$ is a perfect set and hence $\sigma_e(A)$ is also a perfect set.

For the converse suppose that $\sigma_e(A)$ is a perfect set. Then by Theorem 2.3, there is a homeomorphism h such that $K := h(\sigma_e(A))$ has positive area density at each of its points. Using BDF-Theory as above the index function of $h(A)$ will be identically zero. Now by Theorem 2.14, there exists an irreducible hyponormal operator T with $\sigma(T) = \sigma_e(T) = K$. Since $\sigma(T) = \sigma_e(T)$, then the index function of T is identically zero. Thus T and $h(A)$ have the same spectral picture, so by Proposition 2.9, $C^*(A)$ has a generator unitarily equivalent to T. □

6. Irreducible Essentially Normal Operators with Non-Zero Index

When A is an irreducible essentially normal operator and $\mathrm{ind}(A-\lambda I) \neq 0$ for some $\lambda \notin \sigma_e(A)$, then it is much more likely that $C^*(A)$ will have a subnormal or hyponormal generator, then in the case where $\mathrm{ind}(A-\lambda I)=0$ for all $\lambda \in \mathbb{C}\setminus\sigma_e(A)$. For example the following Theorem shows that if $\mathrm{ind}(A-\lambda I) \neq 0$ for some $\lambda \notin \sigma_e(A)$, then the essential spectrum of A need not be connected for $C^*(A)$ to have a subnormal generator (contrast this with Corollary 5.4). However, we shall see that there are still restrictions in many cases.

Theorem 6.1 (Direct sums with disjoint essential spectra)**.** *Suppose that the operators $A_1, A_2, \ldots, A_n$ are essentially normal and such that for each $i \in \{1,\ldots,n\}$, $C^*(A_i)$ has a subnormal generator. Suppose also that $\sigma_e(A_i)\cap\sigma_e(A_j)=\emptyset$ whenever $i \neq j$ and that A_n is irreducible and $ind(A_n-\lambda I) \neq 0$ for some $\lambda \notin \sigma_e(A_n)$. If B is any irreducible essentially normal operator with the same spectral picture as $\bigoplus_{i=1}^n A_i$, then $C^*(B)$ has a subnormal generator.*

Notice that the operators A_i for $1 \le i \le (n-1)$ need not be irreducible, and need not be pure. In fact, they can be normal operators. The Theorem only requires that one of the operators be irreducible with non-zero index. In fact, if all the operators A_i have index functions which vanish identically, then Corollary 5.4 applies to say that the conclusion of the above theorem is not true since the direct sum would have a disconnected spectrum.

Proof. For $i \in \{1,\ldots,n\}$, let S_i be a subnormal generator of $C^*(A_i)$. Thus for $i \in \{1,\ldots,n\}$, there exists continuous functions $\phi_i \in C(\sigma_n(S_i))$ and compact operators K_i such that $A_i = \phi(S_i)+K_i$ and ϕ_i is one-to-one on $\sigma_e(S_i)$.

Since A_n is irreducible it follows that S_n is also irreducible. Also, since $ind(A_n-\lambda I) \neq 0$ for some $\lambda \notin \sigma_e(A_n)$, we get by Proposition 2.7 that $ind(S_n-\lambda I)\neq 0$ for some $\lambda \notin \sigma_e(S_n)$. Thus, since S_n is pure, $\sigma(S_n)\setminus\sigma_e(S_n)$ is a non-empty open set. Choose $n-1$ disjoint closed disks contained in $\sigma(S_n)\setminus\sigma_e(S_n)$, call them $\{B_1,\ldots,B_{n-1}\}$. For each i satisfying $1\le i\le(n-1)$, choose $\epsilon_i>0$ and $b_i\in\mathbb{C}$ such that if $f_i(z)=\epsilon_i z+b_i$, then $\sigma(f_i(S_i))\subseteq B_i$. Then let $T := \left(\bigoplus_{i=1}^{n-1} f_i(S_i)\right)\oplus S_n$. Notice that $\sigma(T)=\sigma(S_n)$ and since S_n is irreducible, it follows that $R(\sigma(T))$ has exactly one non-trivial Gleason part which is dense in $\sigma(T)$. By [6] there exists an irreducible essentially normal subnormal operator S with the same spectral picture as T. Now define a continuous function ϕ on $\sigma_e(S)$ as follows: $\phi(z)=\phi_i(f_i^{-1}(z))$ if $z\in\sigma_e(f_i(S_i))$, $1\le i\le(n-1)$ and $\phi(z)=\phi_n(z)$ if $z\in\sigma_e(S_n)$. Since the $B_i\cap B_j=\emptyset$ if $i\neq j$, it follows that ϕ is well defined and continuous on $\sigma_e(S)$. Also, since $\sigma_e(S_i)\cap\sigma_e(S_j)=\emptyset$ when $i\neq j$, it follows that ϕ is one-to-one on $\sigma_e(S)$. Now extend ϕ to be a continuous function on $\sigma_n(S)$ (we will still denote the extension by ϕ).

We claim that $\phi(S)$ has the same spectral picture as $\bigoplus_{i=1}^n A_i$. To see this, simply note that by Brown-Douglas-Fillmore Theory [2], S is unitarily equivalent to a compact perturbation of T. As ϕ is a continuous function on $\sigma_n(S)$ and $\sigma_n(T)$ (extend ϕ if necessary), it follows that $\phi(S)$ is unitarily equivalent to a compact perturbation of $\phi(T)$. But since $A_i=\phi_i(S_i)+K_i$, it follows that $\phi(T)=\bigoplus_{i=1}^n\phi_i(S_i)=\bigoplus_{i=1}^n(A_i-K_i)$. Hence, $\phi(T)$ has the same spectral picture as $\bigoplus_{i=1}^n A_i$; thus $\phi(S)$ does also.

Since $\phi(S)$ has the same spectral picture as B, it follows by Proposition 2.9 that $C^*(B)$ has a subnormal generator unitarily equivalent to S. $\square$

We give another proof of the following result due to McGuire [12].

Corollary 6.2. *If A is an irreducible essentially normal operator whose essential spectrum, $\sigma_e(A)$, is a disjoint union of a finite number of Jordan curves, then $C^*(A)$ has a subnormal generator if and only if $ind(A-\lambda I) \neq 0$ for some $\lambda \in \mathbb{C} \setminus \sigma_e(A)$.*

Proof. For a Jordan curve γ in the complex plane and for a negative integer n, let $S(\gamma, n)$ be an irreducible essentially normal subnormal operator such that $\sigma_e(S(\gamma,n)) = \gamma$ and $ind(S(\gamma,n) - \lambda I) = n$ for all $\lambda \in inside(\gamma)$ (see [6]). If $n = 0$, then let $S(\gamma, 0)$ be a normal operator with $\sigma(S(\gamma,0)) = \sigma_e(S(\gamma,0)) = \gamma$. If n is a positive integer, then let $S(\gamma,n) = S(\gamma^*, -n)^*$, where γ^* is the reflection of γ about the real axis. Then for every integer n, $S(\gamma,n)$ is an essentially normal operator with essential spectrum equal to γ and whose index function is n on the inside of γ. Furthermore, $S(\gamma,n)$ is irreducible if $n \neq 0$ and for each $n \in \mathbb{Z}$, $S(\gamma,n)$ is either subnormal or cosubnormal. Thus, $C^*(S(\gamma,n))$ has a subnormal generator.

Now suppose that $\sigma_e(A)$ is the union of the disjoint Jordan curves, $\{\Gamma_i : i \in I\}$ where I is a finite index set. We need to choose operators $\{A_i\}$ as in Theorem 6.1. For each curve Γ_i we will choose $A_i = S(\Gamma_i, n_i)$ where n_i is an integer to be chosen such that $\bigoplus_{i=1}^n A_i$ has the same spectral picture as A.

To see how to do this, define a partial order on I as follows: If $i, j \in I$, then $i \prec j$ if and only if $inside(\Gamma_i) \subseteq inside(\Gamma_j)$. If $i \in I$ is a maximal element and $m_i = ind(A - \lambda I)$ for $\lambda \in inside(\Gamma_i) \setminus \bigcup_{j \neq i} inside(\Gamma_j)$, then set $n_i = m_i$ and $A_i = S(\Gamma_i, n_i)$. So, A_i is defined whenever i is a maximal element of I. Now working in an "inductive" manner, suppose that $i \in I$ and that for every $j \in I$ with $i \prec j$, n_j has been chosen and A_j is defined as $S(\Gamma_j, n_j)$. We now choose n_i and A_i as follows: Let

$$n_i = \left[ind(A - \lambda I) - \sum_{\{j: i \prec j\}} n_j \right] \text{ where } \lambda \in inside(\Gamma_i) \setminus \bigcup_{j \prec i} inside(\Gamma_j).$$

Then let $A_i = S(\Gamma_i, n_i)$. In this manner, we have chosen operators $A_i = S(\Gamma_i, n_i)$ for each $i \in I$. Furthermore, it follows that $\bigoplus_{i \in I} A_i$ has the same spectral picture as A. Since $ind(A - \lambda I) \neq 0$ for some $\lambda \in \mathbb{C} \setminus \sigma_e(A)$ we have that $n_i \neq 0$ for some $i \in I$ and thus for that i, A_i is an irreducible essentially normal operator such that $C^*(A_i)$ has a subnormal generator and $\text{ind}(A_i - \lambda I) = n_i \neq 0$ for $\lambda \in inside(\Gamma_i)$. Thus by Theorem 6.1, $C^*(A)$ has a subnormal generator. $\square$

Theorem 6.3 (Attaching a Jordan Curve)**.** *Suppose that A_1 is an irreducible essentially normal operator and that A_2 is an essentially normal operator with essential spectrum equal to a Jordan curve γ. Suppose also that*

1. *$inside(\gamma)$ is contained in the unbounded component of $\mathbb{C} \setminus \sigma_e(A_1)$.*
2. *There is a component G of $\mathbb{C} \setminus \sigma_e(A_1)$ that is bounded by a Jordan curve such that $\gamma \cap \partial G$ is a non-trivial Jordan arc.*

Let B be an irreducible essentially normal operator with the same spectral picture as $A_1 \oplus A_2$. If $C^*(A)$ has a subnormal generator, then $C^*(B)$ also has a subnormal generator.

See Proposition 7.19 for an example of Theorem 6.3.

Proof. Since $C^*(A_1)$ has a subnormal generator, then by Theorem 2.10 there is a compact set $K_e \subseteq \mathbb{C}$ and a homeomorphism $h : \sigma_e(A_1) \to K_e$ such that $\text{ind}(h(A_1) - \lambda I) \leq 0$ for all $\lambda \in \mathbb{C} \setminus K_e$ and if $K := K_e \cup \{\lambda : \text{ind}(h(A_1) - \lambda I) < 0\}$, then $R(K)$ has only one non-trivial Gleason part which is dense in K. Let $I = \gamma \cap \partial G$, then I is a non-trivial Jordan arc (homeomorphic to the unit interval $[0, 1]$). Note $h(I)$ is also a nontrivial Jordan arc contained in the Jordan curve $h(\partial G)$. If $p \in I$ and is not an endpoint of I, then there is an $r > 0$ such that $B(p, r) \cap \sigma_e(A_1) = B(p, r) \cap I$. Thus if $a \in h(I)$ and is not an endpoint of $h(I)$, then there is an $\epsilon > 0$ such that $B(a, \epsilon) \cap K_e = B(a, \epsilon) \cap h(I)$. Now since the non-trivial Gleason part of $R(K)$ is dense in K, then it follows that $h(I)$ is in the closure of $\{\lambda : \text{ind}(h(A_1) - \lambda I) < 0\}$. Since $h(I)$ is "isolated", as described above, in K_e, it follows that there is a component Ω_1 of $\mathbb{C} \setminus K_e$ such that $ind(h(A_1) - \lambda I) < 0$ if $\lambda \in \Omega_1$ and such that $h(I) \subseteq \partial\Omega_1$. There must also be another component Ω_2 of $\mathbb{C} \setminus K_e$ that is not equal to Ω_1 such that $h(I) \subseteq \partial\Omega_2$ (Ω_i are the components "on each side" of $h(I)$.

Let γ_i be a Jordan arc that is contained in Ω_i except for its endpoints which are the same as the endpoints of $h(I)$. Thus $\Gamma_i := h(I) \cup \gamma_i$ is a Jordan curve whose inside is contained in Ω_i. We will extend h from $\sigma_e(A)$ to $\sigma_e(A) \cup \gamma$ by defining h_i to be a homeomorphism from γ onto γ_i. Now either h_1 or h_2 will be the desired extension. Let $a = ind(A_2 - \lambda I)$ for $\lambda \in inside(\gamma)$. Notice that

$$ind(h_i(B) - \lambda I) = \text{ind}(h(A_1) - \lambda I) + a \cdot n(h_i(\gamma), \lambda)$$

for $\lambda \in inside(\Gamma_i)$.

If $a = 0$, then h_1 is the desired extension. Otherwise if $a \neq 0$, then note that the winding numbers $n(h_i(\gamma), \lambda) = \pm 1$ and that $n(h_1(\gamma), \lambda_1) = -n(h_2(\gamma), \lambda_2)$ where $\lambda_i \in inside(\Gamma_i)$. Hence either $a \cdot n(h_1(\gamma), \lambda_1) < 0$ for $\lambda_1 \in inside(\Gamma_1)$ or $a \cdot n(h_2(\gamma), \lambda_2) < 0$ for $\lambda_2 \in inside(\Gamma_2)$.

Let $j \in \{1, 2\}$ be such that $a \cdot n(h_j(\gamma), \lambda_j) < 0$ for $\lambda_j \in inside(\Gamma_j)$. Since we know that $ind(h(A_1) - \lambda I) \leq 0$ for all $\lambda \notin K_e$, then it follows that $ind(h_j(B) - \lambda I) \leq 0$ for all $\lambda \notin K_e \cup \Gamma_j$. Further if $K' := (K_e \cup \Gamma_j) \cup \{\lambda : \text{ind}(h_j(B) - \lambda I) < 0\}$, then either $K' = K$ or $K' = K \cup [clinside(\Gamma_j)]$. In either case it follows that $R(K')$ has only one non-trivial Gleason part which is dense in K'. Thus it follows from Theorem 2.10 that $C^*(B)$ has a subnormal generator. □

The following two results makes precise the most fundamental operation: That of "flipping" about a Jordan curve. The operation maps everything that is outside of the Jordan curve to the inside of the curve. These results follows from Theorem 2.6 and Rouche's Theorem.

Proposition 6.4 (Flipping about a Jordan Curve)**.** *Let A be an irreducible essentially normal operator, G_0 a bounded component of $\mathbb{C} \setminus \sigma_e(A)$ that is bounded by a Jordan curve, and $N = ind(A - \lambda I)$ for $\lambda \in G_0$. Also let $U = (\mathbb{C} \setminus clG_0) \cup \{\infty\}$. If $\psi : U \to G_0$ is a Riemann map, then ψ extends to be continuous on $clU \supseteq \sigma_e(A)$ and the spectral picture of $\psi(A)$ is as follows: $\sigma_e(\psi(A)) = \psi(\sigma_e(A))$ and $ind(\psi(A) - \psi(\lambda)I) = ind(A - \lambda I) - N$, $\lambda \in \mathbb{C} \setminus \sigma_e(A)$.*

Proposition 6.5 (Flipping about more general regions)**.** *Let A be an irreducible essentially normal operator, G_0 a bounded component of $\mathbb{C} \setminus \sigma_e(A)$ and $N = ind(A - \lambda I)$ for $\lambda \in G_0$. Also let Δ be a closed disk contained in G_0. If ψ is a Mobius* *transformation such that $\psi(\mathbb{C} \setminus \Delta) = \Delta$, then the spectral picture of $\psi(A)$ is as follows: $\sigma_e(\psi(A)) = \psi(\sigma_e(A))$ and $ind(\psi(A) - \psi(\lambda)I) = ind(A - \lambda I) - N$, $\lambda \in \mathbb{C} \setminus \sigma_e(A)$.*

Hence the flipping operation effects the values of the Fredholm index by subtracting the value N from each of the other values of the index. Hence if N is a strict maximum, then all the values of the index become negative.

We now give a basic result that will be improved upon later that guarantees that $C^*(A)$ has a subnormal generator.

Theorem 6.6. *Suppose A is an irreducible essentially normal operator such that one of the following hold:*

(1) $M := \sup\{ind(A-\lambda I) : \lambda \in \mathbb{C} \setminus \sigma_e(A)\} < \infty$ *and $int(K_M)$ is connected and dense in K_M where $K_M := \sigma_e(A) \cup \{\lambda \in \mathbb{C} \setminus \sigma_e(A) : ind(A - \lambda I) < M\}$.*
or
(2) $m := \inf\{ind(A - \lambda I) : \lambda \in \mathbb{C} \setminus \sigma_e(A)\} > -\infty$ *and $int(K_m)$ is connected and dense in K_m where $K_m := \sigma_e(A) \cup \{\lambda \in \mathbb{C} \setminus \sigma_e(A) : ind(A-\lambda I) > m\}$.*

Then $C^*(A)$ has a subnormal generator.

Remark. Notice that $M \geq 0$ and $m \leq 0$ since the Fredholm index is always zero in the unbounded component of $\mathbb{C} \setminus \sigma_e(A)$. Also notice that if $M > 0$ (or $m < 0$), then K_M (resp. K_m) will contain the unbounded component of $\mathbb{C} \setminus \sigma_e(A)$ and thus be an unbounded closed set.

Also note that if the supremum (resp. infimum) is attained on a unique component of the complement of the essential spectrum and that component is bounded by a Jordan curve, then it follows by the Jordan curve theorem that $int(K_M)$ (resp. $int(K_m)$) is connected and dense in K_M (resp. K_m).

Proof. Assume that (1) holds and we will verify that Theorem 2.10 holds by applying Proposition 6.5. If $M = 0$, then Theorem 2.10 holds with $h(z) = z$, so $C^*(A)$ has a subnormal generator. Now suppose that $M > 0$. Let G_0 be a bounded component of $\mathbb{C} \setminus \sigma_e(A)$ such that $\text{ind}(A - \lambda I) = M$ for $\lambda \in G_0$ and let $\Delta \subseteq G_0$ be a closed disk and ψ a Mobius transformation such that $\psi(\mathbb{C} \setminus \Delta) = \Delta$. Since M is the maximum value of the Fredholm index function, we see from Proposition 6.5 that $\text{ind}(\psi(A) - \mu I) \leq 0$ for all $\mu \in \mathbb{C} \setminus \sigma_e(\psi(A))$, thus condition (a) of Theorem 2.10 holds. Also in view of condition (b) of Theorem 2.10, let $K = \psi(\sigma_e(A)) \cup \{\lambda \in \mathbb{C} \setminus \sigma_e(\psi(A)) : ind(\psi(A) - \lambda I) < 0\}$. Then by Proposition 6.5, $K = \psi(K_M)$. Now by hypothesis, the interior of K_M is connected and dense in K_M. Hence it follows that $int(K)$ is connected and dense in K. Thus, $R(K)$ has only one non-trivial Gleason part (namely the part that contains $int(K)$) and it is dense in K. Thus by Theorem 2.10 it follows that $C^*(A)$ has a subnormal generator. If A satisfies condition (2), then A^* satisfies condition (1), hence $C^*(A^*) = C^*(A)$ has a subnormal generator. □

We now prove a theorem that will allow us to show that for operators whose essential spectrum has certain types of topological properties a necessary condition that its C^*-algebra has a subnormal or hyponormal generator is that the values of its Fredholm index function are either bounded above or bounded below.

Theorem 6.7. *Suppose that A is an essentially normal operator and every component of $\mathbb{C} \setminus \sigma_e(A)$ is bounded by a Jordan curve and for each component G of*

$\mathbb{C} \setminus \sigma_e(A)$ we have $\sigma_e(A) \setminus \partial G$ is a connected set. If $\psi : \sigma_e(A) \to \mathbb{C}$ is a one-to-one continuous function, then ψ extends to a homeomorphism of $\mathbb{C} \cup \{\infty\}$ onto itself and there exists a $\lambda_0 \in \mathbb{C} \setminus \sigma_e(A)$ and an integer $p \in \{0, 1\}$ such that if $N = ind(A - \lambda_0 I)$, then

$$ind(\psi(A) - \psi(\lambda)I) = (-1)^p[ind(A - \lambda I) - N], \text{ for all } \lambda \in \mathbb{C} \setminus \sigma_e(A).$$

Proof. Let $K = \sigma_e(A)$ and let $\{G_i\}_{i=0}^{\infty}$ be the collection of components of $\mathbb{C} \setminus K$ with G_0 being the unbounded component of $\mathbb{C} \setminus K$. Let $\gamma_i = \partial G_i$ for $i \geq 0$. Let $\Gamma_i = \psi(\gamma_i)$. Let $\Omega_i = \text{inside}(\Gamma_i)$. By Lemma 2.4 for each $i \geq 0$ we have either $\psi(K \setminus \gamma_i) \subseteq \text{inside}(\Gamma_i)$ or $\psi(K \setminus \gamma_i) \subseteq \text{outside}(\Gamma_i)$.

Claim 1: There is a unique $i \geq 0$ such that $\psi(K \setminus \gamma_i) \subseteq \text{inside}(\Gamma_i)$.

Suppose that $i, j \geq 0$ and $\psi(K \setminus \gamma_i) \subseteq \text{inside}(\Gamma_i)$ and $\psi(K \setminus \gamma_j) \subseteq \text{inside}(\Gamma_j)$. Thus $\Gamma_j \subseteq \text{inside}(\Gamma_i)$ and $\Gamma_i \subseteq \text{inside}(\Gamma_j)$, but this is clearly a contradiction. Hence there is at most one $i \geq 0$ such that $\psi(K \setminus \gamma_i) \subseteq \text{inside}(\Gamma_i)$. We now show that such an i exists. Suppose that for each $i \geq 1$ we have $\psi(K \setminus \gamma_i) \subseteq \text{outside}(\Gamma_i)$. Then we will show that $\psi(K \setminus \gamma_0) \subseteq \text{inside}(\Gamma_0)$. Since $\psi(K \setminus \gamma_i) \subseteq \text{outside}(\Gamma_i)$ for all $i \geq 1$, then $\Omega_i \cap \Omega_j = \emptyset$ for all $i, j \geq 1$, $i \neq j$. By Theorem 2.5 for each $i \geq 1$ we may extend ψ to a homeomorphism (still called ψ) from cl[inside(γ_i)] to cl[inside(Γ_i)] $= \text{cl}\Omega_i$. Since $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, $i, j \geq 1$ it follows that $\psi : \hat{K} \to \mathbb{C}$ is one-to-one and continuous where $\hat{K} = K \cup \bigcup_{i=1}^{\infty} G_i$ (is the polynomially convex hull of K). Now $\hat{K} = \text{cl[inside}(\gamma_0)]$. Now since ψ is one-to-one on $\hat{K}$ a simple argument shows that $\partial\psi(\text{inside}(\gamma_0)) \subseteq \Gamma_0 = \psi(\gamma_0)$. Since ψ is continuous, then $\psi(\hat{K})$ is a compact set, hence bounded. These last two conditions imply that $\psi(\text{inside}(\gamma_0)) = \text{inside}(\Gamma_0)$. In particular it follows that $\psi(K \setminus \gamma_0) \subseteq \text{inside}(\Gamma_0)$. The claim now follows.

Claim 2: If $\psi(K \setminus \gamma_0) \subseteq \text{inside}(\Gamma_0)$, then ψ extends to a homeomorphism of $\mathbb{C} \cup \{\infty\}$ onto itself and there exists a $p \in \{0, 1\}$ such that $\text{ind}(\psi(A) - \psi(\lambda)I) = (-1)^p \text{ind}(A - \lambda I)$, for all $\lambda \in \mathbb{C} \setminus \sigma_e(A)$, where $p = 0$ if ψ preserves the orientation of γ_0 and $p = 1$ if ψ reverses the orientation of γ_0.

Using Claim 1 and its proof we may extend ψ to a homeomorphism (still called ψ) $\psi : \text{cl[inside}(\gamma_0)] \to \text{cl[inside}(\Gamma_0)]$. Again by Theorem 2.5 we may extend ψ to be a homeomorphism from outside(γ_0) onto outside(Γ_0), thus ψ extends to be a homeomorphsim of $\mathbb{C} \cup \{\infty\}$ onto itself that fixes ∞. Now suppose that each γ_j, $j \geq 0$ is given the positive (counter-clockwise) orientation. Let $\lambda \in \mathbb{C} \setminus \sigma_e(A)$, and we may suppose that λ is not in the unbounded component, then $\lambda \in G_j$ for some $j \geq 1$. It then follows easily that γ_0 is homotopic to γ_j within cl[inside(γ_0)] $\setminus \{\lambda\}$. Thus, by applying ψ to the homotopy we see that Γ_0 is homotopic to Γ_j within cl[inside(Γ_0)]$\setminus\{\psi(\gamma)\}$. Hence $n(\Gamma_j, \psi(\lambda)) = n(\Gamma_0, \psi(\lambda)) = (-1)^p$, where p is chosen so that $p = 0$ if ψ preserves the orientation of γ_0 and $p = 1$ if ψ reverses the orientation of γ_0. Also, since $\psi(\gamma) \in G_j \subseteq \text{outside}(\Gamma_i)$ for all $i \geq 1$, $i \neq j$, then $n(\Gamma_i, \psi(\lambda)) = 0$. It follows from Theorem 2.6 (part (3)) that $\text{ind}(\psi(A) - \psi(\lambda)I) = (-1)^p \text{ind}(A - \lambda I)$.

Claim 3: The general case.

Suppose that $\psi : K \to \mathbb{C}$ is any one-to-one continuous map. By Claim 1, there exists a unique $i \geq 0$ such that $\psi(K \setminus \gamma_i) \subseteq \text{inside}(\Gamma_i)$. In Claim 2 we dealt with the case $i = 0$. So now assume that $i \geq 1$. Again by Claim 1 we must have that

$\psi(K \setminus \gamma_0) \subseteq \text{outside}(\Gamma_0)$. Now let's do a "flip" about Γ_0 as in Proposition 6.4. That is, let $\phi : \text{outside}(\Gamma_0) \to \mathbb{D}$ be a Riemann map such that $\psi(\infty) = 0$. Notice that ϕ extends to be a homeomorhism of cl[outside(Γ_0)] onto cl$\mathbb{D}$. From there we may use Theorem 2.5 to extend ϕ to be a homeomorphism of inside(Γ_0) onto $\mathbb{C} \setminus \text{cl}\mathbb{D}$. Thus ϕ is a homeomorphism of $\mathbb{C} \cup \{\infty\}$ onto itself satisfying $\phi(\infty) = 0$. Now consider $\Psi = \phi \circ \psi$. Then Ψ maps γ_0 onto $\partial\mathbb{D}$ and $\Psi(K \setminus \gamma_0) \subseteq \mathbb{D}$. It follows from Claim 2 that Ψ extends to a homeomorphism of $\mathbb{C} \cup \{\infty\}$ onto itself. Since $\psi = \phi^{-1} \circ \Psi$ and both ϕ and Ψ are homeomorphisms of $\mathbb{C} \cup \{\infty\}$, then ψ also extends to be a homeomorphism of $\mathbb{C} \cup \{\infty\}$. Also by Claim 2, there exists an integer $p \in \{0, 1\}$ such that $\text{ind}(\Psi(A) - \Psi(\lambda)I) = (-1)^p \text{ind}(A - \lambda I)$ for all $\lambda \in \mathbb{C} \setminus \sigma_e(A)$. Suppose that $p = 0$, then

$$\text{ind}(\Psi(A) - \Psi(\lambda)I) = \text{ind}(A - \lambda I) \text{ for all } \lambda \in \mathbb{C} \setminus \sigma_e(A).$$

Let $\lambda_0 \in \text{inside}(\gamma_i)$ and let $N = \text{ind}(A - \lambda_0 I)$. Since ψ extends to be a homeomorphism of $\mathbb{C} \cup \{\infty\}$, then $\text{ind}(\psi(A) - \lambda I) = n(\Gamma_i, \lambda)\text{ind}(A - \lambda_0 I) = -N$ for $\lambda \in \text{inside}(\Gamma_0)$. This is because we are assuming ($p = 0$) that Ψ preserves the orientation of γ_0, and hence the orientation of γ_i. But ϕ reverses the orientation of both Γ_0 and Γ_i. This implies that ψ also reverses the orientations of γ_0 and γ_i. Now the flip about Γ_0 will subtract the value $(-N)$ from all other values of the index, see Proposition 6.4. Thus $\text{ind}(\Psi(A) - \phi(\mu)I) = \text{ind}(\psi(A) - \mu I) + N$ for $\mu \in \mathbb{C} \setminus \sigma_e(\psi(A))$. Now if $\lambda \in \mathbb{C} \setminus \sigma_e(A)$ and we set $\mu = \psi(\lambda)$, then using the above equations we get

$$\text{ind}(A - \lambda I) = \text{ind}(\Psi(A) - \Psi(\lambda)I) = \text{ind}(\psi(A) - \psi(\lambda)I) + N.$$

Thus, $\text{ind}(\psi(A) - \psi(\lambda)I) = \text{ind}(A - \lambda I) - N$. Thus the theorem holds when $p = 0$.

If $p = 1$, then

$$\text{ind}(\Psi(A) - \Psi(\lambda)I) = -\text{ind}(A - \lambda I) \text{ for all } \lambda \in \mathbb{C} \setminus \sigma_e(A).$$

So, Ψ reverses Γ_0 and thus also Γ_i, hence ψ preserves the orientation of γ_i, thus $\text{ind}(\psi(A) - \lambda I) = n(\Gamma_i, \lambda)\text{ind}(A - \lambda_0 I) = N$ for $\lambda \in \text{inside}(\Gamma_0)$. Now this time the flip about Γ_0 will subtract N from all the other values of the index, so we get $\text{ind}(\Psi(A) - \phi(\mu)I) = \text{ind}(\psi(A) - \mu I) - N$ for $\mu \in \mathbb{C} \setminus \sigma_e(\psi(A))$. Now if $\lambda \in \mathbb{C} \setminus \sigma_e(A)$ and we set $\mu = \psi(\lambda)$, then using the above equations we get

$$-\text{ind}(A - \lambda I) = \text{ind}(\Psi(A) - \Psi(\lambda)I) = \text{ind}(\psi(A) - \psi(\lambda)I) - N.$$

Thus, $\text{ind}(\psi(A) - \psi(\lambda)I) = -\text{ind}(A - \lambda I) + N = -[\text{ind}(A - \lambda I) - N]$. Hence the theorem also holds when $p = 1$. □

For an operator A we say that the (Fredholm) index function is *bounded above* if $\sup\{\text{ind}(A - \lambda I) : \lambda \in \mathbb{C} \setminus \sigma_e(A)\} < \infty$ and we say that the index function is *bounded below* if $\inf\{\text{ind}(A - \lambda I) : \lambda \in \mathbb{C} \setminus \sigma_e(A)\} > -\infty$.

Corollary 6.8 (Index bounded above or below)**.** *Suppose that A is an irreducible essentially normal operator and every component of $\mathbb{C} \setminus \sigma_e(A)$ is bounded by a Jordan curve and for each component G of $\mathbb{C} \setminus \sigma_e(A)$ we have $\sigma_e(A) \setminus \partial G$ is a connected set. If $C^*(A)$ has a generator whose index function is either bounded above or bounded below, then the index function of A is bounded above or bounded below.*

Proof. Suppose that T is a generator of $C^*(A)$ whose index function is bounded above by M, that is $\text{ind}(T - \lambda I) \leq M$ for all $\lambda \in \mathbb{C} \setminus \sigma_e(T)$. Then by Proposition 2.9, there is a homeomorphism $\psi : \sigma_e(A) \to \sigma_e(T)$ such that $\psi(A)$ and T have the same spectral picture. In particular, they have the same index functions. Thus, $\operatorname{ind}(\psi(A)-\mu I) = \operatorname{ind}(T-\mu I) \leq M$ for all $\mu \in \mathbb{C}\setminus\sigma_e(\psi(A))$. Now with Theorem 6.7 we see that there is an integer N and a $p \in \{0,1\}$ such that

$$(-1)^p[\operatorname{ind}(A-\lambda I) - N] = \operatorname{ind}(\psi(A) - \psi(\lambda)I) \leq M \text{ for all } \lambda \in \mathbb{C}\setminus\sigma_e(A).$$

It then follows that either $\operatorname{ind}(A-\lambda I) \leq (M+N)$ or that $\operatorname{ind}(A-\lambda I) \geq (N-M)$ for all $\lambda \in \mathbb{C}\setminus\sigma_e(A)$ (depending on whether $p=0$ or $p=1$). Thus the index function for A is bounded above or bounded below.

If $C^*(A)$ has a generator T whose index function is bounded below, then T^* is a generator whose index function is bounded above and thus the above argument applies. □

For operators A satisfying the hypothesis of Corollary 6.8, if $C^*(A)$ has a subnormal or hyponormal generator, then the index function for A is bounded above or bounded below (because subnormal and hyponormal operators have index functions bounded above by zero). Another way to think of Corollary 6.8 is that if the hypothesis is satisfied, then if one generator of $C^*(A)$ has index function bounded above or below, then all generators have index function bounded above or below.

We now describe some compact sets for which the conditions in Theorem 6.6 become both necessary and sufficient for $C^*(A)$ to have a subnormal generator.

We will give examples of the following Theorem later, but one example to keep in mind is a (finite or infinite) checkerboard. That is, an operator whose essential spectrum is a grid of horizontal and vertical lines.

Theorem 6.9 (Holes only get small near a countable set)**.** *Let $K \subseteq \mathbb{C}$ be a compact set such that the following conditions hold.*

(i) *K is connected.*
(ii) *Each component of $\mathbb{C}\setminus K$ is bounded by a Jordan curve.*
(iii) *There is a countable set $E \subseteq \partial K$ such that for each $a \in \partial K \setminus E$, there is an $r>0$ and a $\delta>0$ such that for each component G of $\mathbb{C}\setminus K$ that intersects $B(a,r)$ we have diam$(G) > \delta$.*
(iv) *For each component G of $\mathbb{C}\setminus K$ we have that $K\setminus\partial G$ is a connected set.*

If A is an irreducible essentially normal operator with $\sigma_e(A) = K$, then $C^*(A)$ has a subnormal generator if and only if one of the following conditions hold:

(1) *$M = \sup\{ind(A-\lambda I) : \lambda \in \mathbb{C}\setminus K\} < \infty$ and int(K_M) is connected and dense in K_M where $K_M = K \cup \{\lambda \in \mathbb{C}\setminus K : ind(A-\lambda I) < M\}$.*
or
(2) *$m = \inf\{ind(A-\lambda I) : \lambda \in \mathbb{C}\setminus K\} > -\infty$ and int(K_m) is connected and dense in K_m where $K_m = K \cup \{\lambda \in \mathbb{C}\setminus K : ind(A-\lambda I) > m\}$.*

Clearly, if $\mathbb{C}\setminus K$ has only finitely many components or if the diameters of the components of $\mathbb{C}\setminus K$ are bounded away from zero, then condition (iii) above holds. Also, see Proposition 7.15 for a class of compact sets with infinitely many holes that Theorem 6.9 applies to.

Remark. In the statement of Theorem 6.9 conditions (ii) and (iv) naturally imply conditions on the *unbounded component* of $\mathbb{C}\setminus K$ as well as the bounded components. Notice that it is always true that $M \geq 0$ and $m \leq 0$, because $\operatorname{ind}(A-\lambda I) = 0$ for λ in the unbounded component of $\mathbb{C}\setminus K$. If $M > 0$ ($m < 0$), then K_M (resp. K_m) will contain the unbounded component of $\mathbb{C}\setminus K$ and is an unbounded closed set. However, if $M = 0$ ($m = 0$), then the unbounded component of $\mathbb{C} \setminus K$ is not contained in K_M (resp. K_m), so K_M (resp. K_m) is compact.

The importance of conditions (i) - (iv) above is that for such sets properties (a) - (d) below hold.

Lemma 6.10. *Let $K \subseteq \mathbb{C}$ be a compact set satisfying (i)-(iv) below.*

- (i) *K is connected.*
- (ii) *Each component of $\mathbb{C} \setminus K$ is bounded by a Jordan curve.*
- (iii) *There is a countable set $E \subseteq \partial K$ such that for each $a \in \partial K \setminus E$, there is an $r > 0$ and a $\delta > 0$ such that for each component G of $\mathbb{C} \setminus K$ that intersects $B(a, r)$ we have $diam(G) > \delta$.*
- (iv) *For each component G of $\mathbb{C} \setminus K$ we have that $K \setminus \partial G$ is a connected set.*

Then the following are true:

- (a) *If K' is any compact set homeomorphic to K, then K' also satisfies properties (i)-(iv).*
- (b) *If L is the union of K and some collection of the bounded components of $\mathbb{C} \setminus K$, then L also satisfies properties (i)-(iv).*
- (c) *A nontrivial Gleason part of $R(K)$ has the form $G \cup F$ where G is a (nonempty) component of the interior of K and $F \subseteq E$.*
- (d) *If $R(K)$ has exactly one nontrivial Gleason part which is dense in K, then the interior of K is connected and the interior of K is dense in K.*

Proof of Lemma 6.10. (a) Suppose that $h : K \to K'$ is a homeomorphism. Then since (ii) and (iv) hold it follows from Theorem 6.7 that h extends to a homeomorphism (still called h) of $\mathbb{C} \cup \{\infty\}$ onto $\mathbb{C} \cup \{\infty\}$. It then easily follows that K' satisfies conditions (i), (ii) and (iv). For condition (iii), let $E' = h(E)$. Then $E' \subseteq \partial K'$ and E' is a countable set. If $b \in \partial K' \setminus E'$, then there is an $a \in \partial K \setminus E$ such that $b = h(a)$. Now by condition (iii) we have an $r > 0$ and a $\delta > 0$ such that for each component G of $\mathbb{C} \setminus K$ that intersects $B(a, r)$ we have $diam(G) > \delta$.

Let $c = h(\infty)$. If $c \in \mathbb{C}$, then let G_0 be the component of $\mathbb{C} \setminus K'$ such that $c \in G_0$, if $c = \infty$, then let G_0 be the unbounded component of $\mathbb{C} \setminus K'$. If $c \in \mathbb{C}$, then choose $\alpha, \beta > 0$ such that $\Omega = \{z \in \mathbb{C} : \alpha \leq |z - c| \leq \beta\} \supseteq \hat{K}' \setminus G_0$, where $\hat{K}'$ is the polynomially convex hull of K'. Thus Ω is a compact annulus that contains K' and all of its "bounded holes" except for the one that contains c. If $c = \infty$, then let Ω be a large closed disk that contains $\hat{K}'$. Then h^{-1} is a continuous complex valued function on Ω (h^{-1} does not assume the value ∞ on Ω), thus h^{-1} is uniformly continuous on Ω.

Let $\epsilon = \min\{r, \delta\} > 0$, then by the uniform continuity of h^{-1} on Ω, there is an $\delta' > 0$ such that, for $w_1, w_2 \in \Omega$,

(*) $$\text{if } |w_1 - w_2| < \delta', \text{ then } |h^{-1}(w_1) - h^{-1}(w_2)| < \epsilon.$$

By choosing δ' a little smaller if necessary, we may also assume that $\delta' < diam(G_0)$. Now let $r' = \delta'$.

Notice that the contrapositive of $(*)$ (which is an equivalent statement to $(*)$) says the following: for $w_1, w_2 \in \Omega$,

(**) $$\text{if } |h^{-1}(w_1) - h^{-1}(w_2)| \geq \epsilon, \text{ then } |w_1 - w_2| \geq \delta'.$$

Claim: If G is any component of $\mathbb{C} \setminus K'$ and $G \cap B(b, r') \neq \emptyset$, then $diam(G) > \delta'$.

Once we prove the claim then we are done proving property (a). So, suppose G is a component of $\mathbb{C} \setminus K'$ and $G \cap B(b, r') \neq \emptyset$. If $G = G_0$, then by assumption we have $diam(G) > \delta'$. So suppose that $G \neq G_0$. Let H be a component of $\mathbb{C} \setminus K$ such that $h(H) = G$. By (*), $h^{-1}(B(b, r')) \subseteq B(a, r)$. Since $G \cap B(b, r') \neq \emptyset$, then $H \cap B(a, r) \neq \emptyset$. Hence by assumption (iii) we have $diam(H) > \delta \geq \epsilon$. Since $G \neq G_0$, then H is a bounded component of $\mathbb{C} \setminus K$ and thus $H \subseteq \hat{K} = h^{-1}(\hat{K}') \subseteq h^{-1}(\Omega)$. Let $z_1, z_2 \in H$ with $|z_1 - z_2| > \epsilon$ and let $w_1, w_2 \in \Omega$ such that $z_1 = h^{-1}(w_1)$ and $z_2 = h^{-1}(w_2)$. Now by $(**)$ it follows that $|w_1 - w_2| \geq \delta'$. Since $w_1, w_2 \in h(H) = G$ we see that $diam(G) \geq \delta'$. Condition (a) now follows.

(b) Notice that if L is the union of K and some collection of the bounded components of $\mathbb{C} \setminus K$, then since K satisfies (i)-(iv), it follows that L will also satisfy the four conditions (i) - (iv) with the same exceptional set E.

(c) First notice that if the interior of K is empty, then $R(K) = C(K)$ and so $R(K)$ has no nontrivial Gleason parts. To see this notice that it follows from assumption (iii) on K and Curtis's Peak Point Criterion (see Conway [3, p. 222]) that if $a \in \partial K \setminus E$, then a is a peak point for $R(K)$. So if $\text{int}(K)$ is empty, then all but countably many points of K are peak points for $R(K)$, hence $R(K) = C(K)$ (see Corollary 11.10 of Conway [3, p. 216]).

Thus we may suppose that $\text{int}(K)$ is nonempty and we will describe the nontrivial Gleason parts of $R(K)$. If we can show that $R(K)$ is (isometrically) pointwise boundedly dense in $H^\infty(intK)$, then a characteristic function of a component of the interior of K may be approximated by functions in $R(K)$, thus Corollary 15.8 of Conway [3, p. 236] implies that distinct components of $\text{int}K$ belong to distinct Gleason parts. To show that $R(K)$ is (isometrically) pointwise boundedly dense in $H^\infty(intK)$ we will apply a theorem of Gamelin and Garnett (see [9] or Corollary 3.22 of Conway [3, p. 332] with $\mu = Area|int(K))$ and a corollary of Davie's Theorem (see Corollary 22.2 of Conway [3, p. 267]). To apply Gamelin and Garnett's Theorem we must show that K is *essentially Dirichlet* (see Conway [3, p. 326]) which means that for all but a countable number of points $a \in \partial K$ we have $\liminf_{r \to 0} \frac{\gamma(B(a,r) \setminus K)}{r} > 0$; where $\gamma(\cdot)$ denotes analytic capacity. We will use the same techniques and notation used in the proofs of Gonchar's Criterion, Lemma 13.4 and Corollary 13.5 of Conway [3, p.223-224]. Our hypothesis (iii) implies that for each $a \in \partial K \setminus E$ we have $\liminf_{r \to 0} \frac{\gamma(B(a,r) \setminus K)}{r} \geq \liminf_{r \to 0} \frac{d(r)}{4r} \geq \frac{1}{(4)(2)} > 0$, where $d(r)$ is the supremum of the diameters of the components of $B(a, r) \setminus K$. Hence (iii) implies that K is essentially Dirichlet. This together with the fact that the points in $\partial K \setminus E$ are peak points for $R(K)$ implies that a nontrivial Gleason part for $R(K)$ has the form $G \cup F$ where G is a component of the interior of K and $F \subseteq E$. Notice that F may be empty in this representation, but G may not be; otherwise the Gleason part would have area zero. Thus property (c) follows.

To prove (d), assume that $R(K)$ has exactly one nontrivial Gleason part, call it Ω, which is dense in K. By part (c), the interior of K must be nonempty and connected and there is a set $F \subseteq E$ such that $\Omega = \text{int}(K) \cup F$. Notice that it follows from the definition of E that E is a closed set and a countable set. Thus, $\partial K \setminus E \subseteq K = cl\Omega = cl[int(K)] \cup E$. Thus, $\partial K \setminus E \subseteq cl[int(K)]$.

To show that $\text{int}(K)$ is dense in K, we need to show that $E \subseteq cl[int(K)]$. Suppose not. Then there is an $a \in E$ and an $r > 0$ such that $clB(a, r) \cap cl[int(K)] = \emptyset$. Hence it follows that $clB(a, r) \cap K = clB(a, r) \cap \partial K = clB(a, r) \cap E$. However, $clB(a, r) \cap E$ is countable and $clB(a, r) \cap K$ is uncountable. To see that $clB(a, r) \cap K$

is uncountable notice that it is a closed set. So it either has an isolated point or is a perfect set. It can't have an isolated point since K is connected. Thus it is perfect, hence uncountable. This contradiction implies that $E \subseteq cl[int(K)]$. Thus the interior of K is dense in K. □

Proof of Theorem 6.9. (⇐) We will first suppose that one of the conditions is satisfied and we will prove that $C^*(A)$ does have a subnormal generator. As in the statement of the Theorem, we will let M and m denote the maximum and minimum values of the Fredholm index function for A on $\mathbb{C} \setminus K$.

If either (1) or (2) holds, then it follows immediately from Theorem 6.6 that $C^*(A)$ has a subnormal generator.

(⇒) Now consider the other direction. That is, suppose $C^*(A)$ has a subnormal generator. There are two cases here: the index function is either identically zero or it is not. First suppose that $\text{ind}(A - \lambda I) = 0$ for all $\lambda \in \mathbb{C} \setminus K$. Since $C^*(A)$ has a subnormal generator, Theorem 5.1 implies that K is homeomorphic to a compact set K' such that $R(K')$ has exactly one nontrivial Gleason part which is dense in K'. By Lemma 6.10(a) K' satisfies conditions (i)-(iv) and thus by Lemma 6.10(d), the interior of K' is nonempty, connected, and dense in K'. Since K is homeomorphic to K', it follows that the interior of K is also nonempty, connected, and dense in K. Hence property (1) holds.

Now suppose that $C^*(A)$ has a subnormal generator and $\text{ind}(A-\lambda I) \neq 0$ for some $\lambda \in \mathbb{C}\setminus K$. Then by Theorem 2.10, there is a compact set K' and a homeomorphism $\phi : K \to K'$ such that

$$(*) \qquad \text{ind}(\phi(A) - \mu I) \leq 0 \text{ for all } \mu \in \mathbb{C} \setminus K'$$

and such that $K'_0 = K' \cup \{\mu \in \mathbb{C} \setminus K' : \text{ind}(\phi(A) - \mu I) < 0\}$ has the property that $R(K'_0)$ has exactly one nontrivial Gleason part which is dense in K'_0. By our assumptions (i) - (iv) on K, Lemma 6.10 implies that both K' and K'_0 also satisfy (i)-(iv). Hence Lemma 6.10(d) implies that the interior of K'_0 is nonempty, connected, and dense in K'_0. Now from our assumptions we may apply Theorem 6.7, thus ϕ extends to a homeomorphism (still called ϕ) of $\mathbb{C} \cup \{\infty\}$ onto $\mathbb{C} \cup \{\infty\}$ and there exists a $\lambda_0 \in \mathbb{C}\setminus\sigma_e(A)$ and an integer $p \in \{0, 1\}$ such that if $N = \text{ind}(A-\lambda_0 I)$, then

$$(**) \qquad \text{ind}(\phi(A) - \phi(\lambda)I) = (-1)^p[\text{ind}(A - \lambda I) - N], \text{ for all } \lambda \in \mathbb{C} \setminus K.$$

Now by combining equations (*) and (**) we get the following:

$$(***) \qquad 0 \geq \text{ind}(\phi(A) - \phi(\lambda)I) = (-1)^p[\text{ind}(A - \lambda I) - N], \text{ for all } \lambda \in \mathbb{C} \setminus K.$$

Now either $p = 0$ or $p = 1$. We shall see that if $p = 0$ then (1) holds and if $p = 1$, then (2) holds. Suppose $p = 0$, then equation (***) implies that $\text{ind}(A - \lambda I) \leq N$ for all $\lambda \in \mathbb{C} \setminus K$. Thus N is an upper bound for the index function, so $M \leq N$. Since $N = \text{ind}(A - \lambda_0 I)$, it follows that $N \leq M$. Thus $M = N$. It follows from (**) above that

$$\phi(\{\lambda \in \mathbb{C} \setminus K : \text{ind}(A - \lambda I) < M\}) = \{\mu \in \mathbb{C} \setminus K' : \text{ind}(\phi(A) - \mu I) < 0\}$$

Thus $\phi(K_M) = K'_0$. Since the interior of K'_0 is nonempty, connected, and dense in K'_0, it follows that the interior of K_M is nonempty, connected, and dense in K_M. Thus property (1) holds. It follows similarly, if $p = 1$ that (2) holds. □

7. Examples

In this section we present several examples. Some are immediate applications of the previous theorems, others require some work.

Theorem 7.1 (Isolated Disks Removed; arbitrary values of the Fredholm index). *Let $\{\Delta_n\}_{n=1}^{\infty}$ be a sequence of open disks each having closures contained inside the open unit disk $\mathbb{D}$, each being isolated from the others — meaning that for each $n \geq 1$, there is a neighborhood U_n of $cl\Delta_n$ such that $U_n \cap cl\Delta_k = \emptyset$ for $k \neq n$ — and such that $\mathbb{D} \setminus \bigcup_{n=1}^{\infty} cl\Delta_n$ is an open connected set. Then let $K = cl[\partial\mathbb{D} \cup \bigcup_{n=1}^{\infty} \partial\Delta_n]$.*

If A is an irreducible essentially normal operator with $\sigma_e(A) = K$, then $C^*(A)$ has a subnormal generator if and only if $ind(A - \lambda I) \neq 0$ for some $\lambda \in \mathbb{C} \setminus K$.

Proof. First assume that $\operatorname{ind}(A-\lambda I) \neq 0$ for some $\lambda \in \mathbb{C}\setminus K$. We may assume that $\operatorname{ind}(A-\lambda I) \neq 0$ where $\lambda \in \mathbb{D}\setminus\bigcup_{n=1}^{\infty} \mathrm{cl}\Delta_n$ otherwise apply a Mobius transformation that maps the exterior of Δ_n to $\mathbb{D}$ where n is such that $\operatorname{ind}(A-\lambda I) \neq 0$ for $\lambda \in \Delta_n$. We may also assume that $\operatorname{ind}(A-\lambda I) < 0$ for $\lambda \in \mathbb{D}\setminus\bigcup_{n=1}^{\infty} \mathrm{cl}\Delta_n$ otherwise we may apply the homeomorphism $z \mapsto \overline{z}$ which will negate all the values of the index. Now assuming this, let $a_n = \operatorname{ind}(A - \lambda I)$ for $\lambda \in \Delta_n$. We will define a homeomorphism $\varphi : K \to K$. Let $\varphi(z) = z$ if $z \in \partial\mathbb{D}$ and for $z \in \partial\Delta_n$, define $\varphi(z) = z$ if $a_n \leq 0$ and $\varphi(z) = \overline{(z - z_n)} + z_n$ if $a_n > 0$ where z_n is the center of the disk Δ_n. Also if $z \in K \setminus \bigcup_{n=1}^{\infty} \partial\Delta_n$, then define $\varphi(z) = z$. Clearly then φ is a homeomorphism, and as $z \mapsto \overline{z}$ negates an index, we see that φ will transform the given spectral picture to one with all non-positive indices. Now, if we let $L = \mathrm{cl}\left[\mathbb{D} \setminus \bigcup_{i\in Z} \Delta_i\right]$ where $Z = \{n \in \mathbb{N} : a_n = 0\}$, then by our assumption, L will have a connected interior which is dense in L, thus (see [6]) there is an irreducible essentially normal subnormal operator S with $\sigma(S) = L$, $\sigma_e(S) = \partial L$, and $\operatorname{ind}(S - \lambda I) = -|a_n|$ for $\lambda \in \Delta_n$, $n \in \mathbb{N} \setminus Z$. Thus, $\varphi(A)$ will have the same spectral picture as S. It then follows by Proposition 2.9 that $C^*(A)$ has a subnormal generator.

Conversely, suppose that $C^*(A)$ has a subnormal generator. If $\operatorname{ind}(A - \lambda I) = 0$ for all $\lambda \notin K$, then by Corollary 5.4, $\sigma_e(A) = K$ must be connected, which it is not. Hence we must have $\operatorname{ind}(A - \lambda I) \neq 0$ for some $\lambda \in \mathbb{C} \setminus K$. □

Interesting examples may be created from the above theorem by having the disks $\{\Delta_n\}$ cluster on various types of sets. For example, if the disks $\{\Delta_n\}$ cluster at each point of $\partial\mathbb{D}$, then we end up with a **champagne bubble set**, if the disks converge to zero, then we get a **road runner type set**, the disks could also cluster on a **Cantor set**, or some other complicated set in cl$\mathbb{D}$. But we are requiring that what is left over after the disks are removed, namely $\mathbb{D} \setminus \bigcup_{n=1}^{\infty} \mathrm{cl}\Delta_n$, is an open connected set; thus a **Swiss-cheese** set is not allowed here.

Example 7.2 (Road Runner Set; arbitrary values of the index). *Let $\{\Delta_n\}_{n=1}^{\infty}$ be a sequence of open disks with pairwise disjoint closures all having closures inside the open unit disk $\mathbb{D}$ and each being centered on positive real axis and having centers that decrease to zero. Let $K = \partial\mathbb{D} \cup \bigcup_{n=1}^{\infty} \partial\Delta_n \cup \{0\}$.*

If A is an irreducible essentially normal operator with $\sigma_e(A) = K$, then $C^*(A)$ has a subnormal generator if and only if $ind(A - \lambda I) \neq 0$ for some $\lambda \notin K$.

In particular, the values of the Fredholm index on the bounded components of $\mathbb{C} \setminus K$ can be arbitrary as along as one is non-zero.

The previous example should be contrasted with the following example where the essential spectrum of A is a thick set (it has interior). This then has the effect of limiting the possible values of the Fredholm index. However there is still a very nice condition for having a subnormal generator, namely the index is either bounded above or bounded below. Notice that the index may be identically zero in this case.

Example 7.3 (Solid Road Runner Set - indices restricted)**.** *Let $\{\Delta_n\}_{n=1}^{\infty}$ be a sequence of open disks with pairwise disjoint closures, all having closures inside the open unit disk $\mathbb{D}$, and each being centered on the positive real axis and having centers that decrease to zero. Let $K = cl\mathbb{D} \setminus \bigcup_{n=1}^{\infty} \Delta_n$.*

If A is an irreducible essentially normal operator with $\sigma_e(A) = K$, then $C^*(A)$ has a subnormal generator if and only if either $\sup\{ind(A - \lambda I) : \lambda \notin K\} < \infty$ *or* $\inf\{ind(A - \lambda I) : \lambda \notin K\} > -\infty$.

Proof. Since $\sigma_e(A)$ satisifies the hypothesis of Corollary 6.8, it implies that if $C^*(A)$ has a subnormal generator, then the index function is either bounded above or bounded below. If the indices are bounded above or below, then simply apply Theorem 6.6 or Theorem 6.9. □

A Swiss-cheese set is a compact set K of the form $K = \text{cl}\mathbb{D} \setminus \bigcup_{n=1}^{\infty} \Delta_n$ where $\{\Delta_n\}_{n=1}^{\infty}$ is a sequence of open disks such that $cl\Delta_n \subseteq \mathbb{D}$, $cl\Delta_n \cap cl\Delta_m = \emptyset$ for $n \neq m$, $\bigcup_{n=1}^{\infty} \Delta_n$ is dense in $\mathbb{D}$. Some Swiss-chesse sets have area zero and others have positive area and are the spectra of irreducible subnormal operators. If $\sum_n diam(\Delta_n) < \infty$ (some authors include this condition as part of the definition of a Swiss-cheese set, but we do not), then $R(K) \neq C(K)$ and in fact K is the spectrum of a pure subnormal operator. For a Swiss-cheese set K, $R(K)$ may have any number of non-trivial Gleason parts, ranging from zero to infinity, inclusive. We note that Anthony O'Farrell, James Thomson, as well as others have produced unpublished examples of Swiss-cheese sets with the number of Gleason parts prescribed. In particular, those Swiss-cheese sets with exactly one non-trivial Gleason part which also have the sum of the diameters of the disks converging will be the spectra of irreducible (essentially normal, rationally cyclic) subnormal operators. We need to know that there is a Swiss-cheese set K with only one non-trivial Gleason part which is dense in K which has an additional property, so we will give the construction of such a Swiss-cheese set, since it could not be found in the literature.

Lemma 7.4. *Let G be a region bounded by a finite number of disjoint analytic Jordan curves. Let $a, b \in G$, $a \neq b$ and choose $r_0 > 0$ such that $B(b, r_0) = \{z \in \mathbb{C} : |z - b| \leq r_0\} \subseteq G \setminus \{a\}$. For $0 < r < r_0$, let $\Omega_r = G \setminus B(b, r)$, and let $\omega_{r,a}$ be harmonic measure for Ω_r at the point a, and $\omega_{G,a}$ harmonic measure for G at the point a. Then the following hold:*

1. *$\omega_{r,a}$ and $\omega_{G,a}$ are absolutely continuous with respect to arc length measure, ds, on $\partial\Omega_r$ and on ∂G, respectively.*
2. *The Radon-Nikodym derivatives $d\omega_{r,a}/ds$ and $d\omega_{G,a}/ds$ are positive continuous functions on $\partial\Omega_r$ and on ∂G, respectively.*
3. *If $d\omega_{G,a} = f ds$ and $d\omega_{r,a} = f_r ds$, where $f \in C(\partial G)$ and $f_r \in C(\partial\Omega_r)$, then $f_r|\partial G \to f$ uniformly on ∂G as $r \to 0$.*
4. *$\omega_{r,a}(\Delta) \to \omega_{G,a}(\Delta)$ as $r \to 0$ for any Borel set $\Delta \subseteq \partial G$.*
5. *$\omega_{r,a}(\partial B(b, r)) \to 0$ as $r \to 0$.*

Proof. Properties (1) and (2) are well known see [8, Theorem 6.4, page 22 and Proposition 6.6, page 24]. In fact we see in [8] that the Radon-Nikodym derivative f is the normal derivative on ∂G of the Green's function $g_G(z,b)$ for the region G at the point b and f_r is likewise the normal derivative on $\partial\Omega_r$ of the Green's function $g_{\Omega_r}(z,b)$ for Ω_r at the point b. Since the regions Ω_r are increasing and their union is G, it follows that $g_{\Omega_r}(z,b) \to g_G(z,b)$ uniformly on compact subsets of $G \setminus \{b\}$ as $r \to 0$, see [19, Theorem 4.4.6, page 108]. Since the Green's functions are zero on the boundary, and the boundary of G consists of analytic curves, we may reflect them across the boundary of G using the Schwarz Reflection principle and thus we have $g_{\Omega_r}(z,b) \to g_G(z,b)$ uniformly on a neighborhood of ∂G as $r \to 0$. Thus their normal derivatives will also converge uniformly on ∂G. This gives property (3). Property (4) follows immediately from (3) and property (5) follows from (4) by choosing $\Delta = \partial G$ and using the fact that harmonic measure is a probability measure. □

Theorem 7.5. *There is a Swiss-cheese set K in $\mathbb{C}$ such that $R(K)$ has only one nontrivial Gleason part which is dense in K, and if L is the union of K and any collection of bounded components of $\mathbb{C} \setminus K$, then L is a compact set and $R(L)$ has only one nontrivial Gleason part which is dense in L.*

Idea of proof by Jim Thomson. We will construct K such that $0 \in K$ and such that there is a representing measure ω for $R(K)$ at the point zero that is mutually absolutely continuous with respect to arc length measure on the "circles" forming K. Let $\{a_n\}_{n=1}^{\infty}$ be a countable dense subset of $\mathbb{D} \setminus \{0\}$. Let Δ_1 be an open disk centered at a_1, whose closure lies in $\mathbb{D}$, and does not contain the point 0, and has radius at most 1/2. For the inductive step assume that the n open disks $\Delta_1, \ldots, \Delta_n$ have been chosen with pairwise disjoint closures. Let $K_n = \mathrm{cl}\mathbb{D} \setminus \bigcup_{k=1}^{n} \Delta_k$ and notice that $0 \in int(K_n)$. Let ω_n be harmonic measure for $int(K_n)$ at the point $a = 0$ and by Lemma 7.4 let f_n be such that $d\omega_n = f_n ds$ where f_n is a positive continuous function on ∂K_n. Let j be the smallest integer such that $a_j \notin \mathrm{cl} \bigcup_{k=1}^{n} \Delta_k$. Then let Δ_{n+1} be an open disk centered at a_j and using Lemma 7.4 choose its radius r_{n+1} such that five things happen:

(1) $0 < r_{n+1} < 1/2^{n+1}$.
(2) $0 \notin \mathrm{cl}\Delta_{n+1}$.
(3) $\mathrm{cl}\Delta_{n+1} \subseteq \mathbb{D} \setminus \bigcup_{k=1}^{n} cl\Delta_k$.
(4) $\omega_{n+1}(\partial\Delta_{n+1}) \le 1/2^{n+1}$ where ω_{n+1} is harmonic measure for $int(K_{n+1})$ at the point 0 and $K_{n+1} = \mathrm{cl}\mathbb{D} \setminus \bigcup_{k=1}^{n+1} \Delta_k$.
(5) If $d\omega_{n+1} = f_{n+1} ds$, then choose r_{n+1} small enough so that
$$\|f_n - f_{n+1}\|_{\infty,\partial K_n} \le \frac{1}{2^{n+1}} \min\{f_n(z) : z \in \partial K_n\}.$$

Recall that the minimum above is positive because $f_n = d\omega_n/ds$ is a positive continuous function on a compact set by Lemma 7.4.

Now let $K = \mathrm{cl}\mathbb{D} \setminus \bigcup_{k=1}^{\infty} \Delta_k$. Note that $0 \in K$. Let $\Delta = \partial\mathbb{D} \cup \bigcup_{n=1}^{\infty} \partial\Delta_n$ and ds will continue to denote arc length measure on Δ. By property (1) above, ds is a finite measure on Δ. We will now construct a measure ω mutually absolutely continuous with respect to ds that is a representing measure for $R(K)$ at the point 0.

For each $n \geq 1$, we have ω_n which is harmonic measure on $int(K_n)$ at the point 0 and positive continuous functions f_n on ∂K_n such that $d\omega_n = f_n ds$. We will also consider f_n to be a function on Δ where $f_n = 0$ on $\Delta \setminus \partial K_n$. By the monotonicity of harmonic measure [4, Corollary 1.14, p. 307] we have that $f_{n+1}(z) \leq f_n(z)$ for all $z \in \partial K_n$. Thus, for $z \in \partial K_n$, $f(z) := \lim_{j\to\infty} f_j(z)$ exists. In fact, it follows by condition (5) above that the functions $\{f_j\}_{j=n}^{\infty}$ are uniformly Cauchy on ∂K_n. It follows that for each $n \geq 1$,

$$f_j \to f \text{ uniformly on } \partial K_n.$$

Furthermore, since $f_{j+1}(z) \leq f_j(z)$ for $z \in \partial K_n$ and $j \geq n$ we have that $\min_{z\in\partial K_j} f_j(z) \leq \min_{z\in\partial K_n} f_n(z)$ for $z \in \partial K_n$. Thus by (5) above we have

$$f_n(z) - f(z) = \sum_{j=n}^{\infty}(f_j(z) - f_{j+1}(z)) \leq \sum_{j=n}^{\infty} \frac{1}{2^{j+1}} \min_{z\in\partial K_j} f_j(z) \leq$$

$$\leq \min_{z\in\partial K_n} f_n(z) \sum_{j=n}^{\infty} \frac{1}{2^{j+1}} = \frac{1}{2^n} \min_{z\in\partial K_n} f_n(z).$$

It follows that $f(z) \geq f_n(z) - \frac{1}{2^n} \min_{z\in\partial K_n} f_n(z) > 0$ for each $z \in \partial K_n$, $n \geq 1$. Thus f is a positive continuous function on ∂K_n for each $n \geq 1$.

We claim that $f \in L^1(\Delta, ds)$. Since $f \geq 0$, the Monotone Convergence Theorem and the fact that $f(z) \leq f_n(z)$ for $z \in \partial K_n$, we have that $\int_\Delta f ds = \lim_{n\to\infty} \int_{\partial K_n} f ds \leq \lim_{n\to\infty} \int_{\partial K_n} f_n ds = \lim_{n\to\infty} \omega_n(\partial K_n) = 1$. Thus, $f \in L^1(\Delta, ds)$; so $\omega := f ds$ is a positive finite measure on Δ that is mutually absolutely continuous with respect to ds since $f > 0$.

We now show that $f_n \to f$ in $L^1(\Delta, ds)$. Let $\epsilon > 0$. Since $f \in L^1(\Delta, ds)$ we may choose $j \geq 1$ such that $\int_{\Delta\setminus\partial K_j} |f| ds < \epsilon/3$ and so that $\frac{1}{2^j} < \epsilon/3$. Then, by property (4) above and the monotonicity of harmonic measure, for $n > j$ we have $\int_{\Delta\setminus\partial K_j} |f_n| ds = \sum_{i=j+1}^{\infty} \int_{\partial\Delta_i} f_n ds = \sum_{i=j+1}^{\infty} \omega_n(\partial\Delta_i) = \sum_{i=j+1}^{n} \omega_n(\partial\Delta_i) \leq \sum_{i=j+1}^{n} \omega_i(\partial\Delta_i) \leq \sum_{i=j+1}^{n} \frac{1}{2^i} \leq \sum_{i=j+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^j} < \epsilon/3$. Now since $f_n \to f$ uniformly on ∂K_j we may choose an $N > j$ such that $\int_{\partial K_j} |f_n - f| ds < \epsilon/3$ for all $n \geq N$. Thus for $n \geq N(> j)$, we have

$$\int_\Delta |f_n - f| ds = \int_{\partial K_j} |f_n - f| ds + \int_{\Delta\setminus\partial K_j} |f_n - f| ds \leq$$

$$\leq \int_{\partial K_j} |f_n - f| ds + \int_{\Delta\setminus\partial K_j} |f_n| ds + \int_{\Delta\setminus\partial K_j} |f| ds < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Thus $f_n \to f$ in $L^1(\Delta, ds)$.

Now let $r \in R(K)$, then

$$\int_\Delta r(z) d\omega = \int_\Delta r(z) f(z) ds = \lim_{n\to\infty} \int_\Delta r(z) f_n(z) ds = \lim_{n\to\infty} \int_{\partial K_n} r(z) f_n(z) ds = r(0).$$

Thus, $\omega = f ds$ is a representing measure for $R(K)$ at the point 0. Furthermore, $\omega \approx ds$; that is, ω is mutually absolutely continuous with respect to ds on Δ.

We now claim that $R(K)$ has only one nontrivial Gleason part which is dense in K. Let G be the Gleason part for $R(K)$ containing the point 0. We claim that G is the only nontrivial Gleason part and that G is dense in K. Let μ be the measure on Δ given by $d\mu = dz$ on $\partial\mathbb{D}$ and $d\mu = -dz$ on $\bigcup_{n=1}^{\infty} \partial\Delta_n$. Then μ is an annihilating measure for $R(K)$. Let $\hat{\mu}(w) = \int \frac{1}{z-w} d\mu$ be the Cauchy transform of μ and $\tilde{\mu}(w) = \int \frac{1}{|z-w|} d|\mu|$. Let $E = \{w \in \mathbb{C} : \tilde{\mu}(w) < \infty \text{ and } \hat{\mu}(w) \neq 0\}$. Since μ is an annihilating measure for $R(K)$, its Cauchy transform is zero off of K, thus $E \subseteq K$. Notice that $\tilde{\mu}(w) < \infty$ almost everywhere $[Area]$ on $\mathbb{C}$. Also, if $\tilde{\mu}(w) < \infty$ and $w \in K \setminus \Delta$, then $\hat{\mu}(w)$ is a sum of winding numbers, thus $\hat{\mu}(w) = 2\pi i \neq 0$. It follows that $Area(E) = Area(K)$. Now by [3, Corollary 8.10, p. 195] if $w \in E$, then there is a representing measure ν for $R(K)$ at the point w such that ν is absolutely continuous with respect to μ. Thus, $\nu << \mu \approx ds \approx \omega$. Since ν represents w and ω represents 0 and ν and ω are not singular, then w and 0 must belong to the same Gleason part. So, $w \in G$. It follows that $E \subseteq G$. Thus, $Area(G) = Area(K)$. Hence G is the only nontrivial Gleason part for $R(K)$, since nontrivial Gleason parts have positive area. Now by property (1) above, ds is a finite measure on Δ, thus the operator S equal to multipication by z, $S = M_z$, on $R^2(K, ds)$, which is the closure of $R(K)$ in $L^2(\Delta, ds)$, is a pure subnormal operator. As such, by the Clancey/Putnam criteria [3, p. 180] on the spectrum of a pure subnormal operator, the union of the nontrivial Gleason parts for $R(\sigma(S)) = R(K)$ must be dense in $\sigma(S) = K$. Thus we have that G is dense in K.

Now suppose that $F \subseteq \mathbb{N}$ and $L = K \cup \bigcup_{k\in F} \Delta_k$. Then clearly, L is compact. Let Ω be the Gleason part for $R(L)$ containing the point 0. It follows easily by the definition of Gleason parts that since $K \subseteq L$ then $G \subseteq \Omega$. If $a \in \Delta_k$ for some $k \in F$ and if λ_a is harmonic measure for the disk Δ_k at the point a, then λ_a is mutually absolutely continuous with respect to arc length measure on $\partial\Delta_k$ and clearly λ_a will be a representing measure for $R(L)$ at the point a. Since λ_a and ω are two representing measures for $R(L)$ that are not mutually singular (because $\omega \approx ds$), then a and 0 must belong to the same Gleason part for $R(L)$. It follows that $a \in \Omega$, thus $\Delta_k \subseteq \Omega$. Thus $G \cup \bigcup_{k\in F} \Delta_k \subseteq \Omega$. It follows that $Area(\Omega) = Area(L)$ and that Ω is dense in L. Since $Area(\Omega) = Area(L)$, Ω is the only nontrivial Gleason part for $R(L)$. □

Definition 7.6. *A set K in the complex plane is a **Swiss-cheese type set** if K has the following properties:*

(1) *K is compact, connected, locally connected, and has empty interior in $\mathbb{C}$.*
(2) *Each component of $\mathbb{C} \setminus K$ is bounded by a Jordan curve.*
(3) *Any two distinct components of $\mathbb{C} \setminus K$ have disjoint closures.*

One may check that a Swiss-cheese set is a Swiss-cheese type set. Another example of a Swiss-cheese type set is the Sierpinski Carpet obtained from the unit square by successively removing smaller squares; the Sierpinski triangle is not a Swiss-cheese type set as condition (3) fails.

Theorem 7.7 (Whyburn's Theorem [20])**.** *If K and L are two Swiss-cheese type sets, then K and L are homeomorphic.*

Theorem 7.8. *If K is a Swiss-cheese type set and A is an irreducible essentially normal operator such that $\sigma_e(A) = K$, then the following are equivalent.*

(1) *$C^*(A)$ has a subnormal generator.*
(2) *$C^*(A)$ has a hyponormal generator.*

(3) *$C^*(A)$ has a generator whose index function is either bounded above or bounded below.*
(4) *$ind(A-\lambda I)$ is either bounded above or bounded below.*

Proof. (1) $\Rightarrow$ (2): This is obvious, since every subnormal operator is also hyponormal.

(2) $\Rightarrow$ (3): This is immediate because the Fredholm index function of a hyponormal operator is always non-positive.

(3) $\Rightarrow$ (4): This follows from Corollary 6.8.

(4) $\Rightarrow$ (1): Suppose that $N := \sup\{\text{ind}(A-\lambda I) : \lambda \in \mathbb{C} \setminus \sigma_e(A)\} < \infty$. We will show that $C^*(A)$ has a subnormal generator. By (Whyburn's) Theorem 7.7, let $h : K \to K'$ be a homeomorphism where $K' = \text{cl}\mathbb{D} \setminus \bigcup_{n=1}^{\infty} \Delta_n$ is the Swiss-cheese set guaranteed from Theorem 7.5. Then by Theorem 6.7, $\text{ind}(h(A)-\lambda I)$ is either bounded above or below. We may assume it is bounded above and then by "flipping" as in Proposition 6.4 about the component with the maximum index we may assume that $\text{ind}(h(A)-\lambda I) \leq 0$ for all $\lambda \in \mathbb{C} \setminus \sigma_e(h(A)) = \mathbb{C} \setminus K'$. Now let L be the compact set obtained by taking the union of K' and each of the bounded compoents of $\mathbb{C} \setminus K'$ where the index of $h(A)$ is non-zero; that is, $L = K' \cup \{\lambda \in \mathbb{C} \setminus K' : \text{ind}(h(A)-\lambda I) < 0\}$. Then by Theorem 7.5, $R(L)$ has a single non-trivial Gleason part which is dense in L. Thus there is an irreducible subnormal operator S with $\sigma(S) = L$, $\sigma_e(S) = K'$ and $\text{ind}(S-\lambda I) = \text{ind}(h(A)-\lambda I)$ for all $\lambda \notin K'$. It follows that $h(A)$ and S have the same spectral picture, thus by Theorem 2.10, $C^*(A)$ has a subnormal generator. If $\text{ind}(A-\lambda I)$ is bounded below, then apply the above argument to A^*. $\square$

Theorem 7.9 (Same spectral picture as a subnormal, but no subnormal generator)**.** *Let $\{\Delta_n\}_{n=1}^{\infty}$ be a sequence of open disks with pairwise disjoint closures all having closures inside the open unit disk $\mathbb{D}$ and each being centered on the positive real axis and having centers that decrease to zero. Also let $\Delta = \{z \in \mathbb{C} : |z-2| \leq 1\}$ and $K = \Delta \cup cl\mathbb{D} \setminus \bigcup_{n=1}^{\infty} \Delta_n$.*

If A is an irreducible essentially normal operator with $\sigma_e(A) = K$, then $C^*(A)$ has a subnormal generator if and only if either

$$0 < \sup\{ind(A-\lambda I) : \lambda \notin K\} < \infty \ \ or \ -\infty < \inf\{ind(A-\lambda I) : \lambda \notin K\} < 0.$$

The point of the previous theorem is that if $\text{ind}(A-\lambda I) < 0$ for all $\lambda \notin K$ and $\inf\{\text{ind}(A-\lambda I) : \lambda \notin K\} = -\infty$, then A will have the same spectral picture as a *pure* subnormal operator, yet $C^*(A)$ will not have a subnormal generator (because for any one-to-one continuous function h on $\sigma_e(A)$, $h(A)$ will not have the same spectral picture as an *irreducible* subnormal operator).

Proof. ($\Leftarrow$) Assume that $0 < M := \sup\{\text{ind}(A-\lambda I) : \lambda \notin K\} < \infty$. Then by applying Theorem 6.6, since $M > 0$, K_M contains the unbounded component of $\mathbb{C} \setminus K$ and thus it follows easily that $\text{int}(K_M)$ is connected and dense in K_M. Thus Theorem 6.6 implies that $C^*(A)$ has a subnormal generator. Similarly if the minimum is negative.

($\Rightarrow$) Now assume that $C^*(A)$ has a subnormal generator and suppose that $-\infty < \inf\{\text{ind}(A-\lambda I) : \lambda \notin K\} < 0$ is not true, then we will show that $0 < \sup\{\text{ind}(A-\lambda I) : \lambda \notin K\} < \infty$ is true. Thus we are supposing that either $\inf_{\lambda \notin K} \text{ind}(A-\lambda I) \geq 0$ or $\inf_{\lambda \notin K} \text{ind}(A-\lambda I) = -\infty$ and we must show that both $\sup_{\lambda \notin K} \text{ind}(A-\lambda I) > 0$

and $\sup_{\lambda\notin K}\operatorname{ind}(A-\lambda I)<\infty$. Thus there are four cases to consider. Two are actually equivalent (by taking adjoints) and all are related.

Since $C^*(A)$ has a subnormal generator, then by Theorem 2.10 there is a compact set $\mathcal{K}\subseteq\mathbb{C}$ and a homeomorphism $h:\sigma_e(A)\to\mathcal{K}$ such that $\operatorname{ind}(h(A)-\lambda I)\le 0$ for all $\lambda\notin\mathcal{K}$ and $R(\mathcal{K}')$ has exactly one non-trivial Gleason part which is dense in $\mathcal{K}'$, where $\mathcal{K}'=\mathcal{K}\cup\{\lambda:\operatorname{ind}(h(A)-\lambda I)<0\}$.

Let $L=[\mathrm{cl}\mathbb{D}\setminus\bigcup_{n=1}^{\infty}\Delta_n]$ and let B be an irreducible essentially normal operator with $\sigma_e(B)=L$ and $\operatorname{ind}(B-\lambda I)=\operatorname{ind}(A-\lambda I)$ for all $\lambda\in\mathbb{C}\setminus K$. Also let C be a normal operator with $\sigma(C)=\sigma_e(C)=\Delta$. Then A has the same spectral picture as $B\oplus C$. Hence by (BDF) Theorem 2.1, A and $B\oplus C$ are unitarily equivalent modulo the compacts. Since h is continuous, it follows that $h(A)$ and $h(B)\oplus h(C)$ are also unitarily equivalent modulo the compacts. Since $h(C)$ is a normal operator, it has a zero index function, thus, $\operatorname{ind}(h(A)-\lambda I)=\operatorname{ind}(h(B)-\lambda I)$ for all $\lambda\notin\mathcal{K}$. Since we are assuming that $\operatorname{ind}(h(A)-\lambda I)\le 0$ for all $\lambda\notin\mathcal{K}$, then we also have that $\operatorname{ind}(h(B)-\lambda I)\le 0$ for all $\lambda\notin\mathcal{K}$. Thus by Theorem 6.7 applied to the operator B, we see that the index function of B (and hence also A) must be either bounded above or bounded below. Hence we cannot have $\sup_{\lambda\notin K}\operatorname{ind}(A-\lambda I)=\infty$ and $\inf_{\lambda\notin K}\operatorname{ind}(A-\lambda I)=-\infty$. Now suppose that $\inf_{\lambda\notin K}\operatorname{ind}(A-\lambda I)=-\infty$ and let $N:=\sup_{\lambda\notin K}\operatorname{ind}(A-\lambda I)$. We need to show that $N>0$. By way of contradiction, suppose that $N\le 0$. There are two cases to consider.

Case 1: $h(L)\subseteq cl[\operatorname{inside}(h(\partial\mathbb{D}))]$, that is $h(\partial\mathbb{D})$ is the outer-boundary of $h(L)$.

In this case it follows, since h is one-to-one, that $h(\Delta)$ is contained in the closure of the outside of the Jordan curve $h(\partial\mathbb{D})$. Thus, the interior of $\mathcal{K}'$ has two components — namely $\operatorname{int}(h(L))\cup\{\lambda:\operatorname{ind}(h(A)-\lambda I)<0\}$ and $\operatorname{int}(h(\Delta))$ — whose closures intersect at a single point. Thus, $R(\mathcal{K}')$ has two non-trivial Gleason parts, contradicting our assumption above.

Case 2: $h(L)\subseteq cl[\operatorname{outside}(h(\partial\mathbb{D}))]$.

In this case, again by the injectivity of h we must have $h(\Delta)$ contained in the closure of the inside of $h(\partial\mathbb{D})$.

Again using Theorem 6.7 (applied to B), we may extend $h|L$ to a homeomorphism of $\mathbb{C}\cup\{\infty\}$ onto itself. Then we have $\operatorname{ind}(h(A)-h(\lambda)I)=\operatorname{ind}(h(B)-h(\lambda)I)=(-1)^p[\operatorname{ind}(B-\lambda I)-M]=(-1)^p[\operatorname{ind}(A-\lambda I)-M]$ for $\lambda\notin\sigma_e(A)$, where $M=\operatorname{ind}(A-\lambda_0 I)\le N\le 0$ for some $\lambda_0\in\mathbb{C}\setminus K$. Since we are assuming that $\operatorname{ind}(h(A)-h(\lambda)I)\le 0$, it follows that either $\operatorname{ind}(A-\lambda I)\le M$ (if $p=0$) or $\operatorname{ind}(A-\lambda I)\ge M$ (if $p=1$). Since we are also assuming that $\inf_{\lambda\notin K}\operatorname{ind}(A-\lambda I)=-\infty$, we must have that $\operatorname{ind}(A-\lambda I)\le M$ for all $\lambda\notin\sigma_e(A)$. Thus, $M=N$ and $p=0$. Thus,

$$\operatorname{ind}(h(A)-h(\lambda)I)=[\operatorname{ind}(A-\lambda I)-N]$$

for $\lambda\notin\sigma_e(A)$. However if we take λ in the unbounded component of $\mathbb{C}\setminus\sigma_e(A)$, then we have $\operatorname{ind}(h(A)-h(\lambda)I)=[0-N]=-N\ge 0$. Thus if $N\ne 0$ we have a contradiction since we are assuming that the index function of $h(A)$ is non-positive. It follows that $N=0$. Thus we have $\operatorname{ind}(h(A)-h(\lambda)I)=\operatorname{ind}(A-\lambda I)$ for $\lambda\notin\sigma_e(A)$. Let G be the unbounded component of $\mathbb{C}\setminus\sigma_e(A)$. If $\lambda\in G$, then we have $\operatorname{ind}(h(A)-h(\lambda)I)=0$. Now one of the bounded components of $\mathbb{C}\setminus h(L)$ is $h(\mathbb{C}_\infty\setminus\mathrm{cl}\mathbb{D})$ and $h(\Delta)\subseteq h(\mathbb{C}_\infty\setminus\mathrm{cl}\mathbb{D})$ and $\operatorname{ind}(h(A)-\lambda I)=0$ if $\lambda\in h(\mathbb{C}_\infty\setminus\mathrm{cl}\mathbb{D})\setminus h(\Delta)$. Thus, $\mathcal{K}'=\mathcal{K}\cup\{\lambda:\operatorname{ind}(h(A)-\lambda I)<0\}=h(\Delta)\cup h(L)\cup\{\lambda:\operatorname{ind}(h(A)-\lambda I)<0\}$ and $h(\Delta)$ intersects $C=[h(L)\cup\{\lambda:\operatorname{ind}(h(A)-\lambda I)<0\}]$ at a single point. Thus $R(\mathcal{K}')$ has two Gleason parts, because the interior of $\mathcal{K}'$ has two components — namely $\text{int}(C)$ and $\text{int}(h(\Delta))$ — whose closures intersect at a single point. This is a contradiction to an earlier assumption. It follows that we must have $N > 0$.

Thus far we have shown that if $\inf_{\lambda \notin K} \text{ind}(A - \lambda I) = -\infty$ holds, then we must have $0 < \sup_{\lambda \notin K} \text{ind}(A - \lambda I) < \infty$. We must now assume that $\inf_{\lambda \notin K} \text{ind}(A - \lambda I) \geq 0$ and prove that $0 < \sup_{\lambda \notin K} \text{ind}(A - \lambda I) < \infty$.

If we have $\inf_{\lambda \notin K} \text{ind}(A - \lambda I) \geq 0$ and $\sup_{\lambda \notin K} \text{ind}(A - \lambda I) = \infty$, then by taking adjoints we would have the infimum being minus infinity and the supremum being at most zero, and the previous case said this cannot happen. Thus if $\inf_{\lambda \notin K} \text{ind}(A - \lambda I) \geq 0$, then $\sup_{\lambda \notin K} \text{ind}(A - \lambda I) < \infty$.

For the final case, assume that $\inf_{\lambda \notin K} \text{ind}(A - \lambda I) \geq 0$ and we must show that $\sup_{\lambda \notin K} \text{ind}(A - \lambda I) > 0$. Suppose not. Then $\sup_{\lambda \notin K} \text{ind}(A - \lambda I) \leq 0$ which implies that $\text{ind}(A - \lambda I) = 0$ for all $\lambda \notin K$. But then reasoning as in "Case 2" above with $M = N = 0$, we reach a contradiction. Thus we must have $\sup_{\lambda \notin K} \text{ind}(A - \lambda I) > 0$. The Theorem now follows. □

Theorem 7.10 (A pair of tangent Swiss cheeses)**.** *Let K_1 be a Swiss-cheese set and let K_2 be the image of K_1 under the map $z \mapsto (z + 2)$, so that K_1 and K_2 are two Swiss-cheese sets tangent at one point. Let $K = K_1 \cup K_2$.*

If A is an irreducible essentially normal operator with $\sigma_e(A) = K$, then $C^*(A)$ has a subnormal generator if and only if either

$$0 < \sup\{\mathit{ind}(A - \lambda I) : \lambda \notin K\} < \infty \ \mathit{or}\ -\infty < \inf\{\mathit{ind}(A - \lambda I) : \lambda \notin K\} < 0.$$

The details of the above theorem are similar to those of Theorem 7.9 and are left to the reader. One point of this example is if we choose $\sup_{\lambda \in \hat{K} \setminus K} \text{ind}(A - \lambda I) < 0$ and $\inf_{\lambda \notin K} \text{ind}(A - \lambda I) = -\infty$, then A will have the *same spectral picture as a pure subnormal operator* and its essential spectrum is small—unlike Theorem 7.9 where the essential spectrum has nonempty interior— in this example *the essential spectrum may have area zero* (if the Swiss-cheese set is chosen to have area zero), yet $C^*(A)$ still has *no subnormal generator.*

Example 7.11 (Non-zero index & a hyponormal, but no subnormal generator)**.** *Let $K_1 = [0, 1] \times [0, 1]$ be the unit rectangle, $K_2 = [1/4, 3/4] \times [0, 1/4]$ another rectangle, $K_3 = \{1/2\} \times [1/4, 1/2]$ a vertical line segment, and K_4 a Jordan arc with positive area density at each point that is contained in $\{z : Im(z) < 0\}$ except for one endpoint on the positive x-axis at $z = 1/2$. Let $K = \partial K_1 \cup K_2 \cup K_3 \cup K_4$. If A is an irreducible essentially normal operator with $\sigma_e(A) = K$ and $\mathit{ind}(A - \lambda I) = -1$ for $\lambda \in (\mathit{int}(K_1) \setminus (K_2 \cup K3))$, then $C^*(A)$ does not have a subnormal generator, but it does have a hyponormal generator.*

Proof. It follows from Theorem 2.14, that there is an irreducible hyponormal operator with rank one self-commutator with the same spectral picture as A. Thus, by Proposition 2.9, $C^*(A)$ has a hyponormal generator. However, if $h : K \to \mathbb{C}$ is any one-to-one continuous function and $\mathcal{K} := h(K) \cup \{\lambda \in \mathbb{C} : \text{ind}(h(A) - \lambda I) < 0\}$, then $R(\mathcal{K})$ has only one nontrivial Gleason part, but it is not dense in $\mathcal{K}$. Thus, $h(A)$ does not have the same spectral picture as an irreducible subnormal operator. Thus $C^*(A)$ does not have a subnormal generator. □

Example 7.12 (Non-zero index, but no hyponormal generator)**.** *Let* $K_1 = \{z \in \mathbb{C} : 1 \le |z| \le 2\}$*, let* $K = K_1 \cup \{1-\frac{1}{n}\}_{n=1}^{\infty} \cup \{2+\frac{1}{n}\}_{n=1}^{\infty}$*. If* A *is an irreducible essentially normal operator with* $\sigma_e(A) = K$ *and* $ind(A-\lambda I) = -1$ *for* $\lambda \in \big(\mathbb{D} \setminus \{1-\frac{1}{n}\}_{n=1}^{\infty}\big)$*, then* $C^*(A)$ *does not have a hyponormal generator.*

Remark. Actually in the previous example regardless of what the index is on the component $\big(\mathbb{D} \setminus \{1-\frac{1}{n}\}_{n=1}^{\infty}\big)$, $C^*(A)$ does not have a hyponormal generator, because $\sigma_e(A)$ will not be homeomorphic to the essential spectrum of a pure hyponormal operator.

Proof. If $h : K \to K'$ is a homeomorphism, then K' will have some isolated points in the unbounded component of $h(K_1)$. Hence K' will not have positive area density at each of its points, so K' cannot be the essential spectrum of a pure hyponormal operator. Thus, $C^*(A)$ has no hyponormal generator. □

The following example may be established using techniques from Theorem 6.7 and Theorem 7.9.

Example 7.13 (Doubly unbounded indices, but not arbitrary indices)**.** *Let*

$$K = \partial\mathbb{D} \cup \bigcup_{n=0}^{\infty} \partial\Delta_n \cup \{z \in \mathbb{D} : Re(z) \le 0\} \setminus \bigcup_{n<0} \Delta_n$$

where $\{\Delta_n\}_{n\in\mathbb{Z}}$ *is a sequence of open disks with* $cl\Delta_n \subseteq \mathbb{D}$ *for all* $n \in \mathbb{Z}$ *and* Δ_n *is centered on the positive real axis for* $n \ge 0$*, and* Δ_n *is centered on the negative real axis for* $n < 0$*, any two of the disks have disjoint closures, and the centers converge to zero. If* A *is an irreducible essentially normal operator with* $\sigma_e(A) = K$*, then* $C^*(A)$ *has a subnormal generator if and only if* $ind(A-\lambda I) \ne 0$ *for some* $\lambda \in \mathbb{C}\setminus K$ *and either* $\sup\{ind(A-\lambda I) : \lambda \in \Delta_n, n < 0\} < \infty$ *or* $\inf\{ind(A-\lambda I) : \lambda \in \Delta_n, n < 0\} > -\infty$.

The previous example shows that the set of irreducible essentially normal operators with $\sigma_e(A) = K$ that have subnormal generators for their C^*-algebras, contains those operators whose Fredholm index function is bounded above or bounded below, and also contains some operators whose Fredholm index function is not bounded above or below, yet does not contain all irreducible essentially normal operators A with $\sigma_e(A) = K$.

The following example shows that even if $\mathbb{C} \setminus \sigma_e(A)$ has only a single bounded component G which is simply connected and $ind(A-\lambda I) \ne 0$ for $\lambda \in G$, then $C^*(A)$ need not have a subnormal generator. In fact it also gives an example of a compact set K such that regardless of what the index is for A, $C^*(A)$ does not have a subnormal generator.

Example 7.14. *Let* $E = \{\pm\frac{1}{n} : n \ge 1\} \cup \{0\}$ *and let*

$$K = \partial\mathbb{D} \cup \{re^{i\theta} : 1/2 \le r \le 1, \theta \in E\} \cup \{x \in \mathbb{R} : 1 \le x \le 2\}.$$

If A *is any irreducible essentially normal operator with* $\sigma_e(A) = K$*, then* $C^*(A)$ *does not have a subnormal generator.*

Proof. If $ind(A) = 0$, then apply Corollary 5.4 to see that $C^*(A)$ does not have a subnormal generator. If $ind(A) \ne 0$, then there is a one-to-one continuous function $h : K \to \mathbb{C}$ such that $h(A)$ has the same spectral picture as an irreducible subnormal operator. However, since K and thus $h(K)$ are finitely connected, this implies that if C is the polynomially convex hull of $h(K)$, then the interior of C is dense in C. However, this will never be true, as one "ray" will always be "sticking out" from $cl[int(C)]$. □

Checkerboards: A nice example of Theorem 6.9 arises with "checkerboards". For two closed sets $X, Y \subseteq [0,1]$ with $\{0,1\} \subseteq X \cap Y$, define the "checkerboard" determined by X and Y as follows

$$C_{X,Y} = (X \times [0,1]) \cup ([0,1] \times Y) =$$
$$= \{(x,y) : x \in X, 0 \leq y \leq 1\} \cup \{(x,y) : 0 \leq x \leq 1, y \in Y\}.$$

If X and Y are finite sets, then the "checkerboard" $C_{X,Y}$ is simply a grid of finitely many horizontal and vertical lines within the unit square. However X and or Y may be infinite sets as well. In what follows ∂X and ∂Y denotes the boundary of X and Y respectively considered as subsets of $\mathbb{R}$.

Proposition 7.15. *Let X and Y be closed subsets of $[0,1]$ that each contain the endpoints $\{0,1\}$. Suppose also that one of the following conditions hold.*

(1) *The cardinality of X and Y is at least three and ∂X and ∂Y each have only countably many limit points, or*
(2) *The cardinality of X and Y is at least three and either X or Y is finite (and the other is arbitrary).*

Then the checkerboard $C_{X,Y}$ satisfies the assumptions (i) - (iv) of Theorem 6.9, which are as follows:

(i) *$C_{X,Y}$ is connected.*
(ii) *Each component of $\mathbb{C} \setminus C_{X,Y}$ is bounded by a Jordan curve.*
(iii) *There is a countable set $E \subseteq \partial C_{X,Y}$ such that for each $a \in \partial C_{X,Y} \setminus E$, there is an $r > 0$ and a $\delta > 0$ such that for each component G of $\mathbb{C} \setminus C_{X,Y}$ that intersects $B(a,r)$ we have $diam(G) > \delta$.*
(iv) *For each component G of $\mathbb{C} \setminus C_{X,Y}$ we have that $C_{X,Y} \setminus \partial G$ is a connected set.*

Proof. Conditions (i), (ii) are easily established. Condition (iv) holds if the cardinality of X and Y are both at least three.

If (1) holds, then for condition (iii), the exceptional set E may be taken to be $(\partial X)' \times (\partial Y)'$, where L' denotes the set of limit points of a set L.

If (2) holds with Y finite, then E is the empty set and if $\delta = \min\{|y_1 - y_2| : y_1, y_2 \in Y, y_1 \neq y_2\}$, then $\delta > 0$ and for each component G of $\mathbb{C} \setminus C_{X,Y}$ we have $diam(G) > \delta$. □

Let's try to better understand conditions (1) and (2) in Theorem 6.9 when **both** X **and** Y **are finite**, actually a slightly more general condition will suffice. So assume that A is an irreducible essentially normal operator with $\sigma_e(A) = C_{X,Y}$. *For convenience let $K = C_{X,Y}$.*

The components (bounded and unbounded) of $\mathbb{C} \setminus K$ will be called the faces of the checkerboard. Let's say that a component G of $\mathbb{C} \setminus K$ has index p if $\text{ind}(A - \lambda I) = p$ for $\lambda \in G$.

As in Theorem 6.9, let $M = \sup\{\ \text{ind}(A - \lambda I) : \lambda \in \mathbb{C} \setminus \sigma_e(A)\}$ and $m = \inf\{\text{ind}(A - \lambda I) : \lambda \in \mathbb{C} \setminus \sigma_e(A)\}$, and we will consider the sets K_M and K_m as defined in Theorem 6.9.

The only way that $int(K_M)$ is not going to be dense in K_M is if there are two "adjacent" components of $\mathbb{C} \setminus K$ that both have index equal to M (here we must consider the unbounded component as well as the bounded ones). What do we mean by "adjacent" components? We shall say that two components G and H of $\mathbb{C} \setminus K$ are *adjacent* if $\mathrm{cl}G \cap \mathrm{cl}H$ contains a line segment of positive length. Note that G and H are both rectangles.

If there are two adjacent components of $\mathbb{C} \setminus K$ both having index equal to M, then the line segment that is common between them will not be in the closure of $int(K_M)$, hence $int(K_M)$ will not be dense in K_M.

When is $int(K_M)$ connected? Since K is connected, then $int(K_M)$ is connected if and only if *the set of faces with index not equal to M is path connected.* By this we mean that for any two faces of K having index strictly less than M, we can find a "*path of faces*" connecting the two given faces where consecutive faces in the path are adjacent and each of the faces in the path has index strictly less than M.

Recall that one must include the unbounded component (which always has index 0) as one of the faces of the checkerboard.

Let $|L|$ denote the cardinality of the set L and L' denote the cluster set or set of limit points of L.

Keeping the terminology given above about adjacent faces and paths of faces we may restate Theorem 6.9 for checkerboards that are either finite or have $X' \cup Y' \subseteq \{0,1\}$.

Example 7.16. *Keeping the above notation, suppose that $X' \cup Y' \subseteq \{0,1\}$ and $|X| \geq 3$ and $|Y| \geq 3$. If A is an irreducible essentially normal operator with $\sigma_e(A) = C_{X,Y}$, then $C^*(A)$ has a subnormal generator if and only if $ind(A-\lambda I) \neq 0$ for some $\lambda \in \mathbb{C} \setminus C_{X,Y}$ and one of the following conditions hold:*

(1) *No two faces with index M are adjacent and the set of faces with index less than M is path connected; or*
(2) *No two faces with index m are adjacent and the set of faces with index greater than m is path connected.*

Remember that one must include the unbounded component of $\mathbb{C} \setminus C_{X,Y}$ as one of the faces in the previous example.

As a more specific example, we consider the case where all the indices are either zero or one. If all the indices are zero, then by Theorem 6.9, $C^*(A)$ does not have a subnormal generator. However, by Theorem 5.6, $C^*(A)$ would have a hyponormal generator. If all the indices are one (except for the unbounded face which has index zero), then clearly $C^*(A)$ has a subnormal generator by Theorem 6.9. So we will consider the case below where there are both zeros and ones.

Example 7.17 (zero-one indices)**.** *Suppose that $X' \cup Y' \subseteq \{0,1\}$ and $|X| \geq 3$ and $|Y| \geq 3$. If A is an irreducible essentially normal operator with $\sigma_e(A) = C_{X,Y}$ and $M = 1$ and $m = 0$, then $C^*(A)$ does have a hyponormal generator. However, $C^*(A)$ has a subnormal generator if and only if one of the following conditions holds:*

(1) *No two faces with index 1 are adjacent and the set of faces with index 0 is path connected; or*
(2) *No two faces with index 0 are adjacent and the set of faces with index 1 is path connected.*

Remark. In condition (1), "the set of faces with index 0" will naturally include the unbounded component. Also in condition (2) above the condition that no two faces with index 0 are adjacent implies that no face with index 0 can be adjacent to the unbounded component, hence all the indices in the faces that are adjacent to the unbounded component must have index equal to 1.

Proof. $C^*(A)$ has a hyponormal generator since the spectral picture of A is homeomorphic to one which has positive area density at each point and, since the indices are 0 or 1, we may apply Theorem 2.14 to find an irreducible hyponormal operator T with rank one self-commutator such that T^* has this new spectral picture (of positive area density). By Proposition 2.9, $C^*(A)$ has a generator unitarily equivalent to T. □

Is it necessary in Example 7.16 that $|X|, |Y| \geq 3$? (we use that to satisfy the hypothesis of Theorem 6.9 that $K \setminus \partial G$ is connected). The following example shows that for Example 7.16 to hold we do need $|X|, |Y| \geq 3$.

Example 7.18. *If $X = \{0, 1/4, 3/4, 1\}$ and $Y = \{0, 1\}$ and A is an irreducible essentially normal operator with $\sigma_e(A) = C_{X,Y}$ and has Fredholm indices $0, -1, -1$ in the three faces, then $C^*(A)$ does have a subnormal generator, but none of the conditions in Example 7.16 are satisfied.*

Proof. Clearly none of the conditions of Example 7.16 are satisfied. However we can "fold" the left face with index zero inside the middle one and get a spectral picture that is the spectral picture of an irreducible subnormal operator. It follows that $C^*(A)$ has a subnormal generator. □

We now show that in fact $n \times 1$ checkerboards are more flexible than $n \times m$ checkerboards where $n, m \geq 2$, in the sense that as soon as the index is non-zero, then there is a subnormal generator.

Proposition 7.19 ($n \times 1$ Checkerboards)**.** *Suppose that X is a finite set satisfying $\{0, 1\} \subseteq X \subseteq [0, 1]$ and $Y = \{0, 1\}$ and consider the checkerboard $C_{X,Y}$. If A is an irreducible essentially normal operator with $\sigma_e(A) = C_{X,Y}$, then $C^*(A)$ has a subnormal generator if and only if ind$(A - \lambda I) \neq 0$ for some $\lambda \in \mathbb{C} \setminus \sigma_e(A)$.*

If X has $(n+1)$ points, then $C_{X,Y}$ is an $n \times 1$ checkerboard; that is, it has n bounded components, or n faces. Also, notice that an $n \times 1$ checkerboard does not satisfy the hypothesis of Theorem 6.9. Because if G is a bounded component of $\mathbb{C} \setminus C_{X,Y}$ that is not one of the "ends", then we have that $C_{X,Y} \setminus \partial G$ is disconnected. However Theorem 6.3 applies nicely in this case.

Proof of Proof 7.19. Start with a face with non-zero index and then use Theorem 6.3 to repeatedly attach additional faces. In applying Theorem 6.3 the new face being attached can only be adjacent to one of the existing faces and this will be the case for an $n \times 1$ checkerboard. □

8. Final Remarks and Questions

Question 8.1 (See Questions 4.3 and 5.2)**.** *Which compact sets in the plane are homeomorphic to the spectrum of a pure (or irreducible) subnormal operator?*

See Questions 4.3 and 5.2 and the results surrounding them for more on the above question.

Question 8.2. *If A is an irreducible essentially normal operator and $\sigma_e(A)$ is the Sierpinski Triangle, then under what conditions on the values of the Fredholm index function of A does $C^*(A)$ have a subnormal generator?*

Question 8.3. *Let X and Y be Cantor sets in $[0,1]$. If A is an irreducible essentially normal operator and $\sigma_e(A)$ is the checkerboard $C_{X,Y}$, then under what conditions on the values of the Fredholm index function of A does $C^*(A)$ have a subnormal generator?*

For the above two questions, $\sigma_e(A)$ is not a Swiss Cheese Type set so Theorem 7.8 does not apply. Also, these operators do not satisfy the hypothesis of Theorem 6.9. Hence the above two operators are not covered by the results of this paper.

Question 8.4. *Can one characterize the spectral pictures of irreducible essentially normal hyponormal operators?*

If one could answer the above question, then much more could be said about when $C^*(A)$ has a hyponormal generator. For example, the authors believe the following question has an affirmative answer.

Question 8.5. *If A is an irreducible essentially normal operator, $\sigma_e(A)$ is a perfect set (has no isolated points), and the Fredholm index function of A is bounded above or bounded below, then does $C^*(A)$ have a hyponormal generator?*

Question 8.6. *If $A \in \mathcal{B}(\mathcal{H})$, $\sigma(A) = \partial\mathbb{D}$, and A is not a unitary, then can $C^*(A)$ have a subnormal generator?*

The above question is looking towards extending Theorem 3.1. If A is essentially normal, then the answer is no by Corollary 4.6. So the issue here is when A is not essentially normal.

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Mathematics Department, Washington & Lee University, Lexington, VA 24450
E-mail address: feldmanN@wlu.edu

Mathematics Department, Bucknell University, Lewisburg, PA 17837
E-mail address: pmcguire@bucknell.edu