Instructions: Answer each of the questions on the pages that follow. Record your solutions in the space provided, attaching extra sheets as necessary. Be certain to include all details to receive full credit for your solution. You have ninety minutes to complete this test. You may use neither your book, nor your notes, nor a calculator on this test. Good luck!

Pledge: Please copy the pledge (below) in your own handwriting and sign it after completing this test and before turning it in.

"On my honor, I have neither given nor received any unacknowledged aid on this test."
Formulae

\[ 1 + 2 + 3 + \cdots + k = \frac{k(k + 1)}{2} \]

\[ 1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k + 1)(2k + 1)}{6} \]

\[ 1^3 + 2^3 + 3^3 + \cdots + k^3 = \left( \frac{k(k + 1)}{2} \right)^2 \]
1. A car braked with a constant deceleration of 12 feet per second per second, producing skid marks measuring 54 feet before coming to a stop. How fast was the car traveling when the brakes were first applied?

\[ a = -12 \text{ ft/sec}^2 \]

\[ v = -12t + v_o \quad \text{we want to find this} \]

\[ d = -6t^2 + v_o t + d_0 \]

\[ d_0 = 0 \text{ since we start measuring when } t=0 \text{ (when the brakes were first applied)} \]

Let \( T \) denote the time when the car stops.

At \( t=T \), \( v = 0 \) (since it has stopped) and \( d = 54 \) (since the skid marks are 54 feet long)

\[ 0 = -12T + v_o \quad \text{and} \quad 54 = -6T^2 + v_o T \]

\[ v_o = 12T \]

\[ 54 = -6T^2 + (12T)T \]

\[ 54 = 6T^2 \]

\[ 9 = T^2 \]

\[ T = 3 \text{ seconds} \]

\[ v_o = 12(T) = 12(3) = 36 \text{ ft/sec} \]
2. (a) Evaluate the left Riemann sum for \( f(x) = 2 + 2x - x^2 \) on [0, 3] with three subintervals of uniform length. Explain with the aid of a diagram what the Riemann sum represents.

\[
L_3 = 1(2) + 1(3) + 1(2) = 2 + 3 + 2 = 7
\]

(b) Use the definition of the definite integral as a limit to calculate the value of the integral \( \int_0^3 (2 + 2x - x^2) \, dx \) using left-hand sums.

\[
L_n = \sum_{k=0}^{n-1} \left( \frac{3}{n} \right) f\left( \frac{3k}{n} \right)
\]

Since it's a left sum,

\[
L_n = \left( \frac{3}{n} \right) \sum_{k=0}^{n-1} \left( 2 + 2 \left( \frac{3k}{n} \right) - \left( \frac{3k}{n} \right)^2 \right)
\]

\[
= \frac{3}{n} \sum_{k=0}^{n-1} 2 + \frac{3}{n} \sum_{k=0}^{n-1} 2 \left( \frac{3k}{n} \right) - \frac{3}{n} \sum_{k=0}^{n-1} \frac{9k^2}{n^2}
\]

\[
= \frac{6}{n} \sum_{k=0}^{n-1} 1 + \frac{18}{n^2} \sum_{k=0}^{n-1} k - \frac{27}{n^3} \sum_{k=0}^{n-1} k^2
\]

\[
= \frac{6}{n} \cdot n + \frac{18}{n^2} \cdot \frac{(n-1)n}{2} - \frac{27}{n^3} \cdot \frac{(n-1)n(2n-1)}{6}
\]

\[
= 6 + 9 \left( \frac{n-1}{n} \right) - \frac{9}{2} \cdot \left( \frac{(n-1)(2n-1)}{n^2} \right)
\]

\[
\text{as } n \to \infty \rightarrow 6 + 9 - \frac{9}{2} \cdot 2 = 6
\]
(c) Use the Fundamental Theorem of Calculus to check your answer to part (b).

\[
\int_{0}^{3} (2 + 2x - x^2) \, dx = \left[ 2x + x^2 - \frac{1}{3}x^3 \right]_{0}^{3}
\]

\[
= 2(3) + (3)^2 - \frac{1}{3}(3)^3 - [0 + 0 - 0]
\]

\[
= 6 + 9 - 9 = 6
\]

(d) Draw a diagram to explain the geometric meaning of the integral in part (b).

\[
\int_{0}^{3} f(x) \, dx = \text{the magnitude of the area under } f(x) \text{ from } x=0 \text{ to } x=3,
\]

minus the magnitude of the area between } f(x) \text{ and the x-axis from } x=a \text{ to } x=3,
3. Evaluate each of the following expressions.

(a) \[ \frac{d}{dx} \int_{\pi/2}^{\pi} \left( \sin \frac{t}{3} \cos \frac{t}{2} \right) \, dt = \sin \frac{x^2}{3} \cos \frac{x^2}{3} \cdot 2x \]  

(b) \[ \frac{d}{dx} \int_{x}^{\pi/2} \left( \sin \frac{t}{3} \cos \frac{t}{2} \right) \, dt = \frac{d}{dx} \left( \int_{\pi/2}^{x} \sin \frac{t}{3} \cos \frac{t}{2} \, dt \right) \]
\[ = - \sin \frac{x}{3} \cos \frac{x}{2} \]

(c) \[ \int_{0}^{\pi/2} \frac{d}{dx} \left( \sin \frac{x}{3} \cos \frac{x}{2} \right) \, dx \]
\[ = \left[ \sin \frac{x}{3} \cos \frac{x}{2} \right]_{0}^{\pi/2} = \sin \frac{\pi}{6} \cos \frac{\pi}{4} - \sin 0 \cos 0 \]
\[ = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - 0 = \frac{\sqrt{2}}{4} \]

(d) \[ \frac{d}{dx} \int_{0}^{\pi/2} \left( \sin \frac{x}{3} \cos \frac{x}{2} \right) \, dx \]
\[ = \frac{d}{dx} \left( \text{a number} = \text{the area under } \sin \frac{x}{3} \cos \frac{x}{2} \text{ from } x=0 \text{ to } x=\pi/2 \right) \]
\[ = 0 \]
4. (a) Find the average value of the even function \( g(x) \) on the interval \([-2, 0]\) given that \( \int_{-2}^{0} g(x) \, dx = 9 \).

Average value \( = \frac{1}{2} \int_{-2}^{0} g(x) \, dx \)

\[
= \frac{1}{2} \cdot \frac{1}{2} \int_{-2}^{2} g(x) \, dx = \frac{1}{4} (9) = \frac{9}{4}
\]

(b) Find the value of \( c \) that gives the average value of the function \( f(x) = 3x + 1 \) on the interval \([0, 4]\).

\[
f(c) = \frac{1}{4 - 0} \int_{0}^{4} (3x + 1) \, dx = \frac{1}{4} \left[ \frac{3}{2}x^2 + x \right]_{0}^{4}
\]

\[
= \frac{1}{4} \left[ \frac{3}{2} (16) + 4 - \frac{3}{2} (0) - 0 \right]
\]

\[
= \frac{1}{4} \left[ 24 + 4 \right] = 7
\]

\[
f(c) = 3c + 1 = 7
\]

\[
3c = 6
\]

\[
c = 2
\]
5. Find a lower bound and an upper bound for \( \int_{\pi/6}^{\pi/3} \sin x \cos x \, dx \) using the extrema of \( \sin x \cos x \) on \([\pi/6, \pi/3]\).

\[
f(x) = \sin x \cos x
\]

\[
f'(x) = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x
\]

\(f' = 0: \) \((\cos x - \sin x)(\cos x + \sin x) = 0\)

\[
\cos x = \sin x \quad \text{or} \quad \cos x = -\sin x
\]

\(x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4, \text{etc.}\)

\[
\frac{\sqrt{3}}{4} \left( \frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{\sqrt{3} \pi}{24} \leq \int_{\pi/6}^{\pi/3} \sin x \cos x \, dx \leq \frac{1}{2} \left( \frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{\pi}{12}
\]

6. Let \( F(x) = \int_0^{\sqrt{x}} f(t) \, dt \) and \( f(x) = \int_0^x y^2 \, dy \). Find the point of inflection of \( F(x) \).

\[
F'(x) = \frac{f(\sqrt{x})}{2\sqrt{x}}, \quad F''(x) = \frac{f'(\sqrt{x}) \cdot 2\sqrt{x} - f(\sqrt{x})}{4x} = \frac{\sqrt{x} f'(\sqrt{x}) - f(\sqrt{x})}{4x^{3/2}}
\]

\[
f'(\sqrt{x}) = x
\]

\[
f(\sqrt{x}) = \int_0^x y^2 \, dy = \frac{1}{3} y^3 \bigg|_0^x = \frac{1}{3} x^3
\]

\[
f(\sqrt{x}) = \frac{1}{3} x \sqrt{x}
\]

\[
F''(x) = \frac{\sqrt{x} \cdot x - \frac{1}{3} x \sqrt{x}}{4x^{3/2}} = \frac{\frac{2}{3} x \sqrt{x}}{4x^{3/2}} = \frac{1}{6}
\]

So \( F(x) \) has no inflection points since it is always concave up.
7. Let \( g(x) = \int_0^x f(t) \, dt \), where \( f \) is the function shown in the graph below.

(a) At what values do the local extrema values of \( g \) occur?

\[ \text{maxima: } x = 2, 6, 8 \]
\[ \text{minima: } x = 4, 7, 10 \]

(b) Where does \( g \) attain its absolute maximum value?

\[ x = 2 \]

(c) On what interval(s) is \( g \) decreasing?

\[ g' < 0 \rightarrow g' = f < 0 \quad (2, 4) \cup (6, 7) \cup (8, 10) \]

(d) On what intervals is \( g \) concave upward?

\[ g'' = f' > 0 \rightarrow f \text{ increasing} \]
\[ (0, 1) \cup (3, 5) \cup (6.5, 7.5) \cup (9, 11) \]
8. Let \( A_n \) be the area of a polygon with \( n \) equal sides inscribed in a circle with radius \( r \). Show that \( A_n = \frac{1}{2}nr^2 \sin(2\pi/n) \) and compute \( \lim_{n \to \infty} A_n \).

\[
A_n = \frac{1}{2} \cdot n \cdot r \cdot \sin \left( \frac{2\pi}{n} \right) \cdot \frac{n}{\text{Each } \Delta} \quad \Downarrow \quad \text{n } \Delta's
\]

\[
A_n = \frac{1}{2} nr^2 \sin \left( \frac{2\pi}{n} \right)
\]

\[
\lim_{n \to \infty} A_n = \lim_{n \to \infty} \frac{1}{2} nr^2 \sin \left( \frac{2\pi}{n} \right)
\]

\[
= \frac{1}{2} r^2 \lim_{n \to \infty} n \cdot \sin \left( \frac{2\pi}{n} \right)
\]

\[
= \frac{1}{2} r^2 \lim_{n \to \infty} \frac{\sin \left( \frac{2\pi}{n} \right)}{\frac{2\pi}{n}} \cdot 2\pi
\]

\[
= \frac{1}{2} r^2 \cdot 2\pi \lim_{u \to 0} \frac{\sin u}{u}
\]

\[
= \pi r^2 \cdot 1
\]

\[
= \pi r^2
\]