Hyperbolic Functions

(1) definitions

• \( \sinh x = \frac{e^x - e^{-x}}{2} \)
• \( \cosh x = \frac{e^x + e^{-x}}{2} \)

(2) relations/why hyperbolic?

• \( \cosh^2 x - \sinh^2 x = 1 \)
• \( t = 2A \) on unit circle; also on hyperbola

(3) derivatives

• \( D_x(\sinh x) = \cosh x \)
• \( D_x(\cosh x) = \sinh x \)

(4) inverses (log form also)

• inverse of \( \sinh x \) is \( \sinh^{-1} x \)
• use \( \sinh x = \frac{e^x - e^{-x}}{2} \) to get log form of \( \sinh^{-1} x \)

(5) derivatives of inverses

• use \( \sinh(\sinh^{-1} x) = x \)
• set \( u = \sinh^{-1} x \) and use chain rule
• use relations and replace \( u \)

(6) catenary
Consider a section \( PQ \) of length \( x \). We’ll obtain a differential equation for the shape \( y = f(x) \) of the hanging cable by balancing horizontal and vertical components of the forces acting on \( PQ \). These forces are:

- \( T_0 \): the horizontal tension pulling on the cable at \( P \)
- \( T \): the tangential tension pulling on the cable at \( Q \)
- \( ws \): the force of gravity pulling downward on the section \( PQ \)

When we equate the horizontal and vertical components, we find that

\[
T \cos \phi = T_0 \quad \text{and} \quad T \sin \phi = ws
\]

We divide the second equation by the first and find that

\[
\frac{dy}{dx} = \tan \phi = \frac{T \sin \phi}{T \cos \phi} = \frac{ws}{T_0}
\]

Since \( s \) is a function of \( x \), this differential equation is not as simple as we might hope. But after differentiating both sides, we see that

\[
\frac{d^2y}{dx^2} = \frac{w}{T_0} \frac{ds}{dx}
\]

We know from our study of arc length that

\[
\frac{ds}{dx} = \sqrt{1 + \left( \frac{dy}{dx} \right)^2}
\]

and so

\[
\frac{d^2y}{dx^2} = \frac{w}{T_0} \sqrt{1 + \left( \frac{dy}{dx} \right)^2}
\]

This is the differential equation that will give us the shape \( y = f(x) \) of the hanging cable. The problem now is to find a solution of the equation. The solution involves a standard trick - we’ll let \( p = \frac{dy}{dx} \), so that \( \frac{dp}{dx} = \frac{d^2y}{dx^2} \). This gives us the simpler equation

\[
\frac{dp}{dx} = \frac{w}{T_0} \sqrt{1 + p^2}
\]

Rewrite this as

\[
\frac{1}{\sqrt{1 + p^2}} \frac{dp}{dx} = \frac{w}{T_0}.
\]

The left side is just the derivative of \( \sinh^{-1} p \)! Integrating both sides, we have

\[
\sinh^{-1} p = \frac{w}{T_0} x + C.
\]

Now \( p = 0 \) when \( x = 0 \) because the cable has a horizontal tangent at its lowest point \( P \). This tells us that \( C = \sinh^{-1}(0) = 0 \). This gives us

\[
\frac{dy}{dx} = p = \sinh \left( \frac{w}{T_0} x \right).
\]

To find \( y \), we simply integrate both sides of this equation:

\[
y = \frac{T_0}{w} \cosh \left( \frac{w}{T_0} x \right) + C
\]

If \( y_0 \) is the height of the point \( P \) above the \( x \)-axis, then \( C = y_0 - \frac{T_0}{w} \), so the shape of the hanging cable is given by

\[
y = \frac{T_0}{w} \cosh \left( \frac{w}{T_0} x \right) + y_0 - \frac{T_0}{w}
\]

We may choose the location of the \( x \)-axis so that \( y_0 = \frac{T_0}{w} \), and then the equation takes the simpler form

\[
y = \frac{T_0}{w} \cosh \left( \frac{w}{T_0} x \right).
\]

If we let \( a = T_0/w \), then the equation becomes even simpler:

\[
y = a \cosh \left( \frac{x}{a} \right).
\]