(1) (a) First calculate the demand at the original price $p_b = 2$

$$b(p_b, m) = \frac{1000}{20} - 5p_b$$

$$\Rightarrow b_0 = b(2) = 40$$

In general $m^c = m + (p^1_b - p^0_b)b_0$. If the price increases $1 per unit $m^c$ must include $40 over Allen's old income. so $m^c = m + 40 = 1040$.

(b) We get $b^c$, the compensated demand for beer by setting price equal to the new price, $p_b = 3$, and income equal to the compensated level of income, $m^c = 1040$.

$$b(3, 1040) = \frac{1040}{20} - 5 \times 3$$

$$= 37$$

(c) Find the substitution effect, income effect and total effect of this price change.

- The **substitution effect** is the difference between compensate demand and the quantity demanded before the price change.

$$\text{S.E.} = b(3, m^c) - b(2, m)$$

$$= 37 - 40$$

$$= -3.$$  

In words the substitution effect of the price change from $2 to $3 is a **decrease** of 3 pints of beer.

- The **income effect** is the difference between the new demand at the higher price and the compensated demand.

$$\text{I.E.} = b(3, m) - b(3, m^c)$$

$$= \frac{1000}{20} - 5 \times 3 - \left( \frac{1040}{20} - 5 \times 3 \right)$$

$$= \frac{1000}{20} - \frac{1040}{20}$$

$$= -2.$$  

In words the income effect of the price change from $2 to $3 is a **decrease** of 2 pints of beer.

- The **total effect** is the sum of the substitution and income effects and is equal to -5.
(2) An inferior but not Giffen good. Consider two goods x and y and a decrease in the price of x. Let \( x(p_x, p_y, m) \) stand for the quantity demanded of good x when prices are \( p_x \) and \( p_y \) and income level is \( m \). Let \( p_H \) be the original high price of good x and \( p_L \) be the new lower price of good x. Let \( m_0 \) be the actual income level and let \( m^c \) be the Slutsky-compensated level of income for the price change, so that \( m^c = m + x(p_H, p_y, m)(p_L - p_H) \). In order to meet the definition of inferior and ordinary (not Giffen) we must have the following conditions met.

\[
\begin{align*}
\bullet & \ SE > 0 \iff x(p_L, p_y, m^c) - x(p_H, p_y, m_0) > 0 \\
\bullet & \ IE < 0 \iff x(x_L, p_y, m_0) - x(p_L, p_y, m^c) < 0 \\
\bullet & \ SE + IE > 0 \iff x(p_L, p_y, m_0) - x(x_H, p_y, m_0) > 0
\end{align*}
\]

The black budget line with shallowest slope (lowest MRT) represents the original budget when the price of good x is low. The red budget line represent the hypothetical Slutsky-compensated budget line reflecting the new price of good x but allow sufficient income so that he can still afford his old consumption bundle. The blue line represents the budget under the new higher price for good x. \( x(p_H) \) in the picture is short-hand for \( x(p_H, p_y, m) \). Similarly, \( x(p_L) = x(p_L, p_y, m) \) and \( x^c = x(p_H, p_y, m^c) \). The picture shows how the larger-in-magnitude substitution effect keeps the inferior good ordinary.

(3) Labor Supply. Charles gets utility \( u(C, L) \) from consumption, \( C \), measured in dollars and leisure, \( L \), measured in hours. There are 168 hours in a week, but assume that Charles has at most 100 hours per week for leisure since the remainder are required for vital activities such as sleeping, eating and doing chores for his wife. Let \( w \) be the wage that Charles receives at the loading dock where he can put in all the hours he wants (up to 100).

(a) If we always assume that the “price” of consumption, \( p_c \), is $1, what is the price of leisure \( p_L \)? [Hint: Consider the MRT; How many dollars of consumption must he give up to get another hour of leisure?]

**ANSWER.** Each of leisure consumed is one less hour of work. Every hour not worked reduces Charles’ income by \( w \). Hence the MRT is \( w \).

(b) Draw a picture of his budget set in \( C \times L \) space. **ANSWER** see below.
(c) Suppose that when his wage is $10 per hour, Charles works 40 hours per week. Plot this point on his budget line. [Hint: (Hours Working) + (Leisure) = 100]

The red budget line is the Slutsky-compensated budget line for the transition from $10 per hour (black budget line) to $15 per hour (blue line). The inc. and subst. effects on the demand for leisure are highlighted. The total effect on the demand for leisure is the change from 60 hours to 50 hours (or put another the change from 40 hours of work to 50 hours of work). Note that the income effect is positive, albeit small. (the little blue arrow) This is in keeping with our assumption that leisure is a normal good. The green line represents the budget when Charles’ wage is $20 per hour. The income and substitution effects are not highlighted for this second wage change (see next figure). The pink line shows a possible wage-expansion path.

(d) Assume that leisure is a normal good. Sketch three points along Charles’ wage expansion path which show him decreasing his consumption of leisure to 50 when his wage goes to $15 and then increasing it when his wage changes to $20. Label income and substitution effects. (It may be easier to draw two new pictures to show the decompositions - one for the wage change from 10 to 15, and the other to show the wage change from 15 to 20.) What is going on here? Explain why it is possible for Charles to increase his consumption of leisure when its price increases even though it is a normal good?

ANSWER
In this second picture we see the transition from a wage of $15 to $20 decomposed into subst. and inc. effects. The red line again represents the Slutsky-compensated budget for this transition. Here I have drawn the income effect as larger than the substitution effect so that Charles works less, now putting in just 30 hours per wee instead of 50, and consuming more leisure time, 70 instead of 50. To answer the question about why it is possible to consume more leisure when its price increases, even though it is not an inferior good - must less a Giffen good, we have to consider around which point the budget line rotated. Typically when the price of a good increase the budget line rotates around the intercept measuring the other good. But that is not what happened here. When wages increase - effectively changing the price of leisure - the rotation is around the leisure intercept. More importantly this so-called price increase is attended by a positive wealth change (more consumption bundles are available, not fewer). Compared to our traditional framework (think beer vs pizza) this increase in the price of leisure looks exactly like a decrease in the price of consumption. Which is to say Charles does not have to spend as many hours working to get a dollar’s worth of consumption. When viewed in that light, the result is perfectly reasonable.

(4) Suppose that if the interest rate is .1, Jack is a borrower. Explain, in terms of income and substitution effects, what happens to his demand for present-day consumption as the interest rate increases. **ANSWER** In Figure 1, we see one possible way to draw this that is consistent with our standard assumptions (in particular we assume both goods normal). Note that the income and substitution effects along the present period axis both point in the negative direction; the effect on present day consumption of the interest rate increase is unambiguously negative. However, the subst and inc effect along the next period consumption axis point in opposite directions, so depending on preference the total effect could be positive (as it is in my diagram) or it could be negative depending on the original position of the endowment and jack’s preferences. In general the bigger the borrower he would have been under the lower interest rate, the larger will be the relative magnitude of the income effect (compared to the subst effect) and the greater the chance that demand for future consumption would also decrease (unlike what is happening in my digram). Finally, note that Jack remains a borrower at the higher interest rate in my diagram. This too
cannot be generalized. For a large enough rate change or a small enough original loan amount, someone who would have been a borrower under the lower rate would switch to be a saver (or lender).

(5) Recall the utility function from HW #3
\[ u(x_1, x_2) = x_2(5x_1)^2 \]

Suppose that price of good 2 is 1 per unit and income is 50. Decompose the income and substitution effects on both goods for an increase in the price of good 1 from 4 per unit to 6 per unit. Be sure to draw a nice diagram to illustrate the decomposition and (as always) to explain how you arrived at your answer and interpret it. **Answer** From the last homework we have the following demands

\[ x_1(p_1, p_2, m) = \begin{cases} 
0 & \text{if } \frac{p_1}{p_2} > 10 \\
5 - \frac{p_1}{2p_2} & \text{if } m > p_1(5 - \frac{p_1}{2p_2}) \\
\frac{m}{p_1} & \text{otherwise} 
\end{cases} \]

\[ x_2(p_1, p_2, m) = \begin{cases} 
m & \text{if } \frac{p_1}{p_2} > 10 \\
\frac{m-p_1(5-\frac{p_1}{2p_2})}{p_2} & \text{if } m > p_1(5 - \frac{p_1}{2p_2}) \\
0 & \text{otherwise} 
\end{cases} \]

Note that at the given budget parameters, we don’t have to worry about corner solutions. 4 ≤ \( \frac{p_1}{p_2} \) ≤ 6, so \( \frac{p_1}{p_2} \) is always less than 10 (which rules out one kind of corner solution). Also at \( p_1 = 4 \), we \( p_1(5 - \frac{p_1}{2p_2}) \) equal to $12 which is comfortably less than $50. And at \( p_1 = 6 \), we have \( p_1(5 - \frac{p_1}{2p_2}) \) also equal to $12, so again, comfortably less than $50 which rules out the other kind of corner solution. Therefore plugging in the budget parameters into our demand equations we determine that original and new demands are

**Original** : \( (x_1, x_2) = (3, 38) \)

**New** : \( (x_1, x_2) = (2, 38) \)

Next we need to calculate the compensated demands. The compensated income is the amount needed to buy the original bundle \( (3, 38) \) at the new prices \( (p_1, p_2) = (6, 1) \) which is \( 6 \times 3 + 1 \times 38 \) or $56. Plugging this and the new price into our demand equations we get

**Compensated** : \( (x_1, x_2) = (2, 44) \)

Hence the substitution and income effects, as illustrated in Figure 2 are

\[ (SE_1, SE_2) = (-1, 6) \]
\[ (IE_1, IE_2) = (0, -6) \]
\[ (TE_1, TE_2) = (-1, 0) \]

(6) **Cobb-Douglas (Optional).** There is an elegant relationship between the income and substitution effects in CD demands which depends on the weight parameter \( \alpha \) which we will explore here. Let \( u(x_1, x_2) = \alpha \log x_1 + (1-\alpha) \log x_2 \). We will consider a price decrease for good 1 from \( p_1 = p_H \) to \( p_1 = p_L \) where \( p_L < p_H \).

(a) Write down the demand for good 1 when the prices are \( p_H \) and \( p_L \). That is write down \( x_1(p_H, m) \) and \( x_1(p_L, m) \).
Figure 2. Decomposition of change in demands due to increase the price of good 1 from $4 to $6 per unit.
ANSWER. We have seen in previous homeworks how to derive CD demand functions. The demands for good 1 at each price are

\[ x_1(p_H, m) = \frac{\alpha m}{p_H} \]
\[ x_1(p_L, m) = \frac{\alpha m}{p_L} \]

(b) What is the total effect of the price change from \( p_H \) to \( p_L \) on the demand for good 1?

ANSWER. The total effect is the simply the difference between the new demand and the old demand.

\[
\text{TotalEffect} = x_1(p_L, m) - x_1(p_H, m) \\
= \frac{\alpha m}{p_L} - \frac{\alpha m}{p_H} \\
= \frac{\alpha m(p_H - p_L)}{p_L p_H}
\]

The problem define \( \theta = \frac{m(p_H - p_L)}{p_L p_H} \). Hence we can write the total effect as simply \( \alpha \theta \).

(c) If income is \( m \), what is \( m^c \) -- the compensating level of income at which the consumer would be able to just afford the bundle she demanded when the price was \( p_H \) after the price drops to the new low price, \( p_L \)?

ANSWER

\[
m^c = m + (p_L - p_H)x_1(p_H, m)
\]

(d) \( x_1(p_L, m^c) \) is the consumer’s demand for good under at the new low price \( p_L \) when he has only the compensating level of income \( m^c \) to spend. Show that

\[
x_1(p_L, m^c) = \frac{\alpha m^c}{p_L}
\]

ANSWER

Substituting our expression for \( m^c \) above we get

\[
x_1(p_L, m^c) = \frac{\alpha \{m + (p_L - p_H)x_1(p_H, m)\}}{p_L}
\]

Substituting \( \frac{\alpha m}{p_H} \) for \( x_1(p_H, m) \) yields

\[
x_1(p_L, m^c) = \frac{\alpha \{m + (p_L - p_H)\left[\frac{\alpha m}{p_H}\right]\}}{p_L}
\]

\[
= \frac{\alpha m[p_H + \alpha(p_L - p_H)]}{p_H p_L}
\]

which is what we were to show. In terms of \( \theta \) we can clean this up a bit to
where $\theta$ is defined as it was in the problem and is equal to $\frac{m(p_H - p_L)}{p_H p_L}$. But this just says that the compensated demand is equal to the new demand minus something equal to $\alpha^2 \theta$. By definition, then, that something must be the income effect. Note that the income effect for a price decrease is always positive and increases with the square of $\alpha$. Since we already learned that the total effect was $\alpha \theta$, it must be that the substitution effect is equal to $\alpha \theta - \alpha^2 \theta$ which answers the next two parts.

(e) Show that the substitution effect is

$$SE : \ x_1(p_L, m^c) - x_1(p_H, m) = \alpha(1 - \alpha)\theta$$

where $\theta = \frac{m(p_H - p_L)}{p_H p_L}$. Interpret this.

**ANSWER.** see above for derivation. **INTERPRETATION:** Note that $\theta > 0$ and that $0 < \alpha < 1$ which verifies for us that the SE must be a positive number. Since we are talking about a price decrease here, we would be concerned if the sign went the other way. The substitution effect depends on $\alpha$ in a peculiar way. It is small for both small values of $\alpha$ and large (approaching 1) values of $\alpha$. It reaches a maximum at $\alpha = \frac{1}{2}$. This can be seen by simply differentiating $\alpha(1 - \alpha)\theta$ w.r.t. $\alpha$...

$$\frac{\partial SE}{\alpha} = \theta - 2\alpha\theta$$

setting this equal to zero we would get $\alpha = \frac{1}{2}$ as claimed. Note that the second derivative is negative so we know that we are reaching a maximum and not a minimum.

(f) Show that the income effect is

$$IE : \ x_1(p_L, m) - x_1(p_L, m^c) = \alpha^2 \theta$$

where again $\theta = \frac{m(p_H - p_L)}{p_H p_L}$.

**ANSWER.** see above.

(g) Draw a graph with $\alpha$ on the horizontal axis (going from 0 to 1) and change in demand for good 1 on the vertical axis. Plot the total effect, from part (b), and the income effect together in the same graph. Do not plot the substitution effect. Label those two plotted lines. Also label the substitution effect (even though you didn’t draw a curve for it). Describe what happens to the income effect as $\alpha$ approaches 1.
Income Effect and Total Effect as a function of $\alpha$. Note the substitution effect can be seen as the difference between the two graphs as indicated and that it reaches its maximum at $\alpha = \frac{1}{2}$. What is the slope of the income effect graph at the maximum of the substitution effect? Note that as $\alpha$ approaches 1 the income effect accounts for the entire change in demand. This is because the consumer would be spending a constant share of income of nearly 100% on good 1 and so the saving from any price decrease in good 1 will go right back into good 1 spending. On the other hand, the income effect accounts for almost none of the total effect (in percentage terms) as $\alpha$ approaches zero, since the saving from a price decrease in good 1 would be mostly spent on good 2.