AK 4
SLUTSKY COMPENSATION

ECON 210
A. JOSEPH GUSE

(1) (a) First calculate the demand at the original price $p_b = 2$

$$\begin{aligned} b(p_b, m) &= \frac{1000}{20} - 5p_b \\ &\Rightarrow b_0 = b(2) = 40 \end{aligned}$$

In general $m^c = m + (p_b^1 - p_b^0)b_0$. If the price increases \$1 per unit m^c must include \$40 over Allen's old income. so $m^c = m + 40 = 1040$.

(b) We get b^c, the compensated demand for beer by setting price equal to the new price, $p_b = 3$, and income equal to the compensated level of income, $m^c = 1040$.

$$\begin{aligned} b(3, 1040) &= \frac{1040}{20} - 5 \times 3 \\ &= 37 \end{aligned}$$

(c) Find the substitution effect, income effect and total effect of this price change.

- The **substitution effect** is the difference between compensate demand and the quantity demanded before the price change.

$$\begin{aligned} \texttt{S.E.} &= b(3, m^c) - b(2, m) \\ &= 37 - 40 \\ &= -3. \end{aligned}$$

In words the substitution effect of the price change from \$2 to \$3 is a *decrease* of 3 pints of beer.

- The **income effect** is the difference between the new demand at the higher price and the compensated demand.

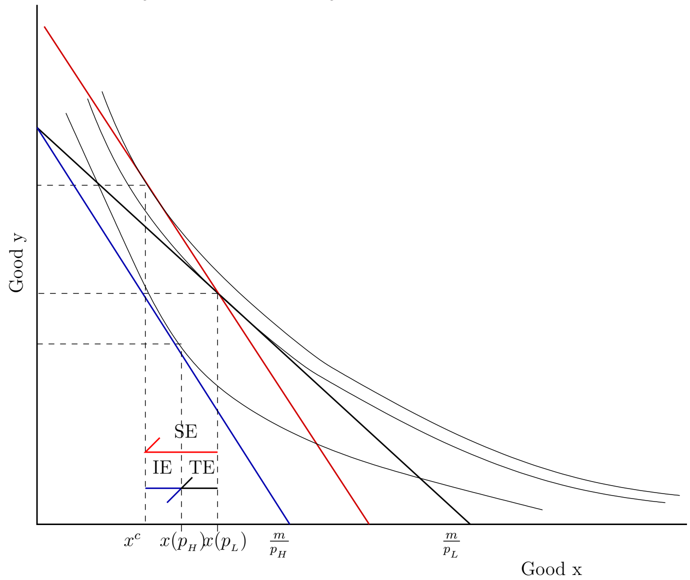
$$\begin{aligned} \texttt{I.E.} &= b(3, m) - b(3, m^c) \\ &= \frac{1000}{20} - 5 \times 3 - \left(\frac{1040}{20} - 5 \times 3\right) \\ &= \frac{1000}{20} - \frac{1040}{20} \\ &= -2. \end{aligned}$$

In words the income effect of the price change from \$2 to \$3 is a *decrease* of 2 pints of beer.

- The **total effect** is the sum of the substitution and income effects and is equal to -5.

(2) An inferior but not Giffen good. Consider two goods x and y and a decrease in the price of x. Let $x(p_x, p_y, m)$ stand for the quantity demanded of good x when prices are p_x and p_y and income level is m. Let p_{H} be the original high price of good x and p_{L} be the new lower price of good x. Let m_0 be the actual income level and let m^c be the Slutsky-compensated level of income for the price change, so that $m^c = m + x(p_{H}, p_y, m)(p_{L} - p_{H})$. In order to meet the definition of inferior and ordinary (not Giffen) we must have the following conditions met.

- $SE > 0 \iff x(p_{L}, p_y, m^c) - x(x_{H}, p_y, m_0) > 0$
- $IE < 0 \iff x(x_{L}, p_y, m_0) - x(p_{L}, p_y, m^c) < 0$
- $SE + IE > 0 \iff x(p_{L}, p_y, m_0) - x(x_{H}, p_y, m_0) > 0$



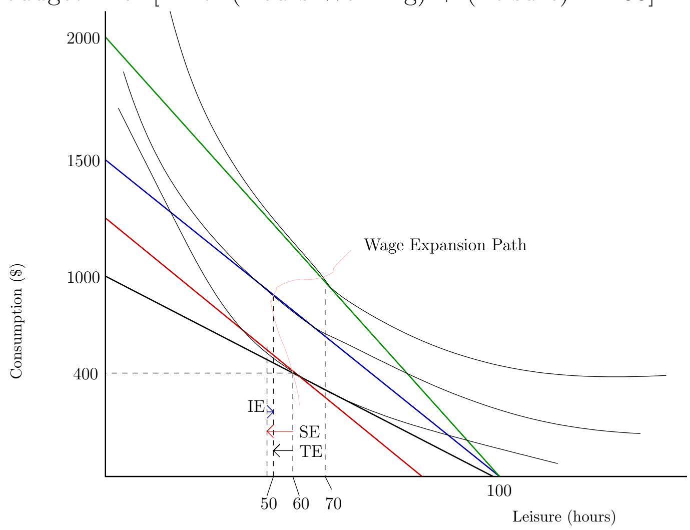
The black budget line with shallowest slope (lowest MRT) represents the original budget when the price of good x is low. The red budget line represent the hypothetical Slutsky-compensated budget line reflecting the new price of good x but allow sufficient income so that he can still afford his old consumption bundle. The blue line represents the budget under the new higher price for good x. $x(p_{H})$ in the picture is short-hand for $x(p_{H}, p_y, m)$. Similarly, $x(p_{L}) = x(p_{L}, p_y, m)$ and $x^c = x(p_{H}, p_y, m^c)$. The picture shows how the larger-in-magnitude substitution effect keeps the inferior good x *ordinary.*

(3) **Labor Supply**. Charles gets utility $u(C, L)$ from consumption, C, measured in dollars and leisure, L, measured in hours. There are 168 hours in a week, but assume that Charles has at most 100 hours per week for leisure since the remainder are required for vital activities such as sleeping, eating and doing chores for his wife. Let w be the wage that Charles receives at the loading dock where he can put in all the hours he wants (up to 100).

  (a) If we always assume that the "price" of consumption, p_c is \$1, what is the price of leisure p_{L}? [Hint: Consider the MRT; How many dollars of consumption must he give up to get another hour of leisure?]
  ANSWER. Each of leisure consumed is one less hour of work. Every hour not worked reduces Charles' income by w. Hence the MRT is w.

  (b) Draw a picture of his budget set in $C \times L$ space. **ANSWER** see below.

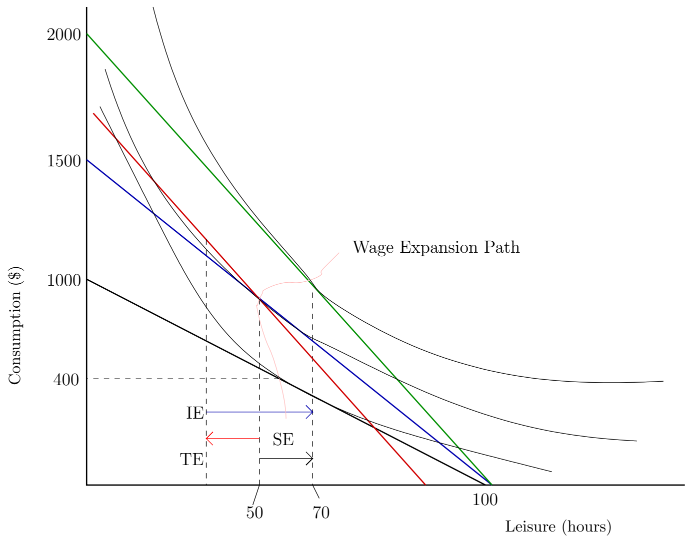
(c) Suppose that when his wage is \$10 per hour, Charles works 40 hours per week. Plot this point on his budget line. [Hint: (Hours Working) + (Leisure) = 100]



The red budget line is the Slutsky-compensated budget line for the transition from \$10 per hour (black budget line) to \$15 per hour (blue line). The inc. and subst. effects on the demand for leisure are highlighted. The total effect on the demand for *leisure* is the change from 60 hours to 50 hours (or put another the change from 40 hours of *work* to 50 hours of *work*). Note that the income effect is positive, albeit small. (the little blue arrow) This is in keeping with our assumption that leisure is a normal good. The green line represents the budget when Charles' wage is \$20 per hour. The income and substitution effects are not highlighted for this second wage change (see next figure). The pink line shows a possible wage-expansion path.

(d) Assume that leisure is a normal good. Sketch three points along Charles' *wage expansion path* which show him *decreasing* his consumption of leisure to 50 when his wage goes to \$15 and then increasing it when his wage changes to \$20. Label income and substitution effects. (It may be easier to draw two new pictures to show the decompositions - one for the wage change from 10 to 15, and the other to show the wage change from 15 to 20.) What is going on here? Explain why it is possible for Charles to *increase* his consumption of leisure when its price increases even though it is a normal good?

ANSWER



In this second picture we see the transition from a wage of $15 to $20 decomposed into subst. and inc. effects. The red line again represents the Slutsky-compensated budget for this transition. Here I have drawn the income effect as larger than the substitution effect so that Charles works less, now putting in just 30 hours per wee instead of 50, and consuming more leisure time, 70 instead of 50. To answer the question about why it is possible to consume more leisure when its price increases, even though it is *not* an inferior good - must less a Giffen good, we have to consider around which point the budget line rotated. Typically when the price of a good increase the budget line rotates around the intercept measuring the *other* good. But that is not what happened here. When wages increase - effectively changing the price of leisure - the rotation is around the *leisure* intercept. More importantly this so-called price increase is attended by a positive wealth change (more consumption bundles are available, not fewer). Compared to our traditional framework (think beer vs pizza) this increase in the price of leisure looks exactly like a *decrease* in the price of consumption. Which is to say Charles does not have to spend as many hours working to get a dollar's worth of consumption. When viewed in that light, the result is perfectly reasonable.

(4) Recall that the present value of an infinite number of annual payments a starting in one year is $\frac{a}{r}$, and a stream of T annual payments starting in one year is $\frac{a}{r} - \frac{\frac{a}{r}}{(1+r)^T}$.

(a) What is the net present value of a project with costs equal to 25, 25 and 300 in one, two and three years and benefits equal to 100, 100 and 100 in years one, two and three? when the interest rate is .05? when the interest rate is .25? How high does one's opportunity cost of capital have to be in order for this project to look like a good investment?

(b) Consider a project with costs equal to 600, 100 and 100 in one, two and three years and benefits equal to 300, 300 and 300 in one, two and three years. Describe the net present value as a function of the opporutnity cost of capital, r. How would you describe the difference between this project and the previous one?

(c) If you have a net worth of 0, but can borrow at r = .1, what is most you would be willing to pay for a piece of farm land that generates $200 in crops (after planting and harvesting costs) and is assessed $40 per year in property taxes?

(d) If you can only afford annual mortgage payments of \$8000, what would the interest rate have to be in order for you to get a 30 year mortgage of \$200,000?

(a) What is the net present value of a project with costs equal to 25, 25 and 300 in one, two and three years and benefits equal to 100, 100 and 100 in years one, two and three? when the interest rate is .05? when the interest rate is .25? How high does one's opportunity cost of capital have to be in order for this project to look like a good investment? **ANSWER.** We want to find the interest rate that makes the present value of this contract just equal to zero - break even. To do this we must calculate the present value as the sum of the discounted net benefits from each year. The table represents how you might set it up.

Year	1	2	3
Benefit	100	100	100
Cost	25	25	300
Net Benefit	75	75	- 200
Discounted Net Benefit	$\frac{75}{(1+r)}$	$\frac{75}{(1+r)^2}$	$\frac{-200}{(1+r)^3}$

Hence we are looking for the r that solves

$$0 = \frac{75}{(1+r)} + \frac{75}{(1+r)^2} - \frac{200}{(1+r)^3}$$

Probably the easiest way to answer this question is to use a spreadsheet program such as MSExcel. Type the following formula into a cell:

```
=75/(1+rate) + 75/(1+rate)^2 - 200/(1+rate)^3
```

Define a different cell to be 'rate'. Notice that this cell is referred in the formular where we want Excel to use the interest rate [1] Now just experiment with different rates until the value in the present value cell is close to zero. I found the break-even interest rate in this way to be about .208.

Note that one could solve the equation analytically using the cubic formular (which I won't write out here). However, that approach quickly become inpractical as the number of year in the project expands beyond 3.

(b) Consider a project with costs equal to 600, 100 and 100 in one, two and three years and benefits equal to 300, 300 and 300 in one, two and three years. Describe the net present value as a function of the opporutnity cost of capital, r. How would you describe the difference between this project and the previous one?

ANSWER. The main thing to notice here is that in the previous example, when the interest rate was below the break even rate, the PV was negative. However, that will *not* be the case for this example. What's going on here?

In the previous example, the benefits arrived before the costs making that project look more attractive at higher interest rates. Examples of the type from the previous question are things such as landfills and mines where a stream of benefits from operating the site preceeds a stream of clean-up and maintenance costs which may persist long after the mining or landingfilling operation has ceased. Such projects are worth investing in when the oppotunity cost of capital is relatively high.

[1] 'r' is a special character in MSExcel so it cannot be used as a variable name.

Examples of the other type might be damns or power plants where there is a huge up-front construction cost followed by years of benefits. Such projects are worth investing in when the oppotunity cost of capital is relatively low.

(c) If you have a net worth of 0, but can borrow at r = .1, what is most you would be willing to pay for a piece of farm land that generates \$200 in crops (after planting and harvesting costs) and is assessed \$40 per year in property taxes? **ANSWER.** Using the PV formula for an infinite stream of benefits we get

$$PV = \frac{200 - 40}{.1} = 1600$$

In other words, you should be indifferent between putting \$1600 in a saving account that pays 10% interest earning you \$160 annually or buying a piece of land, which after taxes earns you \$160 annually.

(d) If you can only afford annual mortgage payments of \$8000, what would the interest rate have to be in order for you to get a 30 year mortgage of \$200,000?
Using the finite stream of idential payment formula, We are looking for the r that solves this equation:

$$200000 = \frac{8000}{r} - \frac{\frac{8000}{r}}{(1+r)^{30}}$$

Again we can use MSExcel to solve this numerically. I found that the interest rate would have to be a very low .0125 in order for this person to afford a 200K mortgage. To get an idea of the effect interest rates has on the size mortgage one can afford consider this table which show the mortgage that 3 different borrowers could afford at three different interest rates. The 3 borrowers can afford to make annual payments of 8000, 12000 and 17000.

Mortgage sizes for 3 interest rates and three annual payments

r / payment :	8000	12000	17000
.0125	199,000	299,000	423,000
.04	138,000	207,000	294,000
.08	90,000	135,000	191,000

It is easy to see why interest rates have such a powerful effect on the housing market. (Try drawing a budget for housing and other consumption. Show what happens to the budget line when rates increase or decrease and think about how optimal consumption bundle choices might change.)

(5) **Cobb-Douglas (Optional)**. There is an elegant relationship between the income and substitution effects in CD demands which depends on the weight parameter α which we will explore here. Let $u(x_1, x_2) = \alpha \log x_1 + (1-\alpha) \log x_2$. We will consider a price decrease for good 1 from $p_1 = p_H$ to $p_1 = p_L$ where $p_L < p_H$.

(a) Write down the demand for good 1 when the prices are p_H and p_L. That is write down $x_1(p_H, m)$ and $x_1(p_L, m)$.
ANSWER. We have seen in previous homeworks how to derive CD demand functions. The demands for good 1 at each price are

$$x_1(p_H, m) = \frac{\alpha m}{p_H}$$
$$x_1(p_L, m) = \frac{\alpha m}{p_L}$$

(b) What is the *total effect* of the price change from p_H to p_L on the demand for good 1?
ANSWER The total effect is the simply the difference between the new demand and the old demand.

$$\begin{aligned} \texttt{TotalEffect} &= x_1(p_L, m) - x_1(p_H, m) \\ &= \frac{\alpha m}{p_L} - \frac{\alpha m}{p_H} \\ &= \frac{\alpha m(p_H - p_L)}{p_L p_H} \end{aligned}$$

The problem define $\theta = \frac{m(p_H - p_L)}{p_L p_H}$. Hence we can write the total effect as simply $\alpha\theta$.

(c) If income is m, what is m^c – the compensating level of income at which the consumer would be able to just afford the bundle she demanded when the price was p_H *after* the price drops to the new low price, p_L?
ANSWER

$$m^c = m + (p_L - p_H)x_1(p_H, m)$$

(d) $x_1(p_L, m^c)$ is the consumer's demand for good under at the new low price p_L when he has only the compensating level of income m^c to spend. Show that

$$x_1(p_L, m^c) = \frac{\alpha m\,[p_H + \alpha(p_L - p_H)]}{p_H p_L}$$

ANSWER

$$x_1(p_L, m^c) = \frac{\alpha m^c}{p_L}$$

(1)

Substituting our expression for m^c above we get

$$x_1(p_L, m^c) = \frac{\alpha\,\{m + (p_L - p_H)x_1(p_H, m)\}}{p_L}$$

Substituting $\frac{\alpha m}{p_H}$ for $x_1(p_H, m)$ yields

$$\begin{aligned} x_1(p_L, m^c) &= \frac{\alpha\left\{m + (p_L - p_H)\left[\frac{\alpha m}{p_H}\right]\right\}}{p_L} \\ &= \frac{\alpha m\,[p_H + \alpha(p_L - p_H)]}{p_H p_L} \end{aligned}$$

which is what we were to show. In terms of θ we can clean this up a bit to

$$x_1(p_L, m^c) = \frac{\alpha m}{p_L} - \alpha^2\theta$$

where θ is define as it was in the problem and is equal to $\frac{m(p_H - p_L)}{p_H p_L}$. But this just says that the compensated demand is equal to the new demand minus something equal to $\alpha^2\theta$. By definition, then, that something must be the income effect. Note that the income effect for a price decease is always positive and increases with the square of α. Since we already learned that the total effect was $\alpha\theta$, it must be that the substitution effect is equal to $\alpha\theta - \alpha^2\theta$ which answers the next two parts.

(e) Show that the substitution effect is

$$SE: \quad x_1(p_{_L}, m^c) - x_1(p_{_H}, m) = \alpha(1-\alpha)\theta$$

where $\theta = \frac{m(p_H - p_L)}{p_H p_L}$. Interpret this.

ANSWER. see above for derivation. *INTERPRETATION:* Note that $\theta > 0$ and that $0 < \alpha < 1$ which verifies for us that the SE must be a positive number. Since we are talking about a price decrease here, we would be concerned if the sign went the other way. The substitution effect depends on α in a peculiar way. It is small for both small values of α and large (approaching 1) values of α. It reaches a maximum at $\alpha = \frac{1}{2}$. This can be seen by simply differentiating $\alpha(1-\alpha)\theta$ w.r.t. α...

$$\frac{\partial SE}{\alpha} = \theta - 2\alpha\theta \tag{2}$$

setting this equal to zero we would get $\alpha = \frac{1}{2}$ as claimed. Note that the second derivative is negative so we know that we are reaching a maximum and not a minimum.

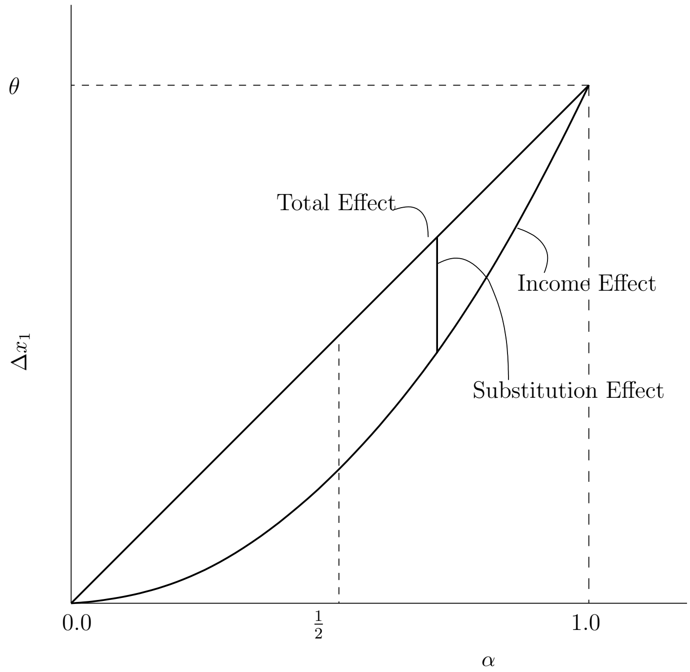
(f) Show that the income effect is

$$IE: \quad x_1(p_{_L}, m) - x_1(p_{_L}, m^c) = \alpha^2\theta$$

where again $\theta = \frac{m(p_H - p_L)}{p_H p_L}$.

ANSWER. see above.

(g) Draw a graph with α on the horizontal axis (going from 0 to 1) and change in demand for good 1 on the vertical axis. Plot the *total effect*, from part (b), and the *income effect* together in the same graph. Do *not* plot the substitution effect. Label those two plotted lines. Also label the substitution effect (even though you didn't draw a curve for it). Describe what happens to the income effect as α approaches 1.



Income Effect and Total Effect as a function of α. Note the substitution effect can be seen as the difference between the two graphs as indicated and that it reaches its maximum at $\alpha = \frac{1}{2}$. What is the slope of the income effect graph at the maximum of the substitution effect?. Note that as α approaches 1 the income effect account for the entire change in demand. This is because the consumer would be spending a constant share of income of nearly 100% on good 1 and so the saving from any price decrease in good 1 will go right back into good 1 spending. On the other hand, the income effect accounts for almost none of the total effect (in percentage terms) as α approaches zero, since the saving from a price decrease in good 1 would be mostly spent on good 2.