

Homework 0.
Preliminary Math Exam
Answer Key

These questions and topics represent a sample of the mathematical skills expected for this course. However, this exam will *not* factor into your final grade and there is no minimum score required to remain in this class.

1 Algebra

Let $f(x) = 5 - 2x$ and $g(x) = 1 + \frac{x}{4}$

1. At what value of x does $f(x)$ equal 0?

Answer Setting $f(x) = 0$ we have

$$\begin{aligned}5 - 2x &= 0 \\ \Rightarrow -2x &= -5 \\ \Rightarrow x &= \frac{5}{2}\end{aligned}$$

2. At what value of x does $g(x)$ equal 11?

Answer. Setting $g(x) = 11$ we have

$$\begin{aligned}1 + \frac{x}{4} &= 11 \\ \Rightarrow 4 + x &= 44 \\ \Rightarrow x &= 40\end{aligned}$$

3. Is it possible to express x as a function of g ? Why or why not? If so, do it.

Answer. Yes it is. g is a *one-to-one* mapping, also known as a *bijective*. The requirement of any mapping or function is that every element in the domain map to a unique element in the range. For a bijective mapping, it must also be the case that for every element in the range, there is exactly one element in the domain which maps to it. (This is not necessarily the case for an arbitrary mapping or function.)

A mapping is invertible *if and only if* it is *one-to-one*. Expressing x as function of g , we have

$$x(g) = 4g - 1 \tag{1}$$

Note that all linear equations (except for constants) are bijective. Therefore they are all invertible (except for constants).

Why it matters for Microeconomics We will often work with the inverse of supply curves and the inverse of demand curves. These curves may be thought of as mapping prices to quantities (horizontal interpretation) or as mapping quantities to prices (vertical interpretation). Both interpretations of supply and demand will be useful depending on the context. For individual firms, we will typically refer to the horizontal interpretation of the relationship between quantity supplied and price as *the supply curve*, while we will refer to the vertical interpretation of the exact same relationship as *the marginal cost curve*. For individual consumers, we will typically refer to the horizontal interpretation of the relationship between quantity demanded and price as *the demand curve*, while we will refer to the vertical interpretation of the exact same relationship as *the marginal willingness to pay curve*.

4. Characterize the intersection of $f(x)$ and $g(x)$.

Answer. Setting $f(x) = g(x)$ we get

$$\begin{aligned} 5 - 2x &= 1 + \frac{x}{4} \\ \Rightarrow 20 - 8x &= 4 + x \\ \Rightarrow -9x &= -16 \\ \Rightarrow x &= \frac{16}{9} \end{aligned}$$

The value of both function at $x = \frac{16}{9}$ is $\frac{13}{9}$. $(\frac{16}{9}, \frac{13}{9})$ is the only element in the intersection of the two graphs.

2 Differentiation

1. Let $f(x)$ be a differentiable function. Express the derivative of this function as a limit.

Answer. The derivative of a function is defined as

$$f'(x) = \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon) - f(x)}{\varepsilon}$$

The assumption that $f(x)$ is differentiable is exactly equivalent to assuming the limit exists. In general, one must check that a function is differentiable. Intuitively the

derivative of a function is its slope at the point being evaluated; it is the rate of change in the value of the function per unit change in x . Note that in the definition above, without the limit operator, the expression $\frac{f(x+\varepsilon)-f(x)}{\varepsilon}$ is simply the average slope of the function between x and $x + \varepsilon$. As ε becomes smaller and smaller, it eventually becomes a better and better estimate of the slope at x .

2. Find the derivatives of the following functions. (i.e. new function of x).

(a) $f(x) = 5 - 2x$. Answer. $f'(x) = -2$

(b) $f(x) = x^2 - 3x$ Answer. $f'(x) = 2x - 3$

(c) $f(x) = e^x$ Answer. $f'(x) = e^x$

(d) $f(x) = \log x$ Answer. $f'(x) = \frac{1}{x}$

Useful Rules of Differentiation. We reviewed several rules of differentiation we are almost certainly going to encounter this semester.

(a) Product Rule. $\frac{\partial(f(x)g(x))}{\partial x} = f(x)\frac{\partial g(x)}{\partial x} + \frac{\partial f(x)}{\partial x}g(x)$

(b) Exponent Rule. $\frac{\partial x^n}{\partial x} = nx^{n-1}$

(c) Sum Rule. $\frac{\partial(f(x)+g(x))}{\partial x} = \frac{\partial f(x)}{\partial x} + \frac{\partial g(x)}{\partial x}$

(d) Chain Rule. $\frac{\partial f(g(x))}{\partial x} = \frac{\partial f(g)}{\partial g} \frac{\partial g(x)}{\partial x}$

3. $f(x)$ is twice differentiable on (a, c) . Let $a < b < c$. Suppose that $f(b) > f(x)$ for all $x \in (a, c) \setminus b$. What can you say about $f'(x)$? What can you say about $f''(x)$?

Answer. b is a maximum point. This means that $f'(b) = 0$ and $f''(b) < 0$. *Rough Proof.* To see this, consider that in order for f to reach a local maximum at b the slope of the function up to b must be positive and slope just after b must be negative. Hence the slope must be zero at b . Furthermore the slope must be decreasing (going from a positive number to a negative number) in the neighborhood of b , which is the same thing as saying the second derivative is negative.

3 Logs and Exponents

Note $\log x$ indicates the *natural log* unless otherwise specified. e indicates the the number e (i.e. 2.71...), the base of the natural log.

1. Solve for x .

(a) $y = e^x$ Answer. $x = \log y$

(b) $y = a^x$ Answer. $x = \frac{\log y}{\log a}$

(c) $y = ba^x$ Answer $x = \frac{\log y - \log b}{\log a}$

- (d) $y = bx^a$ Answer. $x = \left(\frac{y}{b}\right)^{\frac{1}{a}}$
 (e) $y = \log(kx)$ Answer. $x = e^{(y-\log k)} = \frac{e^y}{k}$

Useful Rules. We reviewed some of the following.

- (a) The number e can be written as the following limit

$$e = \lim_{a \rightarrow 0} (1 + a)^{\frac{1}{a}}$$

This way of thinking of e will be useful we discuss present value calculations later in the semester.

- (b) e^x is special in the sense that it is a function whose derivative is itself: $\frac{\partial e^x}{\partial x} = e^x$.
 (c) $\log : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $e : \mathbb{R} \rightarrow \mathbb{R}_+$ are inverses of each other; In other words

$$\begin{aligned} \log e^x &= x \quad \forall x \in \mathbb{R} \\ e^{\log x} &= x \quad \forall x \in \mathbb{R}_+ \end{aligned}$$

- (d) The log of a product is equal to the sum of the logs: $\log(ab) = \log a + \log b$
 (e) $\log(a^b) = b \log a$
 (f) $\frac{\partial \log x}{\partial x} = \frac{1}{x}$

Why it Matters for Microeconomics. Two big reasons. First, as we begin to introduce utility functions, we will learn that *monotonic transformations* preserve the ranking of consumption bundles. That is if $u(x_1, x_2)$ is a consumer's utility function and $u(x_1, x_2) > u(x'_1, x'_2)$ then $f(u(x_1, x_2)) > f(u(x'_1, x'_2))$, if f is a strictly increasing monotonic function. \log happens to be one such very useful monotonic transformation. Second, we when study how to calculate present value, the exponential function will pop up a lot, since, for example, the present value of promise to pay v in one year is exactly ve^{-r} where r is a continuously compounding annual interest rate.

4 Probability

- Suppose that the W&L lacrosse team plays its first game tomorrow. Let p be the probability that it will rain tomorrow. Let q_r be the probability that W&L will win if it rains. Let q_n be the probability that W&L will win if it does not rain. Assume that W&L will either win or lose - no ties or canceled games.
 - If $p = 0$, what is the probability that W&L will win tomorrow?
 - What is the probability that W&L will win *and* it will rain?

- (c) What is the unconditional probability that W&L will win tomorrow? (HINT: It may help to make a 2x2 table of all possible outcomes and their joint probabilities.)
- (d) How much would you expect to earn *on average* from a lottery that paid out y when W&L either loses in the rain or wins in dry weather and pays out nothing otherwise?

Answer. Here is a table of the joint probabilities.

	Win	Lose
Rain	pq_r	$p(1 - q_r)$
No Rain	$(1 - p)q_n$	$(1 - p)(1 - q_n)$

From this table you should be able to answer any of the above questions. For example, the expected payout of the lottery is $y(p(1 - q_r) + (1 - p)q_n)$.

Useful Probability Rules. Let Ω represent the set of all possible outcomes. Let $\omega \in \Omega$ denote an element in this space, called an *outcome* or *sample point*. An *event* is defined as a subset of Ω - it is a set of outcomes, possibly a singleton. If $A \subseteq \Omega$ is some event, I will denote the probability of A as $Pr(A)$. The symbol $Pr(A|B)$ stands for the *conditional probability* of A given B . A *random variable* is a function whose domain is Ω and which maps to the \mathbb{R} . In other words, it is a variable which takes on different values depending on what the outcome is.

- (a) $Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$
- (b) $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$. Note that if A and B are disjoint events, then $A \cap B = \emptyset$ and the probability of either one occurring is simply the sum of their probabilities.
- (c) $Pr(A|B) = \frac{Pr(B|A)Pr(A)}{Pr(B)}$
- (d) The *expected value* of a random variable $x(\omega)$, for finite or countable Ω , is

$$E(x) = \sum_{\omega \in \Omega} x(\omega)Pr(\omega) \quad (2)$$

This is the formula used to calculate the expected value of the lottery above. The expected value is also called the *mean* or the *average* - though sometimes these terms may be used in slightly different ways.