Complexity Estimates for Representations of Schmüdgen Type

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Let $\Phi = \{\phi_1, \ldots, \phi_j\}$ and let $K$ be a closed basic set in $\mathbb{R}^n$ given by the polynomial inequalities $\phi_1 \geq 0, \ldots, \phi_j \geq 0$. Let $\Sigma(\Phi)$ be the semiring generated by the $\phi_i$ and the squares in $\mathbb{R}[x_1, \ldots, x_n]$. For example, if $\Phi = \{\phi_1\}$ then $\Sigma(\Phi) = s_1 + s_2 \phi_1$, where $s_1, s_2$ are sums of squares of polynomials. Schmüdgen has shown that if $K$ is compact then any polynomial strictly positive on $K$ belongs to $\Sigma(\Phi)$. This paper develops a result of Schmüdgen type for functions in one dimension merely non-negative on $K$. For this, it is necessary to add additional hypotheses, such as the proximity of complex zeros, to compensate for the loss of strict positivity necessary for Schmüdgen’s result.

1. INTRODUCTION

From Hilbert, a nullstellensatz is a theorem which characterizes the algebraic conditions for a function to vanish on the common zero set of some collection of other functions. Furthermore, a positivstellensatz is a theorem which describes the conditions when a polynomial is positive or nonnegative on the set of solutions of a system of real equations and inequalities. In 1990, Konrad Schmüdgen [7] gave a result of this kind when characterizing moment sequences of positive Borel measures on subsets of $\mathbb{R}^n$, where $\mathbb{R}$ is the field of real numbers. Schmüdgen discovered an algebraic way to characterize strict positivity for polynomials on compact, basic semialgebraic sets. These compact, basic semialgebraic sets are described by the common nonnegativity set of a finite collection of

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1 This work contains parts of the author’s Ph.D. thesis written by Lehigh University under the supervision of Dr. Gilbert Stengle.
polynomials $\Phi = \{\phi_1, \ldots, \phi_j\}$. To state Schmüdgen’s result, we introduce the notation $\Sigma\{\Phi\}$ in $\mathbb{R}[x_1, \ldots, x_n]$ for the semiring generated by the squares and elements of $\Phi$. Finally, let, $\Phi \geq 0 = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \phi_k(x_1, \ldots, x_n) \geq 0, 1 \leq k \leq t\}$ will mean the system of inequalities obtained by requiring each element of $\Phi$ to be non-negative.

**Theorem 1 (Schmüdgen).** If the set $K = \{\Phi \geq 0\}$ is compact and $f$ is a polynomial function strictly positive on $K$, then $f \in \Sigma\{\Phi\}$.

Using the existence of a greatest lower bound for $f$ on $K$, this is equivalent to:

**Corollary 1 (Schmüdgen Positivstellensatz).** If the set $K = \{\Phi \geq 0\}$ is compact then $f$ is strictly positive on $K$ if and only if $f \in \mathbb{R}^+ + \Sigma\{\Phi\}$.

This is quite remarkable since no powers of $f$ enter into the representation as is usually found in positivstellensatz statements with nonstrict inequalities [8]. Others have made improvements of the above theorem. Acquistapace et al. [1] found a strict analytical positivstellensatz where $K$ needs not be compact, but $f$ remains strictly positive on $K$. We cannot obtain Schmüdgen-type results without strict positivity unless we add additional hypotheses. This paper gives an instance of additional hypotheses which leads to results of Schmüdgen type without strict positivity. In our main result we let $\{\Phi \geq 0\} = [-1, 1]$ where all zeros of $\Phi$, other than $\pm 1$, satisfy $(\text{Im}(z))^2 \geq 2 |\text{Re}(z)| - 1$. Then we suppose $f([-1, 1]) \geq 0$ and obtain that $f \in \Sigma\{\Phi\}$ if and only if the multiplicities of $f$ at $\pm 1$ are even or not smaller than the corresponding multiplicities of $\Phi$. This is a partial result obtained by limiting the form of the representation whereby we obtain nice complexity estimates.

2. Basic Definitions

Let $\Phi = \{\phi_1, \ldots, \phi_j\}$ and $K$ be a closed basic set in $\mathbb{R}^n$ given by the polynomial inequalities $\phi_i \geq 0, \ldots, \phi_j \geq 0$. We let $\Sigma = \{\sigma_1, \ldots, \sigma_r\}$ be sums of squares of polynomials.

**Definition 1.** Let $\Sigma\{\Phi\}$ be the semiring generated by the $\Phi = \{\phi_1, \ldots, \phi_j\}$ which are all greater than or equal to zero on $K$ and $\Sigma = \{\sigma_1, \ldots, \sigma_r\}$ which are sums of squares of polynomials. Thus, if $\Phi$ were composed of one element, $\Sigma\{\Phi\}$ would contain elements of the form $f = \sigma_1 + \sigma_2 \Phi$ where $\sigma_1$ and $\sigma_2$ are sums of squares of polynomials.

**Definition 2.** The representation of a polynomial $f = \sum_{n=0}^p f_n x^n$ is an expression of $f$ in terms of the elements of $\Phi = \{\phi_1, \ldots, \phi_j\}$, the squares, $\sigma_i$, and the semiring operations.
DEFINITION 3. The degree of the representation is the highest degree of any summand, i.e., \( \max \{ \deg \sigma_i \} \).

DEFINITION 4. The norm of the representation is the highest norm of any \( \sigma_i \), i.e., \( \max \| \sigma_i \| \). The norm of a polynomial is the sum of the absolute value of the coefficients when the polynomial is in standard form (i.e., \( f = f_0 + f_1x + f_2x^2 + \cdots + f_nx^n \)). Notice that if \( f \) is a polynomial with nonnegative coefficients, then \( \| f \| = f(1) \) and that if \( \tilde{f}(x) = f(-x) \) then \( \| \tilde{f} \| = \| f \| \). Thus, \( \|(1-x)^m\| = 2^m \) and \( \|(1+x)^n\| = 2^n \), or the norm of any linear term, \( (\alpha_i \pm x)^k \) would be \( (\alpha_i + 1)^k \). As for the norm and degree of representations, if \( \Phi = \{-x^2\} \) and \( f = x+1 \) then \( f = (\frac{1}{2} x + 1)^2 + \frac{1}{2} (-x^2) \) is a representation of norm \( \frac{1}{2} \) and degree 2 since \( \sigma_1 = (\frac{1}{2} x + 1)^2 \) and \( \sigma_2 = \frac{1}{2} \).

The importance of a representation from an algorithmic or constructivist point of view is that, without further reasoning or calculation, we know immediately that \( f \) is semidefinite on \( K \).

3. COMMENTS ON THE ZERO-DIMENSIONAL CASE

By the zero-dimensional case, we mean that the compact set in question is one point. We will assume without loss of generality that this point is \( x = 0 \). It is surprising, at first glance, that there is anything to say about this case. Notice that if \( \Phi = \{-x^2\} \), then it is possible to represent \( x^3 \) in \( \Sigma \{ -x^2 \} \) as

\[
x^3 = \frac{1}{4} x^2 (x + \frac{1}{2})^2 - \frac{1}{4} x^2 (x - \frac{1}{2})^2
\]

but it is not possible to represent \( x \) in \( \Sigma \{ -x^2 \} \). If it were possible to represent \( x \), then \( x = \sigma_1 - x^2 \sigma_2 \), and the implied vanishing of \( \sigma_1 \) at \( x = 0 \) would be of even multiplicity. This implies that \( x \) must vanish at 0 to at least multiplicity 2, a contradiction. As a consequence of Schmüdgen’s theorem, however, it should be possible to represent \( x + \delta \), \( \delta > 0 \) in \( \Sigma \{ -x^2 \} \). In fact, there is a representation of the form

\[
x + \delta = \left( \frac{x}{2 \sqrt{\delta} + \sqrt{\delta}} \right)^2 - \frac{x^2}{4\delta}.
\]

This representation is trivial to obtain and the degree of the representation is bounded. What we consider here is the norm of the representation. Notice that the norm of this representation, \( \frac{1}{4\delta} + 1 + \delta \), grows without bound as \( \delta \) gets small.
4. NONNEGATIVE REPRESENTATION IN THE ONE-DIMENSIONAL CASE

We obtain more interesting results with regard to both the norm and degree when we assume the set in question is one interval. We will assume without loss of generality that the interval is $[-1, 1]$. Powers and Reznick [6] have done similar work in regards to this question. They were able to show the existence of a solution under the conditions of Theorem 2, but have not addressed the possibility that a represented polynomial could vanish at the endpoints. Jacobi [3] and Jacobi and Prestel [4] have also done work in this vein, but have done so where the polynomial in question is strictly positive over the interval.

In the theorem below we will assume that $F$ is one polynomial. Since $\{F \geq 0\} = [-1, 1]$, it must be the case that $F$ is of the form $(1-x)^{2k-1}(1+x)^{a-1}F_1$, $k, a = 1, 2, \ldots$. Furthermore, $F_1 \neq 0$, with equality only if $F_1$ has a zero of even order inside $[-1, 1]$. We will furthermore assume that $f([-1, 1]) \neq 0$. Thus $f$ must be of the form $(1-x)^m(1+x)^ng$ where $m, n$ are nonnegative and $g = b(\sum_{i=1}^{s_i} (a_i \pm x)^s_i)(\prod_{i=1}^{p_i}((x+c_i)^2+d_i^2)^i)$ is such that $a_i \neq \pm 1$; and $(a_i \pm x)^s_i = 0$ inside $[-1, 1]$ only when $s_i$ is even.

**Theorem 2.** Let $\{F \geq 0\} = [-1, 1]$ where $F$ consists of one polynomial and all zeros of $F$, other than $\pm 1$, satisfy $(\text{Im}(z))^2 \geq 2|\text{Re}(z)|-1$. Suppose $f([-1, 1]) \geq 0$. Then $f \in \Sigma\{F\}$ if and only if the multiplicities of $f$ at $\pm 1$ are even or not smaller than the corresponding multiplicities of $F$.

**Proof of Theorem 2.** We will assume throughout that $F$ has multiplicities of $2k-1$ at $x=1$ and $2\ell-1$ at $x=-1$ and $f$ has multiplicities of $m$ at $x=1$ and $n$ at $x=-1$. The critical cases occur if $m \leq 2k-1$ is odd or $n \leq 2\ell-1$ is odd. We may assume without loss of generality that $m=n=1 < 2k-1$, as if, for example $m=3$, then we could split $(1-x)^3 = (1-x)^2(1-x)$ and clearly $(1-x)^2$ would have a representation. Then, if it were the case that $f \in \Sigma\{F\}$ we could find sums of squares $\sigma_1, \sigma_2$ such that $f = \sigma_1 + \sigma_2 F$. However, the implied vanishing of $\sigma_1$ at $x=1$ would be of even order which would imply the contradiction that $(1-x)$ vanishes there to at least order 2. Hence $f \notin \Sigma\{F\}$.

Next we will assume that $m, n$ are even or $m > 2k-1$ and $n > 2\ell-1$ and show that $f \in \Sigma\{F\}$ using an inductive argument. Our first step will be to show that $f \in \Sigma\{1-x^2\}$.

Since $f = (1-x)^m(1+x)^ng$ (with the conditions imposed above) then we can find representations for $(1-x)^m$, $(1+x)^n$ and $g$ and then multiply those representations together to obtain a representation of $f$ in $\Sigma\{1-x^2\}$.

We begin by showing that $1-x \in \Sigma\{1-x^2\}$ and $1+x \in \Sigma\{1-x^2\}$. We will assume without loss of generality that $m = n = 1$. If $m$ or $n$ were even
then clearly \((1-x)^m\in \Sigma\{1-x^2\}\) and \((1+x)^n\in \Sigma\{1-x^2\}\). If \(m\) or \(n\) were greater than one and odd, then either could be broken up in the following way \((1-x)^m = (1-x)(1-x)^{m-1}\). And we could represent the square \((1-x)^{m-1}\in \Sigma\{1-x^2\}\) and multiply the square representation by the representation for \(1-x\) in \(\Sigma\{1-x^2\}\) below.

We have

\[
1-x = \left(\frac{1}{\sqrt{2}} x - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2} (1-x^2) \in \Sigma\{1-x^2\}
\]

\[
1+x = \left(\frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2} (1-x^2) \in \Sigma\{1-x^2\}.
\]

As for \(g = \beta(\prod_{i=1}^{\ell} (\alpha_i \pm x)^i) (\prod_{i=1}^{p} ((x+\gamma_i)^2 + \delta_i^2)^i)\), we need only find representations for linear terms since the quadratic terms are squares. If \(s_i\) is even, then clearly \((\alpha_i \pm x)^i \in \Sigma\{1-x^2\}\). If \(s_i\) is odd, then \(\alpha_i > 1\) or \(\alpha_i < -1\). We can represent

\[
\alpha_i + x = (1+x) + (\alpha_i - 1) = \left(\frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}}\right)^2 + (\alpha_i - 1) + \frac{1}{2} (1-x^2) \quad (1)
\]

\[
\alpha_i - x = (1-x) + (\alpha_i - 1) = \left(\frac{1}{\sqrt{2}} x - \frac{1}{\sqrt{2}}\right)^2 + (\alpha_i - 1) + \frac{1}{2} (1-x^2). \quad (2)
\]

Notice in (1) and (2) that the degree of the representation is 2 and the norm of the representation is \(1 + \alpha_i\). By multiplying the representation for \((1-x)^m\), \((1+x)^n\) and \(g\) together, we obtain a representation for \(f\in \Sigma\{1-x^2\}\).

Our next step in the inductive argument is to assume that \(f\in \Sigma\{1-x^2\}\) and show that \(f\in \Sigma\{(1-x)((x-a)^2 + b^2)\}\) whenever \(b^2 \geq 2 |a| - 1\).

We begin by showing that \(1-x\) can be expressed as \(\sigma_1 + \sigma_2(1-x^2)\) whenever \(b^2 \geq 2 |a| - 1\) where, as usual, \(\sigma_j = \sigma_j(x)\) is a sum of squares.

If we choose \(\sigma_1 = (Ax^2 + Bx + C)(1-x)\) and \(\sigma_2 = k\) we obtain a representation where \((A, B, C, k)\) are found by solving them in the equation

\[
1-x = Ax^2 + Bx + C)(1-x) + k(1-x)((x-a)^2 + b^2).
\]

We then arrive at

\[
A = \frac{1}{2(1-a)^2 + 2b^2} \quad B = \frac{2(1-a)}{2(1-a)^2 + 2b^2}
\]

\[
C = \frac{2(1-a)^2 + b^2 - a^2}{2(1-a)^2 + 2b^2} \quad k = \frac{1}{2(1-a)^2 + 2b^2}.
\]
This will give us a valid representation of $1-x$ in $\Sigma \{(1-x^2)((x-a)^2+b^2)\}$ with conditions opposed on $k$ and $Ax^2+Bx+C$. It will be necessary that $k>0$ and $Ax^2+Bx+C$ is positive semidefinite. Clearly $k>0$ as long as $(a,b)$ is not equal to $(1,0)$. The expression $Ax^2+Bx+C$ will be positive semidefinite as long as its discriminant, $B^2-4AC \leq 0$, and $C>0$. In this case, $B^2-4AC \leq 0$ whenever $b^2 \geq 2a-1$, as the following calculations show,

$$B^2-4AC = -\frac{b^2-(2a-1)}{(1-a)^2+b^2) \leq 0$$

in order to get the condition on $(a,b)$. Thus, we obtain a valid representation for $1-x$ in $\Sigma \{(1-x^2)((x-a)^2+b^2)\}$ whenever $b^2 \geq 2a-1$.

If $(a,x)\mapsto (-a,-x)$, then $(x-a)^2+b^2$ is unchanged. The representation for $1+x$ is valid if $b^2 \geq -2a-1$.

Hence we obtain a valid representation for $1-x^2$ in $\Sigma \{(1-x^2)((x-a)^2+b^2)\}$ whenever $b^2 \geq 2|a|-1$. Finally, to obtain a representation for $f$ in $\Sigma \{(1-x^2)((x-a)^2+b^2)\}$, we recall that by hypothesis $f$ as $\sigma_1+\sigma_2(1-x^2)$. Furthermore, since $1-x^2 \in \Sigma \{(1-x^2)((x-a)^2+b^2)\}$ this implies that $1-x^2=\sigma_1+\sigma_2(1-x^2)((x-a)^2+b^2)$. And, thus $f=\sigma_1+\sigma_2\{\sigma_3+\sigma_4(1-x^2)((x-a)^2+b^2)\} = \sigma_5+\sigma_6(1-x^2)((x-a)^2+b^2)$ where $\sigma_5, \sigma_6$ are sums of squares.

For the next step in the inductive argument, we assume that if $\Phi_i = ((x-a_i)^2+b_i^2)$ then

$$1-x^2 \in \Sigma \{\Phi_1\Phi_2\cdots\Phi_{n-1}(1-x)^2\}$$

and

$$1-x^2 \in \Sigma \{\Phi_n(1-x)^2\}$$

whenever the zeros of $\Phi_1, \Phi_2, \ldots, \Phi_n$ satisfy $b_i^2 \geq 2|a_i|-1$ for $i=1,\ldots,n$.

Then,

$$1-x^2 = \sigma_1 + \sigma_2(1-x^2) \Phi_1\Phi_2\cdots\Phi_{n-1} \quad (3)$$

and

$$1-x^2 = \sigma_3 + \sigma_4(1-x^2) \Phi_n. \quad (4)$$

And if we substitute (3) for $1-x^2$ into (4) we obtain

$$1-x^2 = \sigma_1 + \sigma_2\{\sigma_3 + \sigma_4(1-x^2) \Phi_1\Phi_2\cdots\Phi_{n-1}\} \Phi_n$$
or
\[ 1 - x^2 = \sigma_5 + \sigma_6 \Phi_1 \Phi_2 \cdots \Phi_n, \]
where \( \sigma_5, \sigma_6 \) are sums of squares. Hence \( 1 - x^2 \in \Sigma(\Phi_1, \Phi_2, ..., \Phi_n(1 - x^2)) \) whenever the zeros of \( \Phi_i \) satisfy \( b_i^2 \geq 2 |a_i| - 1 \).

**Corollary 2.** The complexity of the representation in Theorem 2 has the following bounds: the degree of the representation, \( d.r.(f) \), satisfies
\[
\deg f \leq d.r.(f) < 4 \deg \Phi + 2 \deg f.
\]

**Proof.** We will define \( \Phi \) and \( f \) as above. We will make further comment on \( f \) here: Recall that \( f = (1 - x)^m (1 + x)^n g \) where \( m, n \) are nonnegative and
\[
g = \beta \left( \prod_{i=1}^{l} (\alpha_i \pm x)^{s_i} \right) \left( \prod_{i=1}^{r} ((x + \gamma_i)^2 + \delta_i)^{s_i} \right)
\]
is such that \( \alpha_i \neq \pm 1 \); and \( (\alpha_i \pm x)^{s_i} = 0 \) inside \([-1, 1]\) only when \( s_i \) is even. Furthermore, \( \prod_{i=1}^{l} (\alpha_i \pm x)^{s_i} = \prod_{i=1}^{l} (\alpha_i \pm x)^{s_i} \prod_{i=l+1}^{\ell} (\alpha_i \pm x)^{s_i} \) where we may assume that \( s_i \) is odd for \( i = 1 \) to \( \ell \) and \( s_i \) is even for \( i = \ell + 1 \) to \( \ell \). Clearly, if \( f \) is a sum of squares then \( d.r.(f) = \deg f \), hence the lower bound.

To find the upper bound, we first will assume that \( \Phi = (1 - x^2) \) and \( m, n \) are even. Then for each factor of even degree, its degree will be the degree of its representation. And, for each factor of odd degree, we showed in the first part of Theorem 2 that the degree of the representation will be twice the degree of the factor (see (1) and (2)). Hence
\[
d.r.(f) = m + n + 2s_1 + 2s_2 + \cdots + 2s_l + s_{l+1} + \cdots + s_{\ell} + 2t_1 + \cdots + 2t_p.
\]
So,
\[
d.r.(f) = \deg f + s_1 + \cdots + s_{\ell} < 2 \deg f.
\]

Next (as in the induction case above) we will assume that \( \Phi = (1 - x^2) \Phi_1 \), where \( \Phi_1 = (x - a_1)^2 + b_1^2 \). From above
\[
f = \sigma_1 + \sigma_2(1 - x^2), \quad \deg \sigma_i < 2 \deg f
\]
and
\[(1 - x^2) = \sigma_3 + \sigma_4(1 - x^2) \Phi_1, \quad \text{deg } \sigma_3 = 4.\]

Thus,
\[f = \sigma_1 + \sigma_2 \sigma_3 + \sigma_2 \sigma_4(1 - x^2) \Phi_1\]
will be a representation where \(d.r.(f) < 4 + 2 \deg f\).

We continue in this manner with \(\Phi = (1 - x^2) \Phi_1 \cdots \Phi_r\), and find that
\[d.r.(f) < 4r + 2 \deg f\]
and
\[4r + 2 \deg f < 4 \deg \Phi + 2 \deg f.\]

Thus,
\[\deg f < d.r.(f) < 4 \deg \Phi + 2 \deg f.\]

**Corollary 3.** The complexity of the representation in Theorem 2 has the following norm bounds: the norm of the representation, \(n.r.(f)\), satisfies
\[\|f\| \leq n.r.(f) \leq \|f\|^{2^{\deg \Phi \deg f}}.\]

**Proof.** Trivially, \(\|f\| \leq n.r.(f)\). Consider the upper bound (again with \(f\) and \(\Phi\) as above) by first noting that even powered factors are represented in any \(\Sigma\{\Phi\}\) as themselves, hence we have (assuming without loss of generality that \(m, n\) are even)
\[n.r.\{(1 - x)^m\} = 2^m\]
\[n.r.\{(1 + x)^n\} = 2^n\]
\[n.r.\{(a_i \pm x)^{\ell}\} = (a_i + 1)^{\ell}\]
(for \(i = \ell_i + 1 \text{ to } \ell\)) and
\[n.r.\{|\beta| [(1 + \gamma_i)^2 + \delta_i^2]^{\ell_i} = \beta((x + \gamma_i)^2 + \delta_i^2)^{\ell_i}\}.\]

Note that this norm representation works since each of these factors must have nonnegative coefficients, i.e., \(\delta_i > 0\) so that \(\beta |[(x + \gamma_i)^2 + \delta_i^2]^{\ell_i} > 0\). Thus, we must just consider odd-powered factors: \((a_i \pm x)^{\ell}\) where
\(i = 1\) to \(\ell_i\). Our first step will be to represent this in \(\Sigma\{1-x^2\}\). Recall the representation from Theorem 2 of \(\alpha_i \pm x\) (see Eqs. (1) and (2)). Then, the norm of that representation is \(4 + \alpha_i\) and if we raise any of those to the \(s_i\) degree, we obtain a norm of \((4 + \alpha_i)^{s_i}\). And, if \((\alpha_i \pm x)^{s_i}\) is raised to the \(s_i\) power, then \(\|((\alpha_i \pm x)^{s_i})\| = (2^{s_i} + \alpha_i)^{s_i}\). Then, if we consider all \(i = 1\) to \(\ell_i\), we have

\[
\prod_{i=1}^{\ell_i} \|((\alpha_i \pm x)^{s_i})\| \leq \prod_{i=1}^{\ell_i} (2^{s_i} + \alpha_i)^{s_i}
\]

\[
< \prod_{i=1}^{\ell_i} (2^{s_i} + 2^{s_i} \alpha_i)^{s_i}
\]

\[
= \prod_{i=1}^{\ell_i} (2^{s_i}) (1 + \alpha_i)^{s_i}.
\]

Thus, if we want to find \(n.r.(f)\) in \(\Sigma\{1-x^2\}\), we put all the pieces together in the following way:

\[
n.r.(f) \leq 2^{n_2} \prod_{i=1}^{\ell_i} (1 + \alpha_i)^{s_i} \beta \prod_{i=1}^{\ell_i} \left[ (1 + \gamma_i)^2 + \delta_i^2 \right]^{s_i} \prod_{i=1}^{\ell_i} 2^{2s_i} (1 + \alpha_i)^{s_i}
\]

thus

\[
n.r.(f) = \|f\| \prod_{i=1}^{\ell_i} (2^{2s_i}) \leq \|f\| 2^{\deg \Phi \cdot \deg f}.
\]

A similar argument can be made to get a result in general, if \(\Phi = (1-x^2) \psi_1 \cdots \psi_j\).

Hence,

\[
\|f\| < n.r.(f) = \|f\| \prod_{i=1}^{\ell_i} (2^{2s_i}) \leq \|f\| 2^{\deg \Phi \cdot \deg f}.
\]

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