APPLICATION OF LINEAR DIOPHANTINE EQUATIONS IN TEACHING MATHEMATICAL THINKING
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1. INTRODUCTION

The linear Diophantine equations and their solutions are one of the well-known results in number theory. Study of these equations can be found in many works such as Dickson [1], Gallian [2], and so on. A linear Diophantine equation with two variables $x$ and $y$ has the form

$$ax + by = c, \quad (1)$$

where $a$, $b$, and $c$ are all integers. We are interested in integer solutions, that is, integers $x$ and $y$ that satisfy equation (1). In Dickson it is stated that if $a$ and $b$ are relatively prime (that is, $a$ and $b$ have no common divisor other than 1), and if $(u, v)$ is an integer solution of (1), then the whole set of integer solutions of (1) can be expressed as

$$x = u + bw, \quad y = v - aw, \quad (2)$$

where $w$ is an arbitrary integer. In other words, the whole set of integer solutions of (1) is

$$S = \{(u + bw, v - aw) \mid w = 0, \pm 1, \pm 2, \ldots \}. \quad (3)$$

For example, consider the equation

$$3x + 4y = 2. \quad (4)$$

Here, $a = 3$ and $b = 4$. By observation we see that $(2, -1)$ is an integer solution since

$$3(2) + 4(-1) = 2.$$

Hence, we may use $u = 2$ and $v = -1$. Therefore, the integer solution set of (4) is

$$S = \{(2 + 4w, -1 - 3w) \mid w = 0, \pm 1, \pm 2, \ldots \}.$$
\[ x = 2 + 4(2) = 10, \]
\[ y = -1 - 3(2) = -7. \]

It can easily be checked that \((10, -7)\) really is a solution of (4).

The purpose of this paper is to demonstrate how the above result can be applied to teaching certain problem solving concepts and methods in a college-level general mathematics topics course such as Mathematical Thinking. The application is described in the following sections. Under the appropriate guidance of an instructor, this paper can be well appreciated by these students.

### 2. APPLICATION OF LINEAR DIOPHANTINE EQUATIONS IN TEACHING PROBLEM SOLVING

In a highly advanced and technologically minded world, students must be good problem solvers. One example that can be used in teaching problem solving in a Mathematical Thinking course is as follows.

**Suppose you have an unlimited amount of apple cider in a large tank. You want to measure 3 gallons of apple cider for a customer. You have only a 4-gallon and 5-gallon measuring device. How could you measure 3 gallons?**

The instructor may first encourage students to construct a simple table and begin a trial-and-error approach. One possible outcome is given in Table 2.1.

<table>
<thead>
<tr>
<th>Action</th>
<th>Amount in the 4-gallon container (A)</th>
<th>Amount in the 5-gallon container (B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fill container B</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>Pour the cider from B into A</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>until A is full</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Empty container A</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Pour the cider remaining in B</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>into A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fill container B</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>Pour the cider from B into A</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>until A is full</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Empty container A</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Pour the cider remaining in B</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>into A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fill container B</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Pour the cider from B into A</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>until A is full</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Empty container A</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 2.1
Thus, we have measured 3 gallons of apple cider from a 4-gallon and a 5-gallon container. Note that the above table demonstrates only one possible solution. A closer look at this problem reveals that there are only 3 different actions involved in this procedure: (a) filling a container; (b) transferring from one container to another; and (c) emptying a container. Moreover, suppose we use $Q$ to denote the total quantity of cider we have in the containers; then before any action is taken, the value of $Q$ equals zero. During the procedure, each (a) action adds 4 or 5 to $Q$, each (c) action subtracts 4 or 5 from $Q$, and each (b) action does not change the value of $Q$. Our goal in this procedure is to make $Q = 3$ through a sequence of actions. Also note that at any time during this procedure, the value of $Q$ can never go beyond 9, the sum of 4 and 5.

With that in mind, the instructor may point out to students that the above actions are in fact a string of "+" and "-" operations: $+5 - 4 + 5 - 4 + 5 - 4 = 3$. This string, of course, comes with the caveat that we can never have more than 9 gallons (4 gallons + 5 gallons) in total at any given time, or else both containers would be over-full and thus not able to be measured. Therefore, the string $+5 + 5 - 4 - 4 - 4$ cannot be used in this case.

The instructor may also point out that mathematically the string $+5 - 4 + 5 - 4 + 5 - 4 = 3$ is equivalent to the equation $4(-3) + 5(3) = 3$. In fact, in general to find an appropriate string of operations, we may first want to find two integers $x$ and $y$ such that

$$4x + 5y = 3.$$  \tag{5}$$

Note that (5) is in fact a linear Diophantine equation. Therefore, all the solutions of (5) are

$$x = u + 5w, \quad y = v - 4w, \quad w = 0, \pm 1, \pm 2, \ldots,$$  \tag{6}$$

where $(u, v)$ stands for a particular solution of (5). In other words, if we can somehow find a particular solution $(u, v)$ of (5), then all solutions of (5) are given by (6), and correspondingly all possible strings of operations as well as tables of actions can be constructed.

At this point, the instructor may want to briefly summarize the solution procedure for finding all possible ways of measuring 3 gallons from a 4-gallon and a 5-gallon container. The procedure can be described as follows:

**Step 1:** Set up a linear Diophantine equation

$$4x + 5y = 3.$$  \tag{7}$$

**Step 2:** Find a particular integer solution $(u, v)$ of (7).

**Step 3:** Express all integer solutions of (7) as

$$x = u + 5w, \quad y = v - 4w, \quad w = 0, \pm 1, \pm 2, \ldots.$$  \tag{8}$$

**Step 4:** To view any particular way of measuring 3 gallons using these two containers, just assign an integer value to $w$ to get a particular solution $(x, y)$, and then do the following:
Step 4.1: Set up a string of arithmetic operations according to the equation $4x + 5y = 3$. Make sure that we never have more than $4 + 5 = 9$ gallons in total after any given operation in the string.

Step 4.2: Establish an action table correspondingly to demonstrate that particular solution to the problem. The action “transferring from one container into the other” should be inserted appropriately to ensure that at any given time there is no more than 4 or 5 gallons in the corresponding container.

For example, in this case we have seen that $4(-3) + 5(3) = 3$. Thus, $u = -3$ and $v = 3$ yield a particular solution. Hence the set of all integer solutions will be

$$S = \left\{(-3 + 5w, 3 - 4w) | w = 0, \pm 1, \pm 2, \ldots\right\}.$$  

(9)

Note that $w = 0$ leads to the solution $(-3, 3)$ itself. If we let, say, $w = 1$, then we get another solution $x = 2$ and $y = -1$. It is easy to check that $(2, -1)$ does satisfy equation (7). The corresponding string of operations is now (under the condition stated in step 4.1)

$$+4 + 4 - 5 = 3.$$  

The corresponding action table will then be as displayed in Table 2.2. Note that only five actions are taken in Table 2.2. This is much less than the number of actions needed with the string $+5 - 4 + 5 - 4 + 5 - 4 = 3$, as displayed in Table 2.1.

<table>
<thead>
<tr>
<th>Action</th>
<th>Amount in the 4-gallon container (A)</th>
<th>Amount in the 5-gallon container (B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fill container A</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>Pour the cider from A into B</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>Fill container A</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Pour the cider from A into B until B is full</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>Empty container B</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2.2

It is now perhaps time for the instructor to introduce the following two questions to students.

**Question 1:** Assuming that we are given an $a$-gallon container and a $b$-gallon container, with $a$ and $b$ being relatively prime, and that we are to use these two containers to measure $c$ gallons, with $c \leq a + b$, how can we systematically find a particular integer solution $(u, v)$ to the linear Diophantine equation $ax + by = c$?

**Question 2:** After a particular solution $(u, v)$ is found, how can we find the optimal solution from the set

$$S = \{(u + bw, v - aw) | w = 0, \pm 1, \pm 2, \ldots\}.$$  

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so that the corresponding string of operations requires the least number of actions?

The answer to Question 1 is quite simple, and hopefully through discussion can be suggested by students. One way is to observe the remainders of the pair of divisions

\[ \frac{c-bn}{a} \text{ and } \frac{c-an}{b} \text{ for } n = 0, \pm 1, \pm 2, \ldots. \]  

Once a zero remainder is observed, we stop right there and \((u, v)\) is obtained this way: If the zero remainder is observed in the first division then \(u = \frac{c-bn}{a}\) and \(v = n\); otherwise we have \(u = n\) and \(v = \frac{c-an}{b}\).

Students might not consider it necessary to observe both divisions listed in (10), but it is necessary. In fact, a zero remainder may never be observed if only one division is examined. For example, if both \(c\) and \(b\) are even numbers but \(a\) is odd, then the first division will never have a zero remainder.

In order to answer Question 2, the instructor may first mention to students that except for the trivial case when \(c = a + b\), the values of \(x\) and \(y\) in each solution \((x, y)\) cannot both be positive. However, one of them, say \(x\), must be positive. The other number \(y\) will be either negative or zero. Without loss of generality let us assume that \(x > 0\) and \(y \leq 0\). Then the corresponding sequence of actions will have \(x\) times of action (a) (filling a container) and \(y\) times of action (c) (emptying a container). It is also important to note that if \(x \geq b\) and \(y \leq -a\), then at least \(b\) times of action (a) and \(a\) times of action (c) are “wasted” since \(a(b) + b(-a) = 0\). Therefore, if \((x, y)\) yields the optimal solution, then it must be either

\[ |x| < b \]  

or

\[ |y| < a \]

or both. When a solution \((x, y)\) satisfies either (11) or (12), the corresponding sequence of actions will always start with an action (a). Then an action (b) (transferring from one container into the other) will be inserted after each action (a) or action (c). The sequence always ends with either an action (a) or an action (c). Therefore, if a solution \((x, y)\) satisfies either (11) or (12), then the corresponding sequence of actions will contain

\[ 2(|x| + |y|) - 1 \]  

actions.

It is an interesting exercise for students to verify that the particular solution \((u, v)\) obtained from the procedure given above satisfies either (11) or (12), that is, either \(|u| < b\) or \(|v| < a\). It is also easy to see that there exist
two integers \( N \leq 0 \leq M \) such that the solution \((u + bw, v - aw)\) satisfies either (11) or (12) when \( w = N, N + 1, \ldots M \) but fails to satisfy either of them when \( w < N \) or \( w > M \).

This leads us to the answer to the second question. After obtaining the particular solution \((u, v)\) from the above procedure, calculate the value \(|u + bw| + |v - aw|\) for \( w = 0, 1, 2, \ldots \) until \((u + bw, v - aw)\) satisfies neither (11) nor (12). Then do the same for negative values of \( w \). After that, simply examine these values of \(|u + bw| + |v - aw|\), and the smallest one yields the optimal solution.

Let us consider the following example. Assume that we are given a 3-gallon and a 10-gallon container, and we are to use them to measure 8 gallons. In this case, \( a = 3 \), \( b = 10 \), and \( c = 8 \). The following Table 2.3 shows students the steps to finding a particular solution \((u, v)\).

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \frac{c - bn}{a} ) Quotient of ( \frac{c - bn}{a} )</th>
<th>Remainder of ( \frac{c - bn}{a} )</th>
<th>( \frac{c - an}{b} ) Quotient of ( \frac{c - an}{b} )</th>
<th>Remainder of ( \frac{c - an}{b} )</th>
<th>(( u, v )) found?</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8/3</td>
<td>2</td>
<td>2</td>
<td>8/10</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-2/3</td>
<td>0</td>
<td>-2</td>
<td>5/10</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>18/3</td>
<td>6</td>
<td>0</td>
<td>11/10</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2.3

From Table 2.3 we see that the remainder of \( \frac{c - bn}{a} \) equals zero when \( n = -1 \), and hence a particular solution \((u, v) = (6, -1)\) is obtained. Note that this solution does satisfy (11) as well as (12).

Table 2.4 illustrates the steps to finding the optimal solution whose corresponding sequence of actions contains the least number of actions.

| \( W \) | \((u + bw, v - aw)\) | \(|u + bw| + |v - aw|\) | (11) or (12) satisfied? |
|--------|---------------------|-----------------|------------------------|
| 0      | (6, -1)             | 7               | Yes                    |
| 1      | (16, -4)            | 20              | No, stop increasing the value of \( w \) |
| -1     | (-4, 2)             | 6               | Yes                    |
| -2     | (-14, 5)            | 19              | No, stop decreasing the value of \( w \) |

Table 2.4

In this case only \( w = 0, -1 \) yield solutions that satisfy either (11) or (12). Comparing the corresponding value of \(|u + bw| + |v - aw|\), we see that \( w = -1 \) or \((x, y) = (-4, 2)\) is the optimal solution. The corresponding string of operations is \( +10 - 3 -3 -3 + 10 - 3 = 8 \).
It may be left to the students as an exercise to set up the table of the corresponding sequence of actions. In this case, there will be $2(|x| + |y|) - 1 = 11$ actions.

**CONCLUSION**

This paper attempts to use the measuring problem as an example to demonstrate to a college-level general mathematics topics class, such as a Mathematical Thinking class, the links between the sequence of actions, the string of arithmetic operations, and the solution of linear Diophantine equations. It also shows students how a mathematical result in number theory, usually introduced to an upper level college mathematics major class, may be applied in problem solving. This example introduces to students the concepts of "solution set", "particular solution", and "optimal solution". For students with basic computer skills, an interesting exercise may also be to use either spreadsheet software or a programming language to implement the procedure described in Section 2, using values of $a$, $b$, $c$ as the input and obtaining the optimal solution $(x, y)$ as the output.

Let us also mention that this example may be generalized to the case when more than two containers are used to do the measuring. More precisely, assuming that we are given $k$ containers of capacity $a_1$, $a_2$, ... $a_k$ gallons, also assuming that the largest common factor of $a_1$, $a_2$, ... $a_k$ is 1, we want to measure $c$ gallons with $c$ being less than or equal to $a_1 + a_2 + \cdots + a_k$. Using the method given in Dickson for solving the linear Diophantine equation $a_1x_1 + a_2x_2 + \cdots + a_kx_k = c$, we may obtain a particular solution $(u_1, u_2, ... u_k)$ as well as the formula for the whole solution set. To find the optimal solution in this case is a more difficult task. We may first determine the number of actions required by the particular solution $(u_1, u_2, ... u_k)$. This can be done by setting up the corresponding string of operations as well as the sequence of actions. Let us use $N$ to denote this number. Clearly, there are only finitely many solutions $(x_1, x_2, ... x_k)$ that satisfy $|x_1| + |x_2| + \cdots + |x_k| < N$, and any other solution would require at least $N$ actions. Therefore, by examining the number of actions required by these finitely many solutions, we can find the optimal solution. Again, this procedure may be coded into a computer program. In terms of the computational cost, this procedure is not as efficient as the one given in Section 2 for the case when only two containers are used. However, like the one in Section 2, it is guaranteed to deliver the optimal solution. While this generalization may have gone beyond the level of a Mathematical Thinking class, it can very well be used as an example when teaching a Linear Algebra class.
REFERENCES