Binomial Option Pricing Under Stochastic Volatility and Correlated State Variables

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This article develops a method for valuing contingent payoffs for a non-constant volatility process via a simple recombining binomial tree. The direct application of the technique provides a way to price, for example, American calls or puts governed by a stock price process with stochastic volatility.

The stock price and volatility diffusions may have non-zero correlations. This feature allows model prices consistent with the volatility smile. Numerical estimates of the hedge statistics (delta, gamma, and vega) are obtained directly from the tree.

The volatility "smile" is frequently observed in option prices. In a pure Black-Scholes world, there should be no smile, as volatility would be constant across strike price and time. Why does the smile exist?

There are several possible explanations. We choose to develop as an explanation a stochastic volatility model. Pricing models with stochastic volatility have been addressed in the literature by a number of authors, including Hull and White [1987], Scott [1991], and Heston [1993]. The bivariate binomial framework presented here is perhaps more general than those models in that we can value American options and allow non-zero correlation between the underlying process and stochastic volatility.

The same results could be obtained using finite differences, but our method is simpler. For example, we do not require the specification of boundary conditions or mesh ratios. In the main, the only convergence condition we require is that probabilities be well-defined. Thus, the model in this article is simple, robust, and widely applicable.

It is not difficult to use the bivariate binomial tree as a way of including a second state variable such as a stochastic interest rate in the drift term. We develop the mathematically more complex case, where the second state variable is part of the volatility term. The direct application of the technique provides a way to price, for example, derivative securities based on processes with stochastic volatility using a
simple recombining binomial tree with a set of four joint, but possibly non-independent, probabilities. In fact, with the methods developed here, it is no longer necessary to perform transformations to remove correlations, since node probabilities, \( P_u \), can be determined by a simple calculation.

What kind of derivatives can be priced using this technique? We require that the payoff function not be path-dependent (this excludes lookback and Asian options) and that the stochastic processes follow certain functional forms. The payoff functions that can be priced clearly include most of those found in equity, interest rate, commodity, and currency derivatives traded on the exchanges and over-the-counter. The functional forms for the stochastic processes are also consistent with those commonly assumed, either implicitly or explicitly, by practitioners.

For example, although the underlying process has stochastic volatility, it may be otherwise identical with the usual Brownian motion or constant elasticity of variance form. The stochastic volatility process may be in the form of Brownian motion, except that the drift term can be quite general, e.g., mean-reverting.

I. PROCESSES USED FOR PRICING DERIVATIVES

The basis of the pricing algorithm is the bivariate binomial tree. To build the tree, a pair of underlying processes must be assumed. For equities, currencies, and commodities, some version of Brownian motion is generally used.

We consider continuous-time risk-neutralized diffusion processes of the form

\[
\begin{align*}
\text{d}S &= \mu_S \text{d}t + f(S)\text{d}V \quad \text{(1)} \\
\text{d}V &= \mu_V \text{d}t + bV \text{d}Z_v 
\end{align*}
\]

where \( \text{corr}(\text{d}Z_v, \text{d}V) = \rho_{SV} \) and where \( \sigma_S = f(S)\text{d}V \) is typically of the form \( S^\gamma \sqrt{V} \).

The diffusion on \( S \) could be, for example, a constant elasticity of variance asset price and the diffusion on \( V \) stochastic volatility (squared). The process \( \text{d}S \) is the usual Brownian motion if volatility \( (\sqrt{V}) \) is fixed and \( \theta = 1 \). In the risk-neutral representation of a traded asset, \( m \) represents the instantaneous cost of carry, e.g., \( m = S(r - d) \) for an asset with continuous dividends \( d \), \( m = 0 \) for a futures contract, and so on.

Since volatility is a non-traded state variable, the instantaneous drift of Equation (2) is usually of the form \( \xi = (\alpha - \lambda\beta) \), where \( \alpha \) is the unadjusted instantaneous average growth rate of \( V \), and \( \lambda \) is the possible state- and time-dependent market price of risk of \( V \). The function \( \alpha \) is typically constant or mean-reverting in \( V \) or \( \log V \). Examples of mean reversion models can be found in [Scott (1991)] or [Hull and White (1987)].

II. CONSTRUCTING THE LATTICE

The lattice for the underlying asset and volatility must be constructed so that nodes will recombine. In this case, the number of nodes after \( n \) steps is \( (n + 1)^2 \).

If this is not the case, the bivariate binomial model is computationally explosive and cannot be used in realistic applications. Therefore, the processes \( S \) and \( V \) must be transformed to give constant variance and thus a simple grid that recombines at successive nodes. This approach is the key to pricing derivatives with a lattice.

First, transform the relevant economic variables so that a simple, well-behaved lattice results. This requires an inverse transformation at each of the nodes to allow for the evaluation of the payoff (if American) or continuation value of the derivative asset.

For simplicity, we develop transformations that give unit volatility. Thus, the lattice needs to be constructed only once. Different applications are easily handled by changing parameters in the transformation back to the original variables at lattice nodes.

Consider first the volatility transformation. The required transformation is

\[
Y = \ln(V)/b 
\]

which gives a process with unit volatility:

\[
\text{d}Y = (m_Y/bV - 0.5b)dt + bZ_v 
\]

Since the volatility, the coefficient of \( bZ_v \) is
constant (and equal to one), the lattice in Y recombines as required. Unfortunately, the required transformation of S to constant volatility is not straightforward because the volatility of S in its most general form entails both the random variables V and S.

We use a two-step transformation to obtain the needed result. First, we consider a transformation H of the form \( H = H(S, V) \) giving a diffusion

\[
dH = H_s(S)h(V)dZ_S + H_vVdZ_V + m_Sdt \tag{5}
\]

where subscripts s and v on the function H denote partial derivatives, and \( m_s \) is the drift of the H process and depends on \( m_v, m_w \), and second-order partials. After deriving the functional form for H, a second transformation of \( H \) to Q (see Appendix A for detailed derivations of the results of this section) is used to give a diffusion of the form

\[
dQ = m_Qdt + dZ_Q \tag{6}
\]

The diffusion for Q now has unit volatility as required. Since both Y and Q have unit volatility, the bivariate binomial grid is easily constructed in \( Y \times Q \) space, and the corresponding values of the original \( Y \) and \( S \) variables are given by the inverse transforms:

\[
V = \exp(Yb) \tag{7}
\]

\[
H = (2ab)^{-1}[2\rho + (1 - \rho^2) \times 
\exp(-abQ) + \exp(abQ)], \text{ and}
\] or

\[
S = [V^\theta(1 - \theta)H]^{\theta - 1}, \quad \theta \neq 1 \tag{8}
\]

\[
S = \exp(h(v)H), \quad \theta = 1 \tag{9}
\]

Under these transformations, the increments \( dZ_S \) and \( dZ_Q \) have correlation

\[
\text{Corr}(dZ_S, dZ_Q) = (\rho - \alpha b)H/\sigma_Q \tag{10}
\]

and it follows from Equations (4) and (6) that

\[
\text{Corr}(dV, dQ) = \text{Corr}(dZ_S, dZ_Q)
\]

These correlations come into play in computing jump probabilities.

III. APPROXIMATING THE LIMIT DIFFUSION WITH BINOMIAL JUMPS AND PROBABILITIES

Derivatives can be valued using the lattice if the lattice nodes are chosen properly and if appropriate risk-neutral probabilities can be found for each of the nodes. More specifically, if the bivariate binomial is to be used to value derivatives, the jump process and probabilities chosen must give a sensible limiting probability distribution. This requirement means that the binomial distribution must closely approximate the continuous-time distribution.

Therefore, we develop a set of jumps and bivariate binomial probabilities with the property that the limiting first- and second-order moments of the discrete processes match those of the original diffusion processes. Since we model bivariate processes that can be quite general, we do not address the weak convergence of these processes to the bivariate diffusion. In addition, we do not investigate the convergence of functions (e.g., options) defined on the binomial jump process to their corresponding diffusion values (although we find our simulations to be accurate when we restrict our model to special cases found in the literature).

As in the standard univariate model with unit volatility, the binomial jumps for the transformed processes \( Q \) and \( Y \) are defined by:

\[
Y_i^\pm = Y_0 \pm \sqrt{\Delta t}, \text{ and} \tag{11}
\]

and

\[
Q_i^\pm = Q_0 \pm \sqrt{\Delta t} \tag{12}
\]

where \( \Delta t \) is the size of the time step. The associated probabilities for upward jumps for \( Y \) and \( Q \) are, respectively:

\[
p = 0.5(1 + m_y \sqrt{\Delta t}), \text{ and} \tag{13}
\]

and

\[
q = 0.5(1 + m_y \sqrt{\Delta t}) \tag{14}
\]
Joint probabilities are defined by:

\[ P_{11} = \text{prob}(Q^r, Y^r) \]
\[ P_{12} = \text{prob}(Q^r, Y^*) \]
\[ P_{21} = \text{prob}(Q^*, Y^r) \], and
\[ P_{22} = \text{prob}(Q^*, Y^*) \]  \hspace{1cm} (15)

When \( dZ_q \) and \( dZ_q \) are independent, joint probabilities are easily obtained by multiplication, as noted by Hull and White [1990]. For example, \( P_{11} = q(1 - p), P_{12} = pq, P_{21} = (1 - p)(1 - q), \) and \( P_{22} = p(1 - q). \) Under dependence, standard probability rules may be invoked to give the joint probabilities by setting up three constraints on marginal probabilities and cross-product moments:

Constraint on marginal probability of \( \Delta Q = Q_1 - Q_0 \):

\[ P_{11} + P_{12} = q \]  \hspace{1cm} (16)

Constraint on marginal probability of \( \Delta Y = Y_1 - Y_0 \):

\[ P_{11} + P_{21} = 1 - p \] \hspace{1cm} (17)

Constraint on cross-product moments:

\[ E(\Delta Y \Delta Q) = \Delta t(2P_{12} + 2P_{21} - 1) \]  \hspace{1cm} (18)

The last constraint gives the information needed to adjust the probabilities for non-zero correlation. For conciseness, we give the covariance and correlation between \( \Delta Y \) and \( \Delta Q \):

\[ \text{Cov}(\Delta Y, \Delta Q) = 2\Delta t(P_{12} + P_{21} + p + q - 2pq - 1), \text{ and} \]

\[ \text{Corr}(\Delta Y, \Delta Q) = \frac{\text{Cov}(\Delta Y, \Delta Q)}{\sqrt{\text{Var}(\Delta Y) \text{Var}(\Delta Q)}} \]  \hspace{1cm} (19)

Solving Equations (16), (17), and (18) gives

\[ P_{11} = q(1 - p) = \text{Corr}(\Delta Y, \Delta Q) \sqrt{p(1 - p)q(1 - q)} \]  \hspace{1cm} (21)
\[ P_{12} = pq + \text{Corr}(\Delta Y, \Delta Q) \] \hspace{1cm} (22)
\[ P_{21} = (1 - q)(1 - p) + \text{Corr}(\Delta Y, \Delta Q) \] \hspace{1cm} (23)
\[ P_{22} = (1 - q)p - \text{Corr}(\Delta Y, \Delta Q) \sqrt{p(1 - p)q(1 - q)} \] \hspace{1cm} (24)

where \( \kappa = \sqrt{p(1 - p)q(1 - q)}. \) Notice that these equations reduce to the standard expression for independent joint probabilities when the correlation is zero.

We prove in Appendix B that these jumps and probabilities give limiting first- and second-order moments that match those of the diffusion process. The binomial tree is therefore completely defined by the node values [Equations (11) and (12)], the probabilities [Equations (21) through (24)], and the inverse transforms [Equations (7), (8), and (9)].

IV. COMPUTING DELTA, VEGA, AND GAMMA FROM THE GRID

The partial derivatives corresponding to delta, gamma, and vega can be computed from the bivariate grid when transformations are taken into account. Specifically, let \( F(Q, Y) \) be the option value on \( Q \times Y, \) and let \( F(Q + \Delta Q, Y + \Delta Y) = F^{++} \) be the option value one step forward for positive jumps in \( \Delta Q \) and \( \Delta Y. \) Define \( F^{+-}, F^{-+}, \) and \( F^{-+} \) in a similar manner, where the first superscript corresponds to jumps in \( Q \) and the second to jumps in \( Y. \)

We first compute deltas, gammas, and vegas on the equally spaced grid \((Q \times Y),\) and then use the transformations to compute the sensitivities with respect to stock price \((S)\) and volatility \((\sqrt{\sigma})\) changes. Specifically, we let \( C(S, \sigma) = F(Q, Y), \) and then use the chain rule to compute delta, vega, and gamma with respect to \( S \) and \( \sigma. \)

An estimate of the delta of \( Q \) is given by:

\[ \text{delta}(Q) \equiv \frac{p_s}{\text{delta}(-F^{+-} - F^{--})} + \frac{p_{-s}}{\text{delta}(F^{+-} - F^{--})} \] \hspace{1cm} (25)
It can be shown that this approximation has error of the order \( \Delta t \) and thus approaches 0 as the step size, \( \Delta t \), approaches 0.

The delta of the option with respect to the underlying asset \( S \) is, using the chain rule and \( F = C \),

\[
\text{delta}(S) = C_S = F_y Q_y + F_y Y_y 
\]

(26)

but \( Y_y = 0 \) and \( Q_y \) can be computed directly from the transformations in Equations (8) and (9).

Vega can be computed in a similar fashion. From the grid compute

\[
\text{vega}(Y) = F_y = \frac{\left[F^+ - F^-\right] + \left[F^+ - F^-\right]}{4\Delta Y} 
\]

(27)

and vega with respect to \( \sigma \) is

\[
\text{vega}(\sigma) = C_\sigma = F_y Q_\sigma + F_y Y_\sigma 
\]

(28)

where the partials \( F_y \) and \( F_\sigma \) are computed from the grid as in Equations (25) and (27). The partials \( Q_{\sigma} \) and \( Y_{\sigma} \) are computed from the transformations (see Appendix A).

Gamma follows directly using similar techniques. First, use the fact that gamma is the slope of delta. Then, compute the delta of \( Q \) at each of the four node points one time step forward. We denote these estimates by \( \Delta^{++}, \Delta^{-+}, \Delta^{+-}, \) and \( \Delta^{-} \). They depend on node values two steps forward from the point at which gamma is to be computed.

Gamma of \( Q \) is then estimated as:

\[
\text{gamma}(Q) = F_{qq} \equiv \frac{\left[(\Delta^{++} - \Delta^{+-}) + (\Delta^{-+} - \Delta^{-})\right]}{4AQ} 
\]

(29)

The gamma of the underlying follows from \( F = C \) and the chain rule:

\[
\text{gamma}(S) = C_{ss} = F_q Q_{ss} + F_{ss}(Q) 
\]

(30)

Both \( F_q \) and \( F_{ss} \) are obtained from the grid [Equations (25) and (29)], while the analytic partials, \( Q_q \) and \( Q_{ss} \), are computed directly from the transformations in Equations (8) and (9) (see Appendix A for these calculations).

Alternatively, all sensitivity parameters can be estimated directly by imposing a small change in the variable under consideration (e.g., asset price or volatility) and noting the change in option price. Gamma requires two successive increments, because three points are required to estimate the second derivative. This "brute force" methodology is not unreasonable for most applications, because the time requirements for the bivariate binomial are not prohibitive, but we show later that the efficient methods developed earlier are exceedingly accurate.

Scott [1991] discusses hedging option price with respect to changes in the underlying and volatility (delta-sigma hedging). He notes that option price can be hedged with two additional options or with an additional option and the underlying. Hedge ratios depend on deltas and vegas. Tests of delta-sigma hedging are somewhat mixed, although Scott develops scenarios where delta-sigma hedging reduces risk more than risk reduction with a pure delta hedge. Scott's comparison of delta-sigma hedging under Black-Scholes with delta-sigma hedging under the random volatility model reveals little difference in effectiveness.

V. SIMULATIONS

To illustrate the basic model, we consider as a special case the diffusions examined by Hull and White [1987]:

\[
dS = rSdt + SVdZ_s 
\]

(31)

\[
dV = bVdZ_v 
\]

(32)

e.g., \( m_s = rS, \langle S(V) \rangle = SV, \) \( a_s = 0 \). The resulting transformations from \( Q \) and \( Y \) back to the original \( V \times S \) space are then given by Equations (7) through (9) with \( \alpha = 1 \) and \( \sigma = 0.5 \).

The simulations are summarized in the exhibits. Exhibit 1 compares the bivariate binomial calculation of delta, vega, and gamma from the grid with Black-Scholes calculations. Exhibit 2 is an example of the effect of stochastic volatility on European puts when volatility is uncorrelated with the underly-
EXHIBIT 1
ESTIMATES OF DELTA, VEGA, AND GAMMA FROM THE GRID

<table>
<thead>
<tr>
<th>S/X</th>
<th>Delta Black-Scholes</th>
<th>Delta Bivariate</th>
<th>Vega Black-Scholes</th>
<th>Vega Bivariate</th>
<th>Gamma (x 100) Black-Scholes</th>
<th>Gamma (x 100) Bivariate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>0.035</td>
<td>0.034</td>
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<td>0.240</td>
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<td>0.198</td>
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<td>0.613</td>
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<td>0.882</td>
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<td>1.2</td>
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<td>0.978</td>
<td>0.045</td>
<td>0.048</td>
<td>0.418</td>
<td>0.416</td>
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**PANEL A. INITIAL VOLATILITY = 15%**

<table>
<thead>
<tr>
<th>S/X</th>
<th>Delta Black-Scholes</th>
<th>Delta Bivariate</th>
<th>Vega Black-Scholes</th>
<th>Vega Bivariate</th>
<th>Gamma (x 100) Black-Scholes</th>
<th>Gamma (x 100) Bivariate</th>
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<td>0.091</td>
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<td>0.093</td>
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<td>0.369</td>
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<td>0.225</td>
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<td>0.274</td>
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<td>0.104</td>
<td>0.106</td>
<td>0.722</td>
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</table>

**PANEL B. INITIAL VOLATILITY = 20%**

<table>
<thead>
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<th>S/X</th>
<th>Delta Black-Scholes</th>
<th>Delta Bivariate</th>
<th>Vega Black-Scholes</th>
<th>Vega Bivariate</th>
<th>Gamma (x 100) Black-Scholes</th>
<th>Gamma (x 100) Bivariate</th>
</tr>
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<tr>
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<td>0.151</td>
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<td>0.133</td>
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<td>1.655</td>
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<tr>
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<td>0.896</td>
<td>0.153</td>
<td>0.153</td>
<td>0.849</td>
<td>0.850</td>
</tr>
</tbody>
</table>

**PANEL C. INITIAL VOLATILITY = 25%**

The volatility parameter of the volatility diffusion (dV) is $b = 0.0001$, $S/X$ is the moneyness ratio, the interest rate is 5%, exercise price is $100$, and the time to maturity is 0.5 years. The values of delta, gamma, and vega are for European calls. Bivariate is the stochastic volatility model compared with 250 time steps.

The bivariate binomial model typically agrees with the Black-Scholes parameters to the second or third decimal place.

In Exhibit 1, deltas, vegas, and gammas are computed for a one-half year option with volatilities of 15%, 20%, and 25%, and interest rate of 5%, moneyness ($S/X$) ranging from 0.8 to 1.2, and a small value for the volatility parameter of the volatility diffusion ($b = 0.0001$). A small volatility parameter is chosen so we can compare the results with the corresponding parameters from the Black-Scholes model. As can be seen from the table, the estimates from the bivariate binomial model typically agree with the Black-Scholes parameters to the second or third decimal place.

Exhibit 2 depicts European values for a put with the stochastic volatility parameter $b = 25\%$ and zero correlation between volatility and price. The bivariate binomial values estimated with 270 time steps are compared with values generated by the Hull and White stochastic volatility model [1987] and the standard Black-Scholes model [1973].

For this case, the bivariate binomial values are
EXHIBIT 2
EFFECT OF STOCHASTIC VOLATILITY ON EUROPEAN PUT PRICES

<table>
<thead>
<tr>
<th>S/X</th>
<th>Bivariate Binomial</th>
<th>Hull-White</th>
<th>Black-Scholes</th>
</tr>
</thead>
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<td>0.80</td>
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<td>17.645</td>
<td>17.643</td>
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<tr>
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<td>13.878</td>
<td>13.876</td>
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<td>0.88</td>
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<td>10.397</td>
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<tr>
<td>0.92</td>
<td>7.361</td>
<td>7.364</td>
<td>7.365</td>
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<tr>
<td>0.96</td>
<td>4.998</td>
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<td>1.00</td>
<td>3.954</td>
<td>3.953</td>
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<tr>
<td>1.04</td>
<td>1.782</td>
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<td>1.784</td>
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<td>1.08</td>
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<td>0.975</td>
<td>0.975</td>
</tr>
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<td>0.501</td>
<td>0.500</td>
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<tr>
<td>1.16</td>
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<td>0.244</td>
<td>0.241</td>
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<tr>
<td>1.20</td>
<td>0.112</td>
<td>0.112</td>
<td>0.110</td>
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</table>

Bivariate binomial is a stochastic volatility model with 270 time steps. The volatility parameter of the volatility diffusion (\(b\)) is \(b = 25\%\), and there is zero correlation between price and volatility. Hull and White (1987) is the corresponding analytic stochastic volatility approximation, and Black and Scholes (1973) is the standard option pricing model with fixed volatility. European puts are priced with the parameters: risk-free rate = 5%, time to maturity = 0.5 years, stock volatility = 15%, and exercise price = $100.

For stochastic volatility models, the initial volatility, \(\sigma_0\), is equal to 15%, the stock volatility. There are no dividends.

economically indistinguishable from the Hull-White and Black-Scholes values. For example, deviations from Hull-White are less than one-tenth of one cent. Further, the Black-Scholes model differs from both of these models by less than one cent.

Exhibit 2 is constructed primarily to show the convergence of the bivariate binomial to the values produced by the Hull-White analytic approximations. These analytic approximations cannot be used, however, when there is non-zero correlation between stochastic volatility and price. Non-zero correlation causes no problem in bivariate binomial calculations.

Correlation is introduced in Exhibit 3. This exhibit gives the Black-Scholes implied volatility for a number of European calls calculated using bivariate binomial prices and maturities of 0.25 years (Panel A), 0.5 years (Panel B), and 0.75 years (Panel C). The initial volatility is assumed to be 15%, and there is no mean drift (so conditional expected volatility at expiration is also 15%).

The results compare very closely to those of Hull and White (1987), who use Monte Carlo approximation. Zero correlation (\(r = 0\)) gives essentially the same values as Black-Scholes; i.e., the implied volatilities are close to 15%.

Our results show that the Black-Scholes model significantly overprices out-of-the-money calls and underprices in-the-money calls when correlation is negative. For example, in Panel A the implied volatility in the ninety-day case at \(S/X = 0.9\) and \(r = -0.75\) is 12.9% compared to the initial and mean volatility of 15%. Thus, a 15% volatility in the Black-Scholes formula overprices the option relative to the bivariate (true) model. In-the-money calls are underpriced; e.g., \(r = -0.75\), and \(S/X = 1.1\) gives an implied volatility of 16.5%. The results are similar for longer-maturity options (Panels B and C), except that there is more overpricing for the at-the-money calls.

This example is also consistent with the conclusions of Hull and White (1987). Their insight is basically as follows: If a call (put) option is out of the money, its value is largely determined by fit tails in the right (left) side of the probability distribution of terminal asset price. For out-of-the-money calls whose underlying is negatively correlated with volatility, increases in price are associated with decreasing volatility. This contracts the right tail of the distribution and decreases call price relative to that of a constant-volatility model such as Black-Scholes.

Using the stochastic volatility model as the standard, this implies that Black-Scholes tends to overvalue out-of-the-money calls under negative correlation. By analogous reasoning, out-of-the-money puts under negative correlation tend to be underpriced by the Black-Scholes model. Therefore, by put-call parity, in-the-money calls tend to be undervalued. Exhibit 3 demonstrates further that for positive correlations Black-Scholes tends to overprice in-the-money calls and underprice out-of-the-money calls.

These ad hoc statements should be viewed with caution since we can ensure that underpricing occurs in a large region surrounding the at-the-money point (\(S/X = 1\)) if the volatility parameter of the volatility diffusion \(b\) is chosen to be sufficiently large. Thus in-, out-, and at-the-money puts and calls can be shown to
### Exhibit 3
**Effect of Correlation and Moneyness on Black-Scholes Implied Volatilities**

<table>
<thead>
<tr>
<th>S/X</th>
<th>Bivariate ρ = -0.75</th>
<th>Bivariate ρ = -0.50</th>
<th>Bivariate ρ = -0.25</th>
<th>Bivariate ρ = 0.00</th>
<th>Bivariate ρ = 0.25</th>
<th>Bivariate ρ = 0.50</th>
<th>Bivariate ρ = 0.75</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.90</td>
<td>0.129</td>
<td>0.137</td>
<td>0.145</td>
<td>0.151</td>
<td>0.157</td>
<td>0.163</td>
<td>0.168</td>
</tr>
<tr>
<td>0.95</td>
<td>0.138</td>
<td>0.142</td>
<td>0.146</td>
<td>0.149</td>
<td>0.152</td>
<td>0.155</td>
<td>0.158</td>
</tr>
<tr>
<td>1.00</td>
<td>0.148</td>
<td>0.148</td>
<td>0.148</td>
<td>0.149</td>
<td>0.149</td>
<td>0.149</td>
<td>0.149</td>
</tr>
<tr>
<td>1.05</td>
<td>0.156</td>
<td>0.154</td>
<td>0.152</td>
<td>0.149</td>
<td>0.146</td>
<td>0.143</td>
<td>0.139</td>
</tr>
<tr>
<td>1.10</td>
<td>0.165</td>
<td>0.161</td>
<td>0.156</td>
<td>0.151</td>
<td>0.145</td>
<td>0.139</td>
<td>0.131</td>
</tr>
</tbody>
</table>

**Panel A. Maturity = 0.25 Years**

|     |                     |                     |                     |                     |                     |                     |                     |
| 0.90| 0.126               | 0.135               | 0.143               | 0.150               | 0.156               | 0.161               | 0.166               |
| 0.95| 0.135               | 0.140               | 0.144               | 0.148               | 0.151               | 0.154               | 0.156               |
| 1.00| 0.145               | 0.146               | 0.147               | 0.147               | 0.147               | 0.147               | 0.147               |
| 1.05| 0.154               | 0.152               | 0.150               | 0.148               | 0.145               | 0.142               | 0.138               |
| 1.10| 0.162               | 0.158               | 0.154               | 0.149               | 0.144               | 0.137               | 0.130               |

**Panel B. Maturity = 0.5 Years**

|     |                     |                     |                     |                     |                     |                     |                     |
| 0.90| 0.124               | 0.133               | 0.141               | 0.148               | 0.154               | 0.160               | 0.165               |
| 0.95| 0.133               | 0.138               | 0.142               | 0.146               | 0.149               | 0.152               | 0.155               |
| 1.00| 0.142               | 0.141               | 0.145               | 0.146               | 0.146               | 0.146               | 0.145               |
| 1.05| 0.151               | 0.150               | 0.148               | 0.146               | 0.143               | 0.140               | 0.136               |
| 1.10| 0.159               | 0.156               | 0.152               | 0.148               | 0.142               | 0.136               | 0.128               |

**Panel C. Maturity = 0.75 Years**

All values are Black-Scholes implied volatilities where European prices are computed from the bivariate binomial model with 250 steps. Compare to Hull-White (1987) Exhibit 3 with values computed by Monte Carlo simulation. Option prices are for calls with no drift in volatility and price, no dividends, and stock volatility of 15%. The volatility parameter of the volatility diffusion (σV) is b = 1.00. The initial volatility, V0, is equal to 15%, the stock volatility.

Exhibit 3 shows underpricing for large values of b. This is explained in Exhibit 7.

Exhibit 4 gives European put option prices for long-maturity options (two years) under several correlation and moneyness scenarios. Comparison of the Black-Scholes and bivariate models reveals the expected effects. Specifically, the Black-Scholes model overvalues in-the-money European puts when correlation is negative and overvalues out-of-the-money puts when correlation is positive. In-the-money puts with positive correlations tend to be overvalued by Black-Scholes, while out-of-the-money puts with negative correlation show no clear pricing bias pattern in this example.

Exhibit 5 compares the univariate binomial model with the bivariate binomial for an American put option under a variety of correlations, maturities, and moneyness scenarios. These cases could not be addressed by Hull and White because Monte Carlo simulations are not appropriate for the evaluation of the early exercise premium found in American options.

The early exercise premium poses no problem for binomial techniques. At each node, a comparison of the early exercise value and continuation value (the discounted present value of expected node prices one step ahead) of the option is made. Rational agents...
EXHIBIT 4
EFFECT OF STOCHASTIC VOLATILITY FOR EUROPEAN PUTS (LONG MATURITIES)

<table>
<thead>
<tr>
<th>S/X</th>
<th>Black-Scholes</th>
<th>Bivariate ( \rho_w = -0.50 )</th>
<th>Bivariate ( \rho_w = -0.25 )</th>
<th>Bivariate ( \rho_w = 0.00 )</th>
<th>Bivariate ( \rho_w = 0.25 )</th>
<th>Bivariate ( \rho_w = 0.50 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.80</td>
<td>15.716</td>
<td>14.287</td>
<td>14.725</td>
<td>15.145</td>
<td>15.499</td>
<td>15.796</td>
</tr>
<tr>
<td>0.85</td>
<td>12.848</td>
<td>11.485</td>
<td>11.816</td>
<td>12.110</td>
<td>12.351</td>
<td>12.538</td>
</tr>
<tr>
<td>0.95</td>
<td>8.325</td>
<td>7.451</td>
<td>7.512</td>
<td>7.818</td>
<td>7.464</td>
<td>7.338</td>
</tr>
<tr>
<td>1.00</td>
<td>6.611</td>
<td>6.073</td>
<td>6.014</td>
<td>5.890</td>
<td>5.694</td>
<td>5.409</td>
</tr>
<tr>
<td>1.05</td>
<td>5.208</td>
<td>4.999</td>
<td>4.849</td>
<td>4.626</td>
<td>4.319</td>
<td>3.905</td>
</tr>
<tr>
<td>1.10</td>
<td>4.074</td>
<td>4.159</td>
<td>3.945</td>
<td>3.655</td>
<td>3.274</td>
<td>2.775</td>
</tr>
<tr>
<td>1.15</td>
<td>3.167</td>
<td>3.495</td>
<td>3.240</td>
<td>2.910</td>
<td>2.492</td>
<td>1.961</td>
</tr>
<tr>
<td>1.20</td>
<td>2.449</td>
<td>2.965</td>
<td>2.689</td>
<td>2.340</td>
<td>1.912</td>
<td>1.385</td>
</tr>
</tbody>
</table>

European put prices for long maturities computed by the bivariate and Black-Scholes models. The bivariate model uses 100 time steps. The risk-free rate is 5%, maturity is 2.0 years, stock volatility is 20%, the exercise price is $100, and the volatility parameter of the volatility diffusion (\( \Delta V \)) is \( \beta = 1.00 \). There are no dividends. The initial volatility, \( \sqrt{\nu}_0 \), is equal to 20%, the stock volatility.

would choose the maximum of these two values and act accordingly; i.e., they would exercise or continue to hold the option. Thus, the maximum of these two values is chosen to be the node value.

While important for valuation, the effect of early exercise on stochastic volatility does not appear to change the basic implications. We still find that the univariate binomial (fixed volatility) overprices in-the-money puts when there is negative correlation as well as out-of-the-money puts when there is positive correlation.

Put prices under a mean-reverting model of stochastic volatility are given in Exhibit 6. We assume a risk-neutralized model of the form

\[
dV = \kappa(\theta - \sqrt{V})dt + \beta VdZ_v
\]

where \( \kappa \) is the speed of convergence coefficient, and \( \theta \) is the long-run average value of \( \sqrt{V} \).

For European options, the bivariate model produces results well within the standard error of the Monte Carlo method. Comparison of Exhibit 6 to the results of Exhibit 2 illustrates the difference between option prices based on mean-reverting and geometric processes. For uncorrelated processes, the Black-Scholes at-the-money price is $3.06. The mean-reverting model prices the option at about $2.96. When mean reversion is present, the Black-Scholes model tends to overprice at-the-money options. Mispricing of this type is largely due to the reduction in the tail probabilities. The tendency for volatility to pull back to the mean causes a reduction in the tails and in average volatility of the option over its term to maturity.

The Smile

The volatility "smile" is widely accepted by practitioners. The smile describes the convex shape of the implied volatility with respect to moneyness (\( S/X \)) computed via Black-Scholes. Hull and White [1988] use Monte Carlo simulation to evaluate the effects of stochastic volatility and correlations on option prices. Their results are consistent with the smile when implied volatilities are computed by matching Black-Scholes prices to theoretically correct values.

Our simulations also reveal significant convexities, but the under- and overpricing conclusions for puts and calls depend on the assumed volatility parameter in the volatility diffusion. This point can be seen clearly by comparing Exhibits 7 and 8. Both graphs depict implied volatilities for weak correlations and initial stock volatility of 20%. The risk-free rate is 5%; the time to maturity of the put option is 0.5 years; and the volatility drift rate is zero.

In Exhibit 7, the volatility parameter is \( \beta = 250\% \). All points below the implied sigma of 20%
represent overpricing by Black-Scholes. So, for a wide range of correlations and moneyness levels, in-the-money, at-the-money, and out-of-the-money puts are all overpriced. Puts are undervalued by Black-Scholes for all correlations at extreme values of moneyness, e.g., $S/X = 0.8$ or $S/X = 1.20$.

Exhibit 8 is a smile generated from the parameters given in Exhibit 7, except that the volatility parameter is $b = 50\%$. From this graph, we observe the usual phenomenon that far out-of-the-money puts are generally undervalued under negative correlation, while far in-the-money puts are generally undervalued for positive correlations. Furthermore, out-of-the-money puts are overvalued for positive correlations, and in-the-money puts are overvalued for negative correlations.

In general, the results from the bivariate binomial model are completely consistent with the concept of the smile and the set of results for European options given in Hull and White [1988]. The bivariate binomial model used here can also be used to produce the smile for American option prices.

**Convergence and Computational Considerations**

Accurate and timely computations on the

---

**EXHIBIT 5**

**EFFECT OF CORRELATION AND MONEYNESS ON UNIVARIATE AND BIVARIATE AMERICAN PUT PRICES**

<table>
<thead>
<tr>
<th>$S/X$</th>
<th>Univariate $\rho = -0.50$</th>
<th>Bivariate $\rho = -0.50$</th>
<th>Bivariate $\rho = -0.25$</th>
<th>Bivariate $\rho = 0.00$</th>
<th>Bivariate $\rho = 0.25$</th>
<th>Bivariate $\rho = 0.50$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.80</td>
<td>20,000</td>
<td>20,000</td>
<td>20,000</td>
<td>20,000</td>
<td>20,000</td>
<td>20,000</td>
</tr>
<tr>
<td>0.85</td>
<td>15,020</td>
<td>15,000</td>
<td>10,500</td>
<td>10,603</td>
<td>10,767</td>
<td>10,863</td>
</tr>
<tr>
<td>0.90</td>
<td>10,668</td>
<td>10,429</td>
<td>8,701</td>
<td>8,910</td>
<td>9,072</td>
<td>9,180</td>
</tr>
<tr>
<td>0.95</td>
<td>7,226</td>
<td>6,992</td>
<td>7,081</td>
<td>7,160</td>
<td>7,228</td>
<td>7,285</td>
</tr>
<tr>
<td>1.00</td>
<td>4,649</td>
<td>4,566</td>
<td>4,570</td>
<td>4,565</td>
<td>4,549</td>
<td>4,523</td>
</tr>
<tr>
<td>1.05</td>
<td>2,864</td>
<td>2,935</td>
<td>2,865</td>
<td>2,785</td>
<td>2,688</td>
<td>2,581</td>
</tr>
<tr>
<td>1.10</td>
<td>1,675</td>
<td>1,872</td>
<td>1,763</td>
<td>1,642</td>
<td>1,504</td>
<td>1,352</td>
</tr>
<tr>
<td>1.15</td>
<td>0,927</td>
<td>1,195</td>
<td>1,076</td>
<td>0,946</td>
<td>0,806</td>
<td>0,651</td>
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<tr>
<td>1.20</td>
<td>0,500</td>
<td>0,766</td>
<td>0,656</td>
<td>0,539</td>
<td>0,419</td>
<td>0,293</td>
</tr>
</tbody>
</table>

**Panel A. Maturity = 0.5 Years**

<table>
<thead>
<tr>
<th>$S/X$</th>
<th>0.80</th>
<th>0.85</th>
<th>0.90</th>
<th>0.95</th>
<th>1.00</th>
<th>1.05</th>
<th>1.10</th>
<th>1.15</th>
<th>1.20</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.80</td>
<td>20,083</td>
<td>20,000</td>
<td>20,000</td>
<td>20,000</td>
<td>20,145</td>
<td>20,321</td>
<td>20,517</td>
<td>20,517</td>
<td>20,517</td>
</tr>
<tr>
<td>0.85</td>
<td>15,915</td>
<td>15,418</td>
<td>15,608</td>
<td>15,887</td>
<td>16,549</td>
<td>16,943</td>
<td>17,360</td>
<td>17,360</td>
<td>17,360</td>
</tr>
<tr>
<td>0.90</td>
<td>12,990</td>
<td>11,746</td>
<td>12,060</td>
<td>12,327</td>
<td>12,843</td>
<td>12,463</td>
<td>12,699</td>
<td>12,699</td>
<td>12,699</td>
</tr>
<tr>
<td>0.95</td>
<td>9,882</td>
<td>9,111</td>
<td>9,308</td>
<td>9,455</td>
<td>9,523</td>
<td>9,572</td>
<td>9,772</td>
<td>9,772</td>
<td>9,772</td>
</tr>
<tr>
<td>1.00</td>
<td>7,712</td>
<td>7,179</td>
<td>7,226</td>
<td>7,219</td>
<td>7,151</td>
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</tr>
<tr>
<td>1.05</td>
<td>6,002</td>
<td>5,754</td>
<td>5,672</td>
<td>5,526</td>
<td>5,338</td>
<td>5,098</td>
<td>5,098</td>
<td>5,098</td>
<td>5,098</td>
</tr>
<tr>
<td>1.10</td>
<td>4,650</td>
<td>4,687</td>
<td>4,511</td>
<td>4,264</td>
<td>3,930</td>
<td>3,488</td>
<td>3,488</td>
<td>3,488</td>
<td>3,488</td>
</tr>
<tr>
<td>1.15</td>
<td>3,581</td>
<td>3,874</td>
<td>3,637</td>
<td>3,328</td>
<td>2,923</td>
<td>2,405</td>
<td>2,405</td>
<td>2,405</td>
<td>2,405</td>
</tr>
<tr>
<td>1.20</td>
<td>2,735</td>
<td>3,245</td>
<td>2,973</td>
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<td>2,195</td>
<td>1,654</td>
<td>1,654</td>
<td>1,654</td>
<td>1,654</td>
</tr>
</tbody>
</table>

American put prices computed by the bivariate and univariate binomial models. Both models use 100 time steps. The risk-free rate is 5%, stock volatility is 20%, the exercise price is $100$, and the volatility parameter of the volatility diffusion ($\sigma^2$) is $b = 1.00$. There are no dividends. The initial volatility, $\sqrt{\sigma^2_0}$, is equal to 20% the stock volatility.
**EXHIBIT 6**

**Put Prices Under Mean Reversion and Correlation**

\[ dV = \kappa(\Theta - \sqrt{V})dt + \mu VdZ_c \]

<table>
<thead>
<tr>
<th>European</th>
<th>European</th>
<th>American</th>
</tr>
</thead>
<tbody>
<tr>
<td>125 Steps</td>
<td>250,000 Trials</td>
<td>125 Steps</td>
</tr>
<tr>
<td>Bivariate</td>
<td>Monte Carlo</td>
<td>Bivariate</td>
</tr>
<tr>
<td>Binomial</td>
<td>Binomial</td>
<td></td>
</tr>
</tbody>
</table>

**Panel A. \( \rho_{V} = -0.75 \)**

<table>
<thead>
<tr>
<th>( S/X )</th>
<th>0.90</th>
<th>0.95</th>
<th>1.00</th>
<th>1.05</th>
<th>1.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.467</td>
<td>8.473</td>
<td>8.769</td>
<td>3.233</td>
<td>1.782</td>
<td>0.977</td>
</tr>
</tbody>
</table>

**Panel B. \( \rho_{V} = -0.50 \)**

<table>
<thead>
<tr>
<th>( S/X )</th>
<th>0.90</th>
<th>0.95</th>
<th>1.00</th>
<th>1.05</th>
<th>1.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.582</td>
<td>8.583</td>
<td>8.844</td>
<td>3.231</td>
<td>1.721</td>
<td>0.900</td>
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</table>

**Panel C. \( \rho_{V} = -0.25 \)**

<table>
<thead>
<tr>
<th>( S/X )</th>
<th>0.90</th>
<th>0.95</th>
<th>1.00</th>
<th>1.05</th>
<th>1.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.687</td>
<td>8.697</td>
<td>8.913</td>
<td>3.224</td>
<td>1.653</td>
<td>0.815</td>
</tr>
</tbody>
</table>

**Panel D. \( \rho_{V} = 0.00 \)**

<table>
<thead>
<tr>
<th>( S/X )</th>
<th>0.90</th>
<th>0.95</th>
<th>1.00</th>
<th>1.05</th>
<th>1.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.785</td>
<td>8.794</td>
<td>9.074</td>
<td>3.211</td>
<td>1.578</td>
<td>0.732</td>
</tr>
</tbody>
</table>

Put prices by moneyness and correlation. The exercise price is $100, \( \Theta = 0.15 \), \( \kappa = 5 \), risk-free rate = 5%, maturity = 0.5 years, and stock volatility = 15%. The volatility parameter of the diffusion process \( dV = b_{t} \) is \( b = 1.50 \). The initial volatility, \( \sqrt{V_0} \), is equal to 15%, the stock volatility. There are no dividends.

**EXHIBIT 7**

**Volatility Smile — Large Volatility Parameter**

\( b = 250% \)

**EXHIBIT 8**

**Volatility Smile — Small Volatility Parameter**

\( b = 50% \)

European puts are priced by the bivariate binomial model with 250 time steps. The exercise price is $100, the risk-free rate is 5%, the time to maturity is 0.5 years, the volatility drift is zero, and stock volatility is 20%. The volatility parameter of the diffusion process \( dV = b_{t} \) is \( b = 250% \). The initial volatility, \( \sqrt{V_0} \), is equal to 20%, the stock volatility.
transformed grid depend in large measure on the stability of the process and the speed of convergence. The jump process is defined on \( Q \times Y \) space with jump size always the same, \( \sqrt{\Delta t} \). Generally, the transformed values are reasonable in magnitude.

For example, for a 180-step tree and a one-year option, the change in the value of \( Q \) and \( Y \) when all jumps are positive is \( \sqrt{\Delta t} \times 180 = 13.416 \). When the inverse transforms are applied, however, the value of the underlying, \( S \), can be on the order of magnitude \( 10^9 \) or larger.

The most common problems emerge when the asset price is high (> 100), the volatility of the volatility process is large (\( b > 50\% \)), or the options have a long time to maturity. These problems are largely avoidable by appropriate scaling of the price of the underlying asset. Specifically, we adjust the asset and strike price to a new value and maintain well-behaved values in the tree. Option values are then obtained by rescaling the option price obtained from the tree.

Consider, for example, an at-the-money option with an underlying asset price of $100. One might consider dividing stock and strike price by $100 and valuing the option for these scaled values. The option price is then the scaled option price times $100.3

Convergence is guaranteed when the individual jump probabilities remain non-negative and all four jointly sum to one (see Ames [1977]). To provide the quickest convergence, it is postulated that the probabilities of each jump in a branching process should be approximately the same.

Boyle [1988] modifies his trinomial process using a scaling factor selected so that the probabilities of jumping to each of the three branch nodes are approximately one-third. We choose the scaling factor, \( \lambda \), to minimize the deviations from the merit function:

\[
\min_{\lambda} \sum_{i=1}^{4} (p_i - 0.25)^2
\]

Generally, the solution requires that a value of \( \lambda \) be chosen so that the scaled value of \( S \) is close to one. This solution gives values of \( p_i \) near 0.25 and a merit function that approaches zero.

In the one-dimensional case, Carr and Fagnier [1994] take advantage of the oscillatory nature of the convergence to improve their solution. Their technique involves averaging the prices produced by a lattice with an even number of time steps (\( n = 2m \)) and an odd number of time steps (\( n = 2m + 1 \)). As can be seen in Exhibit 9, the average value is fairly stable after about 100 steps.

We apply this technique in two dimensions. The average smoothes out the oscillations and is within one-fifth of a cent of the 700/701 step average at only 100/101 steps (the notation 700/701 means option value is computed by taking the simple average of two bivariate trees; one tree uses 700 steps, and the other 701 steps). This simple technique has a high payoff for the bivariate model. The simple average takes about twice as long to compute as a single value, but doubling the time steps results in an increase of approximately 25% in the amount of CPU time required for bivariate computations.

Typical bivariate binomial run times are shown in Exhibit 10. There is considerable time savings for the average method. The time required for a 700-step solution is about 1,300 times that required to compute the 50/51 average. The price of the 50/51 average is in error by less than one-half cent from the 700/701 answer. This error may be acceptable, depending on the application.

**EXHIBIT 9**

**CONVERGENCE OF THE BIVARIATE BINOMIAL PRICES FOR AMERICAN PUTS**

![Graph showing convergence](image)

The exercise price = stock price is $100, the risk-free rate is 5%, the time to maturity is 0.25 years, stock volatility is 20%, and the volatility parameter of the volatility diffusion \( \sigma \) is \( b = 0.30 \). The initial volatility, \( \sigma_0 \), is 20%, the stock volatility. There are no dividends, and the correlation between stochastic volatility and stock price is zero.
VI. CONCLUSIONS

We develop a stochastic volatility pricing model that is both simple and accurate. It can be used to price the great majority of equity, commodity, and currency options sold on the exchanges and OTC. The basis of the model is a lattice formed from a possibly correlated volatility process and an underlying price process. These processes are then transformed to form a recombining bivariate tree with attractive convergence properties. The lattice, the transformations, and the node probabilities complete the model.

Simulations show that the values given by the bivariate binomial model are economically identical to those reported in Hull and White [1987]. Unlike the Hull and White model, the methods developed here are also appropriate for non-zero correlations and American options. In addition, an effective and accurate technique is developed to obtain estimates of delta, vega, and gamma directly from the bivariate grid.

EXHIBIT 10
EXponential INCREASE in CPU Times for American Put Prices

Calculations made on 486DX2-66 machine. The exercise price is $100, the risk-free rate is 5%, the time to maturity is 0.25 years, stock volatility is 20%, and the volatility parameter of the volatility diffusion (α) is 0.30. The initial volatility, $\sqrt{V_0}$, is 40%, the stock volatility. There are no dividends, and the correlation between stochastic volatility and stock price is zero.

APPENDIX A
SECOND-ORDER DERIVATIVES, DRIFT TERMS, AND TRANSFORMATIONS

Second-Order Derivatives

The transformation $H = H(S, V)$ is defined by:

$$H(S, V) = h^{-1}(V) \frac{dS}{d(S)}$$

(A-1)

Further, let $h(V) = V^\rho$ hereafter, because this functional form is necessary for the transformations developed in the article. Since $\sigma = f(V)^\rho$, derivatives are given by

$$H_s = \frac{1}{\sigma} \frac{\partial V}{\partial S}$$

(A-2)

$$H_v = -\frac{\alpha H}{V}$$

where $\rho = \rho_v$.

$$H_{vv} = \frac{\alpha^2 H}{V^2}$$

(A-3)

The derivatives of $Q = Q(H)$ [see Equation (9)] are:

$$Q_h = \frac{1}{\sigma_h}$$

A-4

$$Q_{hh} = \frac{\alpha^2 H}{\sigma_h^2}$$

where $\rho = \rho_h$.

Drift Terms

Using Itô’s theorem, the equations above, and the exogenously specified risk-neutralized drift terms for $S$ and $V$, $\mu_s$ and $\mu_v$, respectively, we obtain these expressions for the instantaneous drifts of $H$, $Q$, and $V$:

$$m_h = m_s - \frac{\alpha H}{V} - 0.5V^\rho +$$

$$0.5\alpha^2 H(1 + \alpha)^2 - \alpha \rho$$

(A-6)

$$m_v = m_v = \frac{\alpha H}{\sigma_h^2}$$

(A-7)

$$m_{vv} = \frac{5\alpha^2 H}{\sigma_h^2}$$

(A-8)

Transforming from $H$ to $Q$

Let $H = H(S, V)$ to that

$$dH = Hf(S)h(V)dZ_s + H_v VdZ_v + m_h dt$$

(A-9)

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where subscripts $s$ and $v$ on $H$ denote partial derivatives, and $m_s$ is the drift of the $H$ process and depends on $m_v$, $m_s$, and second-order partials. Following the method of Nelson and Ramaswamy [1990], consider the transformation

$$H(s, V) = h(V)^{-1/2} f(y)^{-1} dy$$

(A-16)

with associated first partials $H_s$ and $H_v$. Using the partials in Equation (A-9) gives:

$$dH = m_s dt + dZ_s - h_s^i \text{d}H dZ_v$$

(A-17)

which has instantaneous volatility

$$\sigma_s = [1 - 2h_s h_v^i \text{d}H h_v^i + (h_s h_v^i \text{d}H h_v^i)^2]^{1/2}$$

(A-18)

where $\rho_{sv}$ is the constant correlation between $dZ_s$ and $dZ_v$. The expression in Equation (A-12) is crucial. If $H$ is to be transformed farther to give constant volatility, it is sufficient for the volatility of $h$ to be a function of $H$ only since this allows us to apply the Nelson and Ramaswamy [1990] transformation to give a process with constant volatility.

From Equation (A-12), it follows that $\sigma_s$ will be a function of $H$ if:

$$h_s h_v^i V = \alpha$$

(A-19)

where $\alpha$ is a constant. This requires that

$$\int h_s / h = \alpha \int dV / V,$$

or

$$h(V) = V^\alpha \exp(K)$$

(A-20)

so that $h_s h_v^i V = \alpha V^{-1}$, where $K$ is an arbitrary constant.

Thus, the volatility expression in Equation (A-12) is now written:

$$\sigma_s = [1 - 2\alpha h^i H + (\alpha V^2)^{1/2}]$$

(A-21)

so that the transformed diffusion can be written:

$$dH = m_s dt + [1 - 2\alpha h^i H + (\alpha V^2)^{1/2}] dz_s$$

(A-22)

By equating the random components of (A-11) and (A-17), it follows that

$$dZ_s = (dS - \alpha h H dz_s) / \sigma_s$$

(A-23)

so that the correlation between $dZ_s$ and $dZ_v$ is now dependent on the transformed variable $H$. Specifically, the correlation is

$$\text{Corr}(dZ_s, dZ_v) = \rho_{sv} - \alpha h H / \sigma_s$$

(A-24)

Transforming $H$ to Constant Volatility

A straightforward application of Nelson and Ramaswamy gives a transformation from $H$ to $Q$ of the form

$$Q = \int^t \sigma_s^{-1}(y) dy$$

(A-25)

$$= \left(\alpha h^i \ln(\alpha h H - \rho_{sv} + \sigma_s) \right)^{1/2}$$

(A-26)

with associated diffusion

$$dQ = m_s dt + dZ_s$$

(A-27)

To verify that $Q$ has constant volatility as in (A-21), compute the diffusion

$$dQ = Q_h dz_v$$

(A-28)

where $Q_h = \sigma_s^{-1}$ so that Equation (A-21) reduces to

$$dQ = m_s dt + dZ_s$$

(A-29)

using Equations (A-11) and (A-16).

Partial Derivatives for Delta, Vega, and Gamma

The results below are needed to compute numerical estimates of delta, vega, and gamma from the grid (let $\alpha = 0.5$, $\theta = 1$, and $\sigma = \sqrt{V}$):

$$Q_0$$ [see Equation (A-5)]

(A-30)

$$Q_{bh}$$ [see Equation (A-5)]

(A-31)

$$H_1 = 1/\sigma$$

(A-32)

$$H_2 = -1/(2\sigma^2)$$

(A-33)

$$H_3 = -\ln(\theta)/\sigma^3$$

(A-34)

$$Y_s = 2/(b\sigma)$$

(A-35)

$$Q = Q_h H_s$$

(A-36)
\[ Q_0 = Q_0 H_0 \quad (A-31) \]
\[ Q_n = Q_n H_n + Q_{n+1} H_{n+1}^2 \quad (A-32) \]
\[ C_n = F_n Q_n + F_n Y_n \quad (A-33) \]
\[ C_n = F_n C_n + F_n Y_n \quad (A-34) \]

**APPENDIX B**

**JUMP PROCESS CONVERGENCE TO LOCAL MEANS AND SECOND-ORDER MOMENTS**

The means and second-order moments of the discrete processes match those of the diffusion processes. It is not necessary to consider the jump process for \( dV \) since it is the standard univariate process whose convergence is shown elsewhere by, for example, Nelson and Ramaswamy [1990]. Remembering that \( S = S(Y, Q) \) and \( V = V(Y) \), we must show that the limiting binomial moments match the expected value of the diffusion moments:

\[ E[dS] = [S(Y, Q) + S(q) + 0.5(S_{yy} + S_{qy}) + 2S_{qy} Corr(dY, dQ)] dt \quad (B-1) \]
\[ E[dS^2] = [S(Y, Q) + S(q) + 2S_{qy} Corr(dY, dQ)] dt, \quad (B-2) \]
\[ E[dSdV] = V[S(Y, Q) + S(q) Corr(dY, dQ)] dt \quad (B-3) \]

Equation (B-3) follows from the fact that the local volatility of \( Y \) and \( Q \) is one, so that \( Cov(dY, dQ) = Corr(dY, dQ) dt \).

Consider first the convergence of \( E[\Delta S] \) to \( E[dS] \). The binomial jump process of \( \Delta S \) is defined as:

\[ \Delta S^{\Delta S} = [S(Y \pm \sqrt{\Delta t}), H(Q \pm \sqrt{\Delta t})] - [S(Y), H(Q)] \quad (B-4) \]

where the first superscript on \( \Delta S \) corresponds to changes in \( Y \), and the second superscript corresponds to changes in \( Q \).

The probability assignments are:

\[ P_{11} = \text{Prob}(\Delta S^{\Delta S}) \]
\[ P_{12} = \text{Prob}(\Delta S^{\Delta S'}) \]
\[ P_{21} = \text{Prob}(\Delta S^{\Delta S'}) \]
\[ P_{22} = \text{Prob}(\Delta S^{\Delta S''}) \quad (B-5) \]

The rest of the exercise is primarily algebraic. First, \( \Delta S \) is expanded in a Taylor series about \( Y \) and \( Q \); expected values are calculated. The initial expansion is of the form

\[ \Delta S^{\Delta S} = \sqrt{\Delta t} \left( S_y \pm S_q \right) + \] 

\[ 0.5 \Delta t (S_{yy} + S_{qy} \pm 2S_{qy}) + o(\Delta t^{3/2}) \quad (B-6) \]

where the \( \pm \) preceding \( S_y \) is positive for \( \Delta S^{\Delta S} \) and \( \Delta S^{\Delta S'} \) and negative otherwise. Using the definition of expected values and collecting coefficients gives

\[ E[\Delta S] = \sqrt{\Delta t} \left( g_y S_y + g_q S_q \right) + \]

\[ 0.5 \Delta t (g_{yy} + g_{qy} \pm 2g_{qy}) + o(\Delta t^{3/2}) \quad (B-7) \]

where \( g_y = P_{11} + P_{12} \), \( g_q = P_{11} - P_{12} \), \( g_{yy} = P_{11} - P_{12} - P_{21} + P_{22} \), \( g_{qy} = P_{11} + P_{12} - P_{21} - P_{22} \) using Equations (19), (20), (27) through (30), and (B-7) gives

\[ E[\Delta S] = \Delta t \left( S_{yy} + S_{qy} \pm 2S_{qy} \text{ Corr}(dY, dQ) \right) + o(\Delta t^{3/2}) \quad (B-8) \]

since \( \Delta t = 0.5 + o(\sqrt{\Delta t}) \). Thus, \( E[\Delta S] = E[dS] \) as \( \Delta t \) approaches zero so that the mean of the binomial jump process matches that of the diffusion [see Equation (B-1)].

Next, we show that second moments match, i.e., that \( E[\Delta S^2] \) approaches \( E[dS^2] \) as \( \Delta t \) approaches zero. For conciseness, we define \( a = S_y + S_q \), and \( b = S_y - S_q \). Proceeding as before, and after considerable algebra, we get

\[ E[\Delta S^2] = \Delta t \left( a^2 (P_{11} + P_{12}) + b^2 (P_{11} - P_{12}) + o(\Delta t^{3/2}) \right) \quad (B-9) \]

but from the definitions of \( a \) and \( b \), and Equations (19), (30), and (27) through (30):

\[ E[\Delta S^2] = \Delta t \left( a^2 S_{yy} + S_{qy} \pm 2S_{qy} \text{ Corr}(dY, dQ) \right) + o(\Delta t^{3/2}) \quad (B-10) \]
which is identical to \( E[dS^2] \), Equation (B-2), as \( \Delta t \) approaches zero.

Finally, we show that \( E[\Delta S \Delta V] \) converges to \( E[dSdV] \), where jumps in \( dV \) are defined by \( dV^2 = V(Y - \sqrt{\Delta t}) - V(Y) \). Omitting several steps, the expected joint product is

\[
E[\Delta S \Delta V] = \frac{\partial}{\partial Y} \right( \Delta t + o(\Delta t^{3/2}) \right) \tag{B-11}
\]

and, using Equations (19), (20), and (27) through (30),

\[
E[\Delta S \Delta V] = \\
V[Y_1 + Y_2 \Corr(Y, Q)\Delta t + o(\Delta t^{3/2})] \tag{B-12}
\]

which is identical to \( E[dSdV] \), Equation (B-3), as \( \Delta t \) approaches zero.

END NOTES

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1. Alternatively, Equation (1) could be the Cox-Ingersoll-Ross [1985] diffusion for short rates when \( \alpha = 0.5 \) and \( \beta = 0 \). In this case the diffusion on \( V \) could represent the price of the underlying asset. This latter case has already been addressed in the literature.

2. To show that the error in the approximation of \( \delta \Corr(Q) \) is of first order, first expand \( F^* \), \( F^+ \), \( F^- \), and \( F \) in a Taylor series about the point \( F(Q, V) \). Adding the first two expressions and subtracting the last two results in an expression of the form \( 4 \delta \Corr(Q) = (F^+ - F^-) + o(\Delta t) \). Similarly, the result for the delta of \( Y \) is obtained by adding the expressions for \( F^+ \) and \( F^- \) and subtracting the expressions for \( F^* \) and \( F^+ \).

3. Scaling asset price and exercise price is permissible when option price is linear-homogeneous in the asset price and exercise price (Merton [1973]). The price function is linear-homogeneous if the payoff function is linear-homogeneous and the fundamental partial differential equation is left unchanged under the substitution \( S = kX \), where \( S \) is asset price and \( k \) is a constant. For calls and puts on traded assets, the payoff function is linear-homogeneous of degree one.

Furthermore, the fundamental partial differential equation is linear-homogeneous when the volatility of \( dS/S \) is independent of \( S \). In the applications here, this requires that \( f(S) = S \) [see Equation (1)], that volatility (\( \sigma \)) is exogenous to \( S \), and that the correlation \( \rho_{u,v} \) be independent of \( S \). These conditions are satisfied in most applications and in the examples we give. The local volatility of \( S \) in our model is \( f(S) \sqrt{V} \), so the condition \( f(S) = S \) would not be satisfied for the generalized constant elasticity of variance process, for example.

REFERENCES


