RANDOM ITERATION IN $\mathbf{P}^k$

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Abstract. We develop a pluripotential theory for random iteration on $\mathbf{P}^k$. We show the existence of a positive closed $(1,1)$ current and of a measure on $\mathbf{P}^k$ which are invariant, in a certain sense, and which attract all positive closed $(1,1)$ currents and all measures, respectively, under normalized pull-back and averaging by the maps. Thus the concept of an exceptional set disappears as soon as we allow a slight amount of randomness in our system. We also consider the problem of push-forward of measures, and describe certain limiting measures in this case also, supported near the attractors for the perturbed map.

1. Introduction

In this paper we consider a dynamical system formed by perturbations of a holomorphic map $f_0 : \mathbf{P}^k \to \mathbf{P}^k$. It is known that for $0 \leq i \leq k$ there exists an invariant, closed, positive, current of bidegree (i,i) on $\mathbf{P}^k$, which attracts most closed, positive bidegree (i,i) currents under the operation of normalized pullback by f_0. Explaining what is meant by "most" in this sentence gives rise to the study of the so-called exceptional set of f_0. See for example [3], [9]. The success of this theory results from the fact that f_0 is generically many-to-one, so that the operation of normalized pull-back has a good chance of stabilizing to something which is independent of the initial current. Since points only have one forward image, however, the operation of pushing forward converges to something which is highly dependent on the initial current.

In sections two and three of this paper, we consider an *average* pull-back of closed, positive $(1,1)$ currents and of (k,k) currents (or measures), respectively, over a certain family of perturbations of f_0. We show that there exists a unique limit current in each case, with no exceptional set. Moreover these limit currents converge to their usual counterparts for the single map f_0 as the family of perturbations degenerates to f_0. In section four, we give a fuller description of the limit $(1,1)$ current in terms of normality. In section five, we consider average push-forwards of measures (a special case of a *diffusion;* see [8] and [4]), and show that the limit measures obtained in this case are supported near the *attractors* for f_0. We give some quantitative description of these limit measures.

Let

$$\{f_\lambda\}_{\lambda \in B(0,\delta)}$$

be a family of holomorphic maps of degree $d > 1$ on $\mathbf{P}^k$ varying holomorphically in the parameter $\lambda \in B(0,\delta) \subset \mathbf{C}^k$. We also assume that the maps $\lambda \to f_\lambda(z)$ are finite (i.e., finite-to-one) and hence open. Occasionally we will use the notation $f(\lambda, z)$ for

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$f_\lambda(z)$. Shrinking δ if necessary, we may assume that $\partial f/\partial\lambda$ is uniformly bounded in $B(0,\delta)\times\mathbf{P}^k$, and thus shrinking δ further, we may assume that for each $z\in\mathbf{P}^k$, $f(\cdot,z)(B(0,\delta))$ is relatively compact in some copy of $\mathbf{C}^k\subset\mathbf{P}^k$. We will denote by $\tilde{f}_\lambda$ a holomorphic family of homogeneous liftings of the f_λ to $B(0,\delta)\times\mathbf{C}^{k+1}$.

Let ν be normalized Lebesgue measure on $B(0,\delta)$; then

$$\overline{\nu}:=\nu\times\nu\times\cdots\times\nu\times\dots$$

is a probability measure on

$$X:=\pi_{i=1}^{\infty}B(0,\delta).$$

We give X the product topology. Note that open sets in X then have positive measure with respect to $\overline{\nu}$. Given $\Lambda:=(\lambda_1,\lambda_2,\dots)\in X$ and $\lambda\in B(0,\delta)$, define

$$\lambda\Lambda=(\lambda,\lambda_1,\lambda_2,\dots)\in X.$$

For $\Lambda=(\lambda_1,\lambda_2,\dots)$, set

$$F_{\Lambda,n}=f_{\lambda_n}\circ\cdots\circ f_{\lambda_1}.$$

Let $\tilde{F}_{\Lambda,n}$ denote homogeneous polynomial lifts $\tilde{f}_{\lambda_n}\circ\cdots\circ\tilde{f}_1$ of $F_{\Lambda,n}$, respectively, to $\mathbf{C}^{k+1}\setminus\{0\}$.

2. Convergence of positive, closed (1, 1) currents

For each $\Lambda\in X$, define $G_n(\Lambda,\cdot):\mathbf{C}^{k+1}\to\mathbf{R}$ by

$$G_n(\Lambda,z)=\frac{1}{d^n}\log|\tilde{F}_{\Lambda,n}(z)|,$$

and let

$$G=\lim_n G_n.$$

Since the maps $\tilde{f}_\lambda$ are homogeneous, there is a constant C independent of λ such that

$$\frac{1}{C}|z|^d\le|\tilde{f}_\lambda(z)|\le C|z|^d.$$

Thus

$$\frac{1}{C}|\tilde{F}_{\Lambda,n}(z)|^d\le|\tilde{F}_{\Lambda,n+1}(z)|\le C|\tilde{F}_{\Lambda,n}(z)|^d.$$

Taking logarithms, we get

$$|G_{n+1}(\Lambda,z)-G_n(\Lambda,z)|\le\frac{\log C}{d^{n+1}}.$$

Thus

$$|G_{n+q}(\Lambda,z)-G_n(\Lambda,z)|\le\frac{2\log C}{d^{n+1}}.$$

So $G_n\to G$ uniformly. Thus G is plurisubharmonic and continuous in $X\times\mathbf{C}^{k+1}$. More precisely, let $f_i:\Delta\to B(0,\delta)$, $\phi:\Delta\to\mathbf{C}^{k+1}$ be holomorphic on the uniut disc Δ, then $G((f_i(z)),\phi(z))$ is a continuous subharmonic function on Δ. Let $G_\Lambda=G(\Lambda,\cdot)$. Clearly each G_Λ satisfies

$$G_\Lambda(cz)=\log|c|+G_\Lambda(z)$$

for $c\in\mathbf{C}$, and

$$G_\Lambda\circ\tilde{f}_\lambda=d\cdot G_{\Lambda'},$$

where $\Lambda'=\lambda\Lambda$.

It follows from Theorem 5.9 of [3] that there exists a unique positive closed (1,1) current T_Λ in $\mathbf{P}^k$, of total mass one, such that $\pi^*T_\Lambda=dd^cG_\Lambda$ where $\pi:\mathbf{C}^{k+1}\setminus\{0\}\to$

$\mathbf{P}^k$ is the canonical projection. Locally, if s is a holomorphic section of π, we have $T_\Lambda = dd^c(G_\Lambda \circ s)$. T_Λ depends continuously on Λ in the weak topology of currents, since $G_\Lambda \circ s$ and $G_{\Lambda'} \circ s$ are locally uniformly close if Λ and Λ' are close in the topology on X. Furthermore, by the Chern-Levine-Nirenberg (CLN) estimate ([2]), T_Λ puts no mass on locally pluripolar sets, since it has a locally bounded potential $G_\Lambda \circ s$.

Now, define

$$EG(z) = \int_X G_\Lambda(z) d\overline{\nu}(\Lambda).$$

Proposition 1. *EG is plurisubharmonic and continuous on* $\mathbf{C}^{k+1} \setminus \{0\}$.

Proof. Given $z \in \mathbf{C}^{k+1} \setminus \{0\}$, let Δ be any small linear disk centered at z. Since each G_Λ satisfies the sub-mean value property, we have

$$\begin{aligned} EG(z) &= \int_X G_\Lambda(z) \\ &\leq \int_X \frac{1}{|\partial\Delta|} \int_{\partial\Delta} G_\Lambda(\zeta) \\ &= \frac{1}{|\partial\Delta|} \int_{\partial\Delta} \int_X G_\Lambda(\zeta) \\ &= \frac{1}{|\partial\Delta|} \int_{\partial\Delta} EG(\zeta), \end{aligned}$$

where we have applied Fubini's Theorem to obtain the second-to-last equality. This is justified, since G is continuous in $X \times (\mathbf{C}^{k+1} \setminus \{0\})$, so that if V is an open set in $\mathbf{R}$, $G^{-1}(V)$ is open and hence clearly measurable with respect to $\overline{\nu} \times m$. Thus EG is plurisubharmonic. To obtain the continuity, note that if $z_n \to z$, then

$$\int_X G_\Lambda(z_n) \to \int_X G_\Lambda(z)$$

by the Lebesgue convergence theorem. □

Note that EG satisfies $EG(cz) = \log|c| + EG(z)$ for $c \in \mathbf{C} \setminus \{0\}$. Let T be the unique positive closed (1,1) current on $\mathbf{P}^k$ satisfying $\pi^* T = dd^c(EG)$ (see Theorem 5.9, [3]). Again T has mass one, and puts no mass on locally pluripolar sets, since it has a locally bounded potential.

In general, given a current S_Λ on $\mathbf{P}^k$ varying continuously with $\Lambda \in X$ and with uniformly bounded mass, we define the current $\int_X S_\Lambda$ by

$$< \int_X S_\Lambda, \phi >= \int_X < S_\Lambda, \phi > d\overline{\nu}.$$

Note that, if f is a submersion and ϕ a test form,

$$\begin{aligned} < f^* \int_X S_\Lambda, \phi >: &=< \int_X S_\Lambda, f_*\phi > \\ &= \int_X < S_\Lambda, f_*\phi > \\ &= \int_X < f^* S_\Lambda, \phi > \\ &=< \int_X f^* S_\Lambda, \phi >, \end{aligned}$$

so f^* and $\int_X$ commute.

Proposition 2.

$$T = \int_X T_\Lambda .$$

Proof. Since we may work locally, let ϕ be a test form with support contained in an open set where a section s of π is defined. Then

$$\begin{aligned} < T, \phi > &=< dd^c(EG \circ s), \phi > \\ &=< EG \circ s, dd^c\phi > \\ &= \int_X < G_\Lambda \circ s, dd^c\phi > \\ &= \int_X < dd^c(G_\Lambda \circ s), \phi > \\ &= \int_X < T_\Lambda, \phi > \\ &=< \int_X T_\Lambda, \phi > . \end{aligned}$$

□

Proposition 3.

$$\int_{B(0,\delta)} EG \circ \tilde{f}_\lambda(z) d\nu(\lambda) = d \cdot EG(z).$$

Proof. We have

$$G_\Lambda \circ \tilde{f}_\lambda = d \cdot G_{\lambda\Lambda}.$$

Integrating over $\Lambda \in X$ and then over $\lambda \in B(0,\delta)$ gives the result. □

Proposition 4.

$$\int_{B(0,\delta)} f_\lambda^* T = d \cdot T.$$

Remark. Since f_λ is not a submersion, we must take care in defining $(f_\lambda)^*S$, S a positive, closed, $(1,1)$ current. If $\pi^*S = dd^c u$, define $(f_\lambda)^*S$ by

$$\pi^*(f_\lambda)^*S = dd^c(u \circ \tilde{f}_\lambda).$$

This agrees with calculated value of pullback when f_λ is a submersion, since then

$$\begin{aligned} \pi^*(f_\lambda)^*S &= (\tilde{f}_\lambda)^*\pi^*S \\ &= (\tilde{f}_\lambda)^* dd^c u \\ &= dd^c(u \circ \tilde{f}_\lambda). \end{aligned}$$

Proof. It suffices to work locally. Let ϕ be a test form with support in an open set where a local section s of π is defined. Then

$$(f_\lambda)^*T = dd^c(EG \circ \tilde{f}_\lambda \circ s),$$

and

$$\begin{aligned} < \int_{B(0,\delta)} (f_\lambda)^* T, \phi > &= \int_{B(0,\delta)} < dd^c(EG \circ \tilde{f}_\lambda \circ s), \phi > \\ &= \int_{B(0,\delta)} < EG \circ \tilde{f}_\lambda \circ s, dd^c \phi > \\ &=< \int_{B(0,\delta)} EG \circ \tilde{f}_\lambda \circ s, dd^c \phi > \\ &=< d \cdot EG \circ s, dd^c \phi > \\ &= d < dd^c(EG \circ s), \phi > \\ &= d < T, \phi > . \end{aligned}$$

□

Remark 1. Actually all of the observations so far hold if we replace $B(0,\delta)$ with any index set I, equipped with a probability measure and a topology so that

1. The maps f_λ vary continuously with $\lambda \in I$ in the topology of uniform convergence on compact sets
2. Open sets in I have positive measure.

Example 1. Let $I = \{0,1\}$, give I the discrete topology, and let ν assign mass $1/2$ to each point of I. Let $f_0([z:t]) = [z^2 : t^2]$ and $f_1([z:t]) = [2z^2, t^2]$. Then

$$G_\Lambda = \max(\log|2^{-r}z|, \log|t|),$$

where

$$r = r(\Lambda) = \sum_{i=1}^{\infty} (\lambda_i/2^i).$$

In affine coordinates $t = 1$, supp $T_\Lambda = \{|z| = 2^{r(\Lambda)}\}$, and the support of T is the closed annulus $\{1 \le |z| \le 2\}$. This example was prompted by a question of Shishikura.

From now on we will make use of the assumption, made in the introduction, that $I = B(0,\delta) \subset \mathbf{C}^k$ and $\lambda \mapsto f_\lambda(z)$ is a finite map for each $z \in \mathbf{P}^k$.

Lemma 1. *For each $w \in \mathbf{P}^k$,*

$$U := \{f_\lambda^{-1}(w) : \lambda \in B(0,\delta)\}$$

is open in $\mathbf{P}^k$.

Proof. Consider $f : B(0,\delta) \times \mathbf{P}^k \to \mathbf{P}^k$ given by $f(\lambda, z) = f_\lambda(z)$. Let $V = f^{-1}(w)$. Then V is a $k-$dimensional branched cover over $B(0,\delta)$. Let π be the projection from V to $\mathbf{P}^k$. Note that $U = \pi(V)$. Pick a $z_0 \in U, \lambda_0 \in B(0,\delta)$, $f_{\lambda_0}(z_0) = w$. Since the map $\lambda \to f_\lambda(z_0)$ is finite, (λ_0, z_0) is an isolated point in $V \cap \{z = z_0\}$. Hence π is proper in a neighborhood of (λ_0, z_0). Since V has dimension k, it follows that $z_0 \in \text{int}\, U$. □

Let S be a positive, closed, $(1,1)$ current on $\mathbf{P}^k$ with $||S|| = 1$. Then, as above, there is a plurisubharmonic function u on $\mathbf{C}^{k+1}$ which satisfies the homogeneity condition $u(cz) = \log|c| + u(z)$, $c \in \mathbf{C} \setminus \{0\}$, such that $\pi^* S = dd^c u$. Define the operator Θ on the space of all such currents as follows. Let

$$\Theta(S) = \frac{1}{d} \int_{B(0,\delta)} f_\lambda^* S.$$

Then clearly $\Theta(S)$ is again a positive, closed, $(1,1)$ current with mass one, and thus it has a plurisubharmonic potential u_1 satisfying the same homogeneity condition as u. We wish to show that u_1 is locally bounded.

Lemma 2. *Let S and Θ be as above. Then there is a locally bounded plurisubharmonic function u_1 on $\mathbf{C}^{k+1} \setminus \{0\}$ satisfying $u_1(cz) = \log|c| + u_1(z)$, $c \in \mathbf{C} \setminus \{0\}$ and*

$$\pi^*\Theta(S) = dd^c u_1.$$

Proof. Let $U \subset \mathbf{P}^k$ be an open set where a local section s of π is defined. Let ϕ be a test form supported in U. Let u be a potential for S. It is straightforward to verify that then

$$u_1(z) = \frac{1}{d} \int_{B(0,\delta)} u \circ \tilde{f}_\lambda(z)\, d\nu(\lambda)$$

satisfies all of the conditions of the theorem except possibly the local boundedness. This we now show. Since u_1 is upper semicontinuous, it suffices to show that it is locally bounded from below.

Since u is plurisubharmonic and $\tilde{f}$ is holomorphic, their composition is plurisubharmonic or identically $-\infty$ in $B(0,\delta) \times \mathbf{C}^{k+1} \setminus \{0\}$. Since f varies openly in λ, however, $v := u \circ \tilde{f}(\cdot, z)$ is not identically $-\infty$ for any z. In particular, it is locally integrable with respect to ν, and hence finite at every point. (Recall that $u \circ \tilde{f}$ is defined on a slightly larger ball $\times\, \mathbf{C}^{k+1} \setminus \{0\}$.)

Fix $z^0 \in \mathbf{C}^{k+1} \setminus \{0\}$, let U be a neighborhood of z^0 small enough that $f(B(0,\delta) \times \pi(U))$ is relatively compact in some copy of $\mathbf{C}^k \subset \mathbf{P}^k$, which we assume without loss of generality to be $\{z_0 \neq 0\}$, where $[z_0 : \cdots : z_k]$ are homogeneous coordinates on $\mathbf{P}^k$. Then $z_0 \circ \tilde{f}(B(0,\delta) \times U)$ is uniformly bounded away from $0 \in \mathbf{C}$. Let

$$\begin{aligned} p : \mathbf{C}^{k+1} \setminus \{z_0 = 0\} &\to \{z_0 = 1\} \subset \mathbf{C}^{k+1} \\ (z_0, \ldots, z_k) &\mapsto (1, z_1/z_0, \ldots, z_k/z_0). \end{aligned}$$

Then $p \circ \tilde{f}$ is uniformly bounded in $B(0,\delta) \times U$, (in fact even on a slightly larger ball). The map $p \circ \tilde{f}(\cdot, z^0) : B(0,\delta) \to \mathbf{C}^{k+1} \cap \{z_0 = 1\}$ is finite. Hence we can find a $\lambda^0 = \lambda(z^0)$ where $p \circ \tilde{f}(\cdot, z)$ has rank k and $u(p \circ \tilde{f}(\lambda^0, z^0)) > -\infty$. By the implicit function theorem there exists a holomorphic map $\Phi : V(z^0) \to B(0,\delta)$ defined on a small neighborhood $V \subset U$ of z^0 such that $p \circ \tilde{f}(\Phi(z), z) \equiv p \circ \tilde{f}(\lambda^0, z^0)$.

By the subaveraging principle it follows that for some $\epsilon > 0$ and any $z \in V$,

$$v(\Phi(z), z) \leq \frac{1}{|B(\Phi(z), \epsilon)|} \int_{\lambda \in B(\Phi(z),\epsilon)} v(\lambda, z) dm(\lambda).$$

Since

$$\begin{aligned} v(\Phi(z), z) &= u \circ \tilde{f}(\Phi(z), z) \\ &= u \circ [(z_0 \circ \tilde{f}(\Phi(z), z))(p \circ \tilde{f}(\Phi(z), z))] \\ &= \log|z_0 \circ \tilde{f}(\Phi(z), z)| + u(p \circ \tilde{f}(\lambda^0, z^0)) \end{aligned}$$

it follows that there exists a constant C so that

$$\int_{\lambda \in B(\Phi(z),\epsilon)} v(\lambda, z) dm(\lambda) \geq -C$$

for all $z \in V$.

Since averages of plurisubharmonic functions increase with the radius of the ball, we may replace the domain of integration above by $B(\Phi(z), \delta - |\Phi(z)|)$. Shrinking V if necessary, we may assume that

$$B(\lambda^0, (1/2)(\delta - |\lambda^0|)) \subset B(\Phi(z), \delta - |\Phi(z)|$$

for all $z \in V$. Note that $u \circ \tilde{f}$ is uniformly bounded above on $V \times B(0,\delta)$, say by K. Thus for any sets $B' \subset B \subset B(0,\delta)$,

$$\int_B v(\lambda, z) dm(\lambda) > -C$$

implies that

$$\begin{aligned}\int_{B'} v(\lambda, z) dm(\lambda) \quad &> \quad -C - K|B - B'| \\ &> \quad -C - K|B(0,\delta)|.\end{aligned}$$

Thus

$$\int_{B(\lambda^0,(1/2)(\delta-|\lambda^0|))} v(\lambda, z) dm(\lambda) > -C - K|B(0,\delta)|$$

for all $z \in V$.

Now, by alternately expanding this ball and taking sub-balls, we may replace $B(\lambda^0, (1/2)(\delta - |\lambda^0|))$ with $B(0,\delta)$ after a finite number of steps. Expanding leaves our lower bound on the integral unchanged, while taking a sub-ball decreases it by at most $K|B(0,\delta)|$. Thus we arrive after a finite number of steps at an estimate of the form

$$\int_{\lambda \in B(0,\delta)} u \circ \tilde{f}(\lambda, z) dm(\lambda) > -C'$$

for all $z \in V$. Since $\nu = m/|B(0,\delta)|$, we are done with the proof. □

Lemma 3. *Let u be a locally bounded plurisubharmonic function in $\mathbf{C}^{k+1} \setminus \{0\}$ satisfying $u(cz) = \log|c| + u(z)$, $c \in \mathbf{C} \setminus \{0\}$. Then for every $\Lambda \in X$,*

$$\frac{1}{d^n} u \circ \tilde{F}_{\Lambda,n} \to G_\Lambda$$

locally uniformly as $n \to \infty$.

Proof. Let $u_n = u \circ \tilde{F}_{\Lambda,n}/d^n$. Note that

$$u_n = \frac{1}{d^n} u \left(\frac{\tilde{F}_{\Lambda,n}}{\|\tilde{F}_{\Lambda,n}\|} \right) + G_n(\Lambda, \cdot).$$

Since u is bounded on $\{\|z\| = 1\}$, the first term on the right converges uniformly to zero as $n \to \infty$. Thus

$$\frac{1}{d^n} u \circ \tilde{F}_{\Lambda,n} \to G_\Lambda$$

uniformly on $\{\|z\| = 1\}$, and hence by homogeneity on compact subsets of $\mathbf{C}^{k+1} \setminus \{0\}$. □

Theorem 1. *Let S be any positive, closed, $(1,1)$ current on $\mathbf{P}^k$ of mass 1. Then $\Theta^n(S) \to T$ in the weak topology on currents as $n \to \infty$.*

Proof. We have

$$\Theta^n(S) = \int_{B(0,\delta)^n} \frac{(F_{\Lambda,n})^* S}{d^n}.$$

Since we may work locally, suppose that ϕ is a test form with support in an open set U on which a section s of π is defined. Proving the convergence of the currents is equivalent to showing that

$$< \Theta^n(S), \phi > \to < T, \phi > .$$

Replacing S by $\Theta(S)$, we may assume that S has a locally bounded potential u, by Lemma 2. Then

$$\frac{\pi^*(F_{\Lambda,n})^* S}{d^n} = \frac{dd^c(u \circ \tilde{F}_{\Lambda,n})}{d^n},$$

so in the support of ϕ

$$\frac{F^*_{\Lambda,n} S}{d^n} = \frac{dd^c(u \circ \tilde{F}_{\Lambda,n} \circ s)}{d^n}.$$

Thus

$$\begin{aligned} < \Theta^n(S), \phi > &= \int_{B(0,\delta)^n} < \frac{(F_{\Lambda,n})^* S}{d^n}, \phi > \\ &= \int_X < \frac{(F_{\Lambda,n})^* S}{d^n}, \phi > \\ &= \int_X < \frac{dd^c(u \circ \tilde{F}_{\Lambda,n} \circ s)}{d^n}, \phi > \\ &= \int_X < \frac{u \circ \tilde{F}_{\Lambda,n} \circ s}{d^n}, dd^c \phi > . \end{aligned}$$

Since for each $\Lambda \in X$ we have

$$\frac{u \circ \tilde{F}_{\Lambda,n} \circ s}{d^n} \to G_\Lambda \circ s$$

uniformly in the support of s, by Lemma 3, we have

$$g_n(\Lambda) :=< \frac{u \circ \tilde{F}_{\Lambda,n} \circ s}{d^n}, dd^c \phi > \to g(\Lambda) :=< G_\Lambda \circ s, dd^c \phi >$$

pointwise in X. Also, since u satisfies the homogeneity property

$$u(cz) = \log |c| + u(z),$$

it is clear that $u_n := u \circ \tilde{F}_{\Lambda,n}/d^n$ also satisfies

$$u_n(cz) = \log |c| + u_n(z)$$

for each n, since $\tilde{F}_{\Lambda,n}$ is homogeneous of degree d^n. Thus all currents have total mass one (see Thm. 5.9, [3]), and so the functions g_n are bounded by $1 \cdot \|\phi\|_U$. We may thus apply the Lebesgue convergence theorem to conclude that

$$\begin{aligned} < \Theta^n(S), \phi >= \int_X g_n \to \int_X g \\ &= \int_X < dd^c(G_\Lambda \circ s), \phi > \\ &= \int_X < T_\Lambda, \phi > \\ &=< \int_X T_\Lambda, \phi > \\ &=< T, \phi > . \end{aligned}$$

□

Example 2. For $\lambda = (\lambda_1, \lambda_2) \in B(0, \delta)$, δ small, let

$$f_\lambda([z : w : t]) = [(z - \lambda_1 t)^2 : (w - \lambda_2 t)^2 : t^2].$$

Let S be given by $\pi^* S = dd^c(\log|z - t/2|)$. $\Theta^n(S)$ converges to a current T', invariant under Θ, which is supported near the cylinder $\{|z| = 1\} \subset \mathbf{P}^2$. But $T' \neq T$, since supp T contains a neighborhood of

$$\operatorname{supp} T_0 = \{|w| = |t|\} \cup \{|z| = |t|\} \cup \{|w| = |z|\}.$$

Thus S is an exceptional current in this example (as is, more obviously, the current $[(t = 0)]$), a consequence of the fact that f_λ fails to vary openly in λ at a single point of supp S.

Remark 2. From the continuity of T_Λ in Λ, it is clear that if we take the limit of $T = T(\delta)$ as $\delta \to 0$, we recover T_0, where 0 here denotes the sequence $(0, 0, \dots) \in X$.

3. Convergence of measures

Since G_Λ is locally bounded, the Monge-Ampere operator $(dd^c)^k$ is well-defined on it [1]. Define the measure μ_Λ, considered as a (k, k) current, on $\mathbf{P}^k$, by the equation

$$\pi^* \mu_\Lambda = (dd^c G_\Lambda)^k.$$

Then μ_Λ is locally defined by

$$\mu_\Lambda = (dd^c(G_\Lambda \circ s))^k,$$

which is independent of the choice of local section s.

For f a holomorphic map on $\mathbf{P}^k$ of degree $d \geq 2$ and any continuous function ϕ on $\mathbf{P}^k$ we may define

$$f_* \phi(x) = \sum_{y \in f^{-1}(x)} \phi(y),$$

where if x is a critical value we take into account the multiplicity. Then given a measure ν on $\mathbf{P}^k$, we may define its pullback by

$$< f^* \nu, \phi >=< \nu, f_* \phi > .$$

Therefore, in the special case of pulling back a current which is a measure, this formula makes sense even when f is not a submersion. Let ω be the standard Kahler form on $\mathbf{P}^k$, with $\int_{\mathbf{P}^k} \omega^k = 1$.

Proposition 5.

$$\mu_\Lambda = \lim_{n\to\infty} \frac{F^*_{\Lambda,n}\omega^k}{d^{kn}}.$$

Proof. We have

$$\begin{aligned}
\frac{\pi^* F^*_{\Lambda,n}\omega^k}{d^{kn}} &= \frac{\tilde{F}^*_{\Lambda,n}\pi^*\omega^k}{d^{kn}} \\
&= \tilde{F}^*_{\Lambda,n}\frac{(dd^c \log \|z\|)^k}{d^{kn}} \\
&= \left(\frac{dd^c \log \|\tilde{F}_{\Lambda,n}\|}{d^n}\right)^k \\
&= (dd^c G_n(\Lambda,\cdot))^k.
\end{aligned}$$

By the CLN estimate, since $G_n(\Lambda,\cdot) \to G_\Lambda$ locally uniformly in $\mathbf{C}^{k+1} \setminus \{0\}$, we have $(dd^c G_n(\Lambda,\cdot))^k \to (dd^c G_\Lambda)^k$ as currents. □

Note also that, by the CLN estimate [2], μ_Λ puts no mass on locally pluripolar sets, since G_Λ is locally bounded.

Proposition 6.

$$f^*_\lambda \mu_\Lambda = d^k \mu_{\lambda\Lambda}.$$

Proof. On $\mathbf{P}^k \setminus f_\lambda^{-1}(f_\lambda(C_\lambda))$, C_λ the critical set of f_λ, f_λ is a submersion, and we have

$$\begin{aligned}
\pi^* f^*_\lambda \mu_\Lambda &= \tilde{f}^*_\lambda \pi^* \mu_\Lambda \\
&= \tilde{f}^*_\lambda (dd^c G_\Lambda)^k \\
&= (dd^c (G_\Lambda \circ \tilde{f}_\lambda))^k \\
&= d^k (dd^c G_{\lambda\Lambda})^k \\
&= d^k \pi^* \mu_{\lambda\Lambda}.
\end{aligned}$$

Thus the result holds on $\mathbf{P}^k \setminus f_\lambda^{-1}(f_\lambda(C_\lambda))$, and since μ_Λ puts no mass on complex curves for any $\Lambda \in X$, it holds everywhere. □

Define $\mu = \int_X \mu_\Lambda$.

Proposition 7.

$$\int_{B(0,\delta)} f^*_\lambda \mu = d^k \mu.$$

Proof.

$$\begin{aligned}
\int_{B(0,\delta)} f^*_\lambda \mu &= \int_{B(0,\delta)} \int_X f^*_\lambda \mu_\Lambda \\
&= \int_{B(0,\delta)} \int_X d^k \mu_{\lambda\Lambda} \\
&= d^k \int_X \mu_\Lambda \\
&= d^k \mu.
\end{aligned}$$

□

Given any probability measure η on $\mathbf{P}^k$, let

$$\Omega(\eta) = \frac{1}{d^k} \int_{B(0,\delta)} f_\lambda^* \eta.$$

Theorem 2.

$$\lim_{n\to\infty} \Omega^n(\omega^k) = \mu.$$

Proof. We have

$$\begin{aligned} < \Omega^n(\omega^k), \phi > &= \int_{B(0,\delta)^n} < \frac{F^*_{\Lambda,n}\omega^k}{d^{kn}}, \phi > \\ &= \int_X < \frac{F^*_{\Lambda,n}\omega^k}{d^{kn}}, \phi > \\ &= \int_X g_n(\Lambda), \end{aligned}$$

where

$$g_n(\Lambda) = < \frac{F^*_{\Lambda,n}\omega^k}{d^{kn}}, \phi > .$$

By Proposition 5, $g_n(\Lambda) \to < \mu_\Lambda, \phi >$ for each $\Lambda \in X$. Since all currents have uniformly bounded mass, we may apply the Lebesgue convergence theorem to conclude that

$$< \Omega^n(\omega^k), \phi > \to \int_X < \mu_\Lambda, \phi > = < \mu, \phi > .$$

Thus $\Omega^n(\omega^k) \to \mu$, as desired. □

Lemma 4. *For $w \in \mathbf{P}^k$, $\Omega([w])$ is absolutely continuous with respect to ω^k.*

Proof. Let $E \subset \mathbf{P}^k$, $\omega^k(E) = 0$. Define $V := \{(\lambda, z); f_\lambda(z) = w\}$. Let π_1 and π_2 be the projections from V to $B(0,\delta)$ and $\mathbf{P}^k$ respectively. Let

$$W = \pi_1 \circ \pi_2^{-1}(E).$$

Now, we may define

$$\begin{aligned} \Omega([w])(E) &= < \Omega([w]), \chi_E > \\ &= 1/d^k \int_{B(0,\delta)} < f_\lambda^*[w], \chi_E > d\nu \\ &= 1/d^k \int_{B(0,\delta)} < [w], (f_\lambda)_* \chi_E > d\nu \\ &= 1/d^k \int_{B(0,\delta)} (f_\lambda)_* \chi_E(w) d\nu \\ &= 1/d^k \int_{B(0,\delta)} \left(\sum_{y \in f_\lambda^{-1}(w)} \chi_E(y) \right) d\nu \\ &= 1/d^k \int_{B(0,\delta)} g(\lambda) d\nu, \end{aligned}$$

where $g(\lambda) = (\#\{f_\lambda^{-1}(w) \cap E\})$. Thus $g(\lambda)$ is an integer between 0 and d^k. It is supported on W. Thus the final integral above is less than or equal to

$$1/d^k \cdot d^k \int_W 1 \cdot d\nu = \nu(W).$$

Thus it suffices to show that $\nu(W) = 0$. Let Z be the union of the branching loci of the projections of V to $\mathbf{P}^k$ and $B(0, \delta)$. Then clearly $\nu(W) \leq \nu(W \setminus \pi_1(Z)) + \nu(\pi_1(Z)) = 0$. □

Lemma 5. *Let η be any probability measure on $\mathbf{P}^k$. Then $\Omega(\eta)$ is absolutely continuous with respect to ω^k.*

Proof. Write

$$\eta = \int_{\mathbf{P}^k} [w] d\eta(w).$$

Pulling this back by f_λ^* gives

$$f_\lambda^* \eta = \int_{\mathbf{P}^k} f_\lambda^*[w] d\eta(w).$$

Integrating with respect to λ, and changing the order of integration, we obtain

$$\Omega(\eta) = \int_{\mathbf{P}^k} \Omega([w]) d\eta(w).$$

Suppose that E is a measurable subset of $\mathbf{P}^k$ with $\omega^k(E) = 0$. Then we have

$$\begin{aligned}
\Omega(\eta)(E) &=< \Omega(\eta), \chi_E > \\
&= \int_{\mathbf{P}^k} < \Omega([w]), \chi_E > d\eta(w) \\
&= \int_{\mathbf{P}^k} 0 \; d\eta(w) \\
&= 0,
\end{aligned}$$

where we have used Lemma 4 in the second-to-last equality. □

We cite the following result from [10]; see also [9].

Theorem 3. *Let $f_n : \mathbf{P}^k \to \mathbf{P}^k$ be a sequence of holomorphic maps, f_n of degree d^n. Then there exists a pluripolar subset $\mathcal{E}$ of $\mathbf{P}^k$ such that for all probability measures η on $\mathbf{P}^k$ with $\eta(\mathcal{E}) = 0$,*

$$\frac{1}{d^{nk}} (f_n^* \eta - f_n^* \omega^k) \to 0$$

in the weak topology of currents as $k \to \infty$.

Theorem 4. *Let η be any probability measure on $\mathbf{P}^k$. Then $\Omega^n(\eta) \to \mu$ in the weak topology of currents.*

Proof. Replacing η by $\Omega(\eta)$, if necessary, we may assume that η is absolutely continuous with respect to ω^k, by Lemma 5, and thus that it has no mass on locally pluripolar sets. Fix $\Lambda \in X$. Then applying Theorem 3 with $f_n = F_{\Lambda,n}$, we have that

$$u_n := \frac{1}{d^{kn}} (F_{\Lambda,n}^* \eta - F_{\Lambda,n}^* \omega^k) \to 0$$

in the sense of measures as $n \to \infty$. Fix a test function ϕ, and let $g_n(\Lambda) = < u_n, \phi >$. Then $g_n \to 0$ pointwise on X. But

$$< \Omega^n(\eta) - \Omega^n(\omega^k), \phi > = \int_{B(0,\delta)^n} g_n$$
$$= \int_X g_n.$$

Since the measures have bounded mass, we may apply the Lebesgue convergence theorem to conclude that the limit of this last quantity as $n \to \infty$ is zero. But we have also from Theorem 2 that

$$\Omega^n(\omega^k) \to \mu$$

as $n \to \infty$. □

Remark 3. As in Remark 2, we note that the continuity of μ_Λ in Λ implies that $\mu = \mu(\delta)$ converges to μ_0 as $\delta \to 0$, where again 0 here denotes the sequence $(0, 0, \dots) \in X$.

4. Description of supp T

Let $\mathcal{F}_\Lambda \subset \mathbf{P}^k$ be the largest open set on which the family $\{F_{\Lambda,n}\}_n$ is normal, and let $\mathcal{J}_\Lambda$ be its complement.

Proposition 8. *The support of T_Λ is equal to $\mathcal{J}_\Lambda$.*

Proof. We have the estimate

$$|G_\Lambda(z) - G_n(\Lambda, z)| \leq \frac{C}{d^n}$$

for some $C > 0$. The rest of the proof is the same as that of Theorem 6.2, [3]. □

Proposition 9.

$$\text{supp } T = \overline{\bigcup \text{supp } T_\Lambda}.$$

Proof. The inclusion $\subset$ is obvious, since $T = \int_X T_\Lambda$. For the other inclusion, recall that T_Λ varies continuously in Λ. So if some T_Λ has mass in an open set U, then so does $T = \int_X T_\Lambda$. □

Proposition 10.

$$\text{supp } \mu = \overline{\bigcup \text{supp } \mu_\Lambda}.$$

Proof. The proof is the same, since μ_Λ also varies continuously in Λ. □

We now wish to show that the complement of supp T is relatively compact in the basins of attraction to attracting periodic orbits for f_0. If f_0 has no attracting periodic orbits, then supp $T = \mathbf{P}^k$. We recall first some general notions from the theory of dynamical systems. See Ruelle [7] for background.

Let (X, d) be a compact metric space and f a continuous map from X to X. The sequence $(x_j)_{1 \leq j \leq n}$ is an ϵ-pseudo-orbit if $d(f(x_j), x_{j+1}) < \epsilon$ for $j = 1, \dots, n-1$. For $a, b \in X$, we write $a \succ b$ if for every $\epsilon > 0$ there is an ϵ-pseudo-orbit from a to b. We also write $a \succ a$. We write $a \sim b$ if $a \succ b$ and $b \succ a$, and denote by $[a]$ the equivalence class of a under this relation. Define an *attractor* to be a minimal equivalence class for $\sim$. The following proposition is an easy consequence of Zorn's lemma.

Proposition 11. *Let $f : X \to X$ be a continuous map on a compact metric space X. Then given any $x \in X$, there is an attractor $[a]$ such that $x \succ a$.*

It is also easy to show (see [7]) that an attractor K is compact and satisfies $f(K) = K$. The following theorem is proved in [5].

Theorem 5. *Let $f : \mathbf{P}^k \to \mathbf{P}^k$ be a holomorphic map of degree at least two. Suppose that K is an attractor for f. Then either K is an attracting periodic orbit for f, or K contains a nonconstant, entire image of $\mathbf{C}$.*

Corollary 1. *Let $f : \mathbf{P}^k \to \mathbf{P}^k$ be a holomorphic map of degree at least two, and let K be an attractor for f. Then either K is an attracting periodic orbit for f, or $K \cap J \neq \emptyset$, where J is the Julia set for f.*

Proof. It is a result of Ueda [11] that the Fatou set for f is Kobayashi hyperbolic. By the theorem, if K is not an attracting periodic orbit, it contains an entire nonconstant image of $\mathbf{C}$, in which case the hyperbolicity of the Fatou set implies that $K \cap J \neq \emptyset$. □

Lemma 6. *Let f_λ be the holomorphic family of maps on $\mathbf{P}^k$ described in the introduction. For the map f_0, suppose that $x \succ y$. Then there exist $\lambda_1, \ldots, \lambda_n$ such that*

$$f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}(x) = y.$$

Proof. Let ϵ be so small that $f(\cdot, z)(B(0, \delta)) \supset B(f_0(z), \epsilon)$ for each $z \in \mathbf{P}^k$. Let $(x_1, \ldots, x_{n+1})$ be an ϵ-pseudo-orbit for f_0 with $x_1 = x$, $x_{n+1} = y$. Since $d(f_0(x_j), x_{j+1}) < r$ for each $j < n$, there exists $\lambda_j \in B(0, \delta)$ such that $f_{\lambda_j}(x_j) = x_{j+1}$. □

Lemma 7. *Let V be the full basin of the attracting orbits for f_0, and suppose that $z \in \mathbf{P}^k \setminus V$. Then there exist n and $(\lambda_1, \ldots, \lambda_n) \in B(0, \delta)^n$ such that*

$$f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}(z) \in J,$$

where J is the Julia set for f_0. The same holds for $z \in \mathbf{P}^k \setminus W$, for some $W \subset\subset V$ by continuity.

Proof. We have $z \succ A$ for at least one attractor A. If A is nontrivial, we are done by the previous two lemmas. If A is an attracting periodic orbit, since $f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}(z) \in V$ for some $(\lambda_1, \ldots, \lambda_n) \in B(0, \delta)^n$, but $f_0^n(z) \notin V$, and since $f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}(z)$ varies continuously in $(\lambda_1, \ldots, \lambda_n) \in B(0, \delta)^n$, there must exist some $(\lambda_1', \ldots, \lambda_n') \in B(0, \delta)^n$ such that $f_{\lambda_n'} \circ \cdots \circ f_{\lambda_1'}(z) \in \partial V \subset J$. □

Theorem 6. *The support of T contains a neighborhood of the complement of V, where V is the full basin of the attracting orbits for f_0 (possibly $V = \emptyset$).*

Proof. Given $z \in \mathbf{P}^k \setminus V$, there exist n and $(\lambda_1, \ldots, \lambda_n) \in B(0, \delta)^n$ such that

$$f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}(z) \in J,$$

where J is the Julia set for f_0. (Again, note that by continuity this is true for z in an open neighborhood of $\mathbf{P}^k \setminus V$.) But then $z \in \mathcal{J}_\Lambda$, where $\Lambda = (\lambda_1, \ldots, \lambda_n, 0, 0, \ldots)$. Thus $z \in \operatorname{supp} T$. □

Remark 4. In general, f_0 may have infinitely many attracting periodic orbits. But since only finitely many have basins with diameter larger than some fixed real number, supp T contains all but finitely many of these basins. How many escape depends on δ.

Example 3. Suppose that $f_0 : \mathbf{P}^1 \to \mathbf{P}^1$ has a Siegel disk U. Then $U \subset$ supp T. But it follows from Remark 2 that the amount of mass which T gives to U depends on δ and approaches zero as $\delta \to 0$.

We can now give a characterization of supp T in terms of normality. Let $\mathcal{J}$ be the complement of the largest open set on which the full family

$$\mathcal{G} := \{F_{\Lambda,n}\}_{\Lambda,n}$$

(that is, the family of all possible compositions $f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}$) is normal.

Corollary 2. *The support of T is equal to $\mathcal{J}$.*

Proof. By Propositions 8 and 9, it suffices to show that $\overline{\bigcup \mathcal{J}_\Lambda} = \mathcal{J}$. The inclusion $\subset$ is clear, since $\mathcal{G}$ fails to be normal if the particular subfamily $\{F_{\Lambda,n}\}_n$ fails to be normal. For the other inclusion, it suffices to note that the complement of the left-hand side is Kobayashi hyperbolic, by Theorem 6 and by Ueda's result on the hyperbolicity of Fatou components [11], and that it is invariant under $\mathcal{G}$. Thus $\mathcal{G}$ is normal there. □

5. Limit measures near attractors

We now wish to prove some results on forward iteration, and find some limiting measures in this case.

The following theorem is a generalization of Theorem 0.1, [4].

Theorem 7. *Fix $\delta > 0$. Let the family f_λ be as described in the introduction. Then there exist finitely many attractors A_i for f_0, disjoint open sets $V_i \supset A_i$, and continuous functions g_i such that*

1. $0 \le g_i \le 1$, $\sum_i g_i = 1$
2. *For $z \in \mathbf{P}^k$, there exist disjoint open sets $U_{i,z} \subset X$ with $\overline{\nu}(U_{i,z}) = g_i(z)$ such that if $\Lambda \in U_{i,z}$, then $F_{\Lambda,n}(z) \in V_i$ for all n sufficiently large.*

Proof. By compactness, there exists $\epsilon > 0$ such that for each $z \in \mathbf{P}^k$, $f(\cdot, z)(B(0,\delta))$ contains a ball of radius ϵ (as in the proof of Lemma 6). Since for each $z \in \mathbf{P}^k$ there exists an attractor A such that $z \succ A$, again by the compactness of $\mathbf{P}^k$ there exist finitely many attractors $A_1, \ldots, A_l$ such that for each $z \in \mathbf{P}^k$ there exists an orbit $\{f_{\Lambda,k}(z)\}_{k=0}^n$ from z to some A_i. Let $V_i = \{f_{\Lambda,n}(z); \Lambda \in X, n \ge 0, z \in A_i\}$. Then V_i is an open neighborhood of A_i and $f_\lambda(V_i) \subset V_i$ for all $\lambda \in B(0,\delta)$. Suppose that $V_i \succ A_k$. Then it follows that $V_i \supset V_k$. If this happens for some $i \ne k$, we remove A_k. After finitely many steps we can assume that no $V_i \succ A_k$ if $k \ne i$. Suppose next that $V_i \cap V_j \ne \emptyset$ for some $i \ne j$. Pick $z \in V_i \cap V_j$. Then $z \succ A_k$ for some k. Since k must be different from i or j we get a contradiction. Hence $V_i \cap V_j = \emptyset$ if $i \ne j$. There exist for each $z \in \mathbf{P}^k$ an N and a $(\lambda_1, \ldots, \lambda_N) \subset B(0,\delta)^N$ such that

$$f_{\lambda_N} \circ \cdots \circ f_{\lambda_1}(z) \in \bigcup V_i =: V.$$

Since $\mathbf{P}^k$ is compact, N may be chosen to be independent of z.

Define
$$E_i^j(z) = \{(\lambda_1, \dots, \lambda_j) \in B(0,\delta)^j; f_{\lambda_j} \circ \cdots \circ f_{\lambda_1}(z) \in V_i\},$$
and
$$S^j(z) = \{(\lambda_1, \dots, \lambda_j) \in B(0,\delta)^j; f_{\lambda_j} \circ \cdots \circ f_{\lambda_1}(z) \notin \overline{V}\}.$$
Let $g_{i,j}(z) = \nu_j(E_i^j(z))$, where $\nu_j = \nu \times \cdots \times \nu$ is the product measure on $B(0,\delta)^j$ obtained from ν. Clearly $\nu_j(S^j(z))$ and the $g_{i,j}(z)$ are lower semicontinuous on $\mathbf{P}^k$. Since ∂V has zero volume and f_λ is a finite map in λ,
$$\nu_j(S^j(z)) + \sum_i g_{i,j}(z) = 1.$$
Thus
$$g_{i,j}(z) = 1 - \nu_j(S^j(z)) - \sum_{k \neq i} g_{k,j}(z)$$
is continuous. Hence so is $\nu_j(S^j(z))$.

By the choice of N, there is a constant $\mu > 0$ such that for all $z \in \mathbf{P}^k$, $\sum_i g_{i,N}(z) > \mu$. Clearly $g_{i,j+1}(z) \geq g_{i,j}(z)$. Now,
$$\begin{aligned}\nu_{n+N}(S^{n+N}(z)) &= \int_{(\lambda_1,\dots,\lambda_n) \in S^n(z)} \nu_N(S^N(f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}(z))) d\nu_n(\lambda_1, \dots, \lambda_n) \\ &\leq (1-\mu)\nu_n(S^n(z)).\end{aligned}$$
Thus $\nu_n(S^n(z))$ converges geometrically to zero as $n \to \infty$, uniformly in z. We also have
$$\begin{aligned}g_{i,n+N}(z) &= g_{i,n}(z) + \int_{(\lambda_1,\dots,\lambda_n) \in S^n(z)} \nu_N(E_i^N(f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}(z))) d\nu_n(\lambda_1, \dots, \lambda_n) \\ &\leq g_{i,n}(z) + \nu_n(S^n(z)),\end{aligned}$$
so the $g_{i,n}$ converge uniformly to a continuous limit g_i. □

Remark: Note that each V_i is contained in the compact set $\{g_i = 1\}$. Thus the closures of the V_i are also pairwise disjoint.

Given $\mu \in M$, the space of probability measures on $\mathbf{P}^k$, let
$$a_i(\mu) = \int_{\mathbf{P}^k} g_i(z) d\mu(z).$$
(g_i as in the previous theorem, $i = 1, \dots, l$, where l is the number of attractors from the previous theorem). Say $\mu_1 \sim \mu_2$ if $(a_1, \dots, a_l)(\mu_1) = (a_1, \dots, a_l)(\mu_2)$. Let M_A be the equivalence class under $\sim$ given by $A := (a_1, \dots, a_l)$. Let $T : M \to M$ be given by $T(\mu) = \int_{B(0,\delta)} (f_\lambda)_* \mu$. Then T is an ϵ ω^k*-diffusion,* with ϵ as in the proof of Lemma 6. See [4] for the definition of a diffusion.

Theorem 8. *T has a unique fixed point μ_A in each M_A, $A \in [0,1]^l$, and for each $\mu \in M_A$, $T^n(\mu) \to \mu_A$ strongly. Furthermore, if $A = (a_1, \dots, a_l)$, the limit measure has a decomposition*
$$\mu_A = \sum_{i=1}^{l} a_i \mu_i,$$
where the μ_i have mass one and are independent of A, with μ_i supported on $\overline{V_i}$.

Proof. Fix A, and let $\mu \in M_A$. Write

$$\mu = \int_{\mathbf{P}^k} \delta_z d\mu(z).$$

Let $B = B(0,\delta) \subset \mathbf{C}^k$. Let $P_{n,z}(\zeta)$ be the density of $T^n(\delta_z)$ with respect to ω^k on $\mathbf{P}^k$. For a test function g, we have

$$\begin{aligned}(T^n(\mu), g) &= \int_{\mathbf{P}^k} (T^n(\delta_z), g) d\mu(z)\\ &= \int_{\mathbf{P}^k}\int_{B^n} ((f_{\lambda_n} \circ \cdots \circ f_{\lambda_1})_* \delta_z, g) d\nu_n(\lambda_1, \dots, \lambda_n)\, d\mu(z)\\ &= \int_{\mathbf{P}^k}\int_{B^n} (\delta_z, g \circ (f_{\lambda_n} \circ \cdots \circ f_{\lambda_1})) d\nu_n(\lambda_1, \dots, \lambda_n)\, d\mu(z)\\ &= \int_{\mathbf{P}^k}\int_{B^n} g \circ (f_{\lambda_n} \circ \cdots \circ f_{\lambda_1})(z) d\nu_n(\lambda_1, \dots, \lambda_n)\, d\mu(z)\\ &= \int_{\mathbf{P}^k}\int_{\mathbf{P}^k} g(\zeta) P_{n,z}(\zeta) d\omega^k(\zeta) d\mu(z).\end{aligned}$$

Now, the results of the previous theorem are equivalent to the following statements about $P_{n,z}(\zeta)$:

1. $P_{n,z} \to 0$ in $L^1(V^c)$.
2. $\int_{V_i} P_{n,z}(\zeta) d\omega^k(\zeta) = g_{i,n}(z) \to g_i(z)$ uniformly in $\mathbf{P}^k$.

From (1), we conclude that the mass of $T^n(\mu)$ in V^c converges to zero. From (2), we have that

$$\int_{\mathbf{P}^k} \left(\int_{V_i} P_{n,z}(\zeta) d\omega^k(\zeta) \right) d\mu(z) = \int_{\mathbf{P}^k} g_{i,n}(z) d\mu(z) \to \int_{\mathbf{P}^k} g_i(z) d\mu(z) = a_i.$$

Thus the mass of $T^n(\mu)$ in each V_i converges to a_i. Thus we may assume that μ has mass a_i on each V_i, and no mass elsewhere.

Note that the family $\mathcal{G}$ leaves invariant each V_i, and therefore also each $\overline{V_i}$. We have thus reduced the original problem to the problem of T acting on measures of mass a_i on the compact space $X := \overline{V_i}$, which contains a unique attractor. It was proved in Theorem 3.5 of [4] that in this case there is a unique measure μ_i fixed by T which attracts strongly all measures of mass one on X. Since T is linear, $a_i \mu_i$ attracts all measures of mass a_i. Thus T has a fixed point

$$\mu_A := \sum a_i \mu_i$$

in M_A, which attracts strongly all measures in M_A. □

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