

QUANTIZATIONS OF LINEAR SELF-MAPS OF \mathbf{R}^2

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ABSTRACT. We investigate the dynamics and spectral properties of the unitary operators $U_\lambda := e^{i\lambda x^2} F$, where $\lambda \in \mathbf{R}$ and F is the Fourier transform. We show that U_λ is a quantization of the classical map

$$\begin{aligned} f_\lambda : \mathbf{R}^2 &\rightarrow \mathbf{R}^2 \\ (x, y) &\mapsto (y, 2\lambda y - x), \end{aligned}$$

and that the phase transition at $|\lambda| = 1$ for f_λ corresponds to a similar phase transition for U_λ , which changes at those values from having a pure point to a continuous spectrum.

0. INTRODUCTION

In studying complex dynamics on infinite-dimensional spaces, it is natural, in order to reduce the complexity of the problem, to place some restrictions on the map being iterated. We might begin by requiring that the map be a linear self-map of, say, a Banach space or a separable Hilbert space. However, this restriction does not suffice to produce an interesting theory. If we let M be a compact metric space, $f : M \rightarrow M$ a continuous self-map, and B the backward shift on the Banach space $l^\infty(M)$, for instance, it is well-known that the usual symbolic dynamics trick of identifying a point with its orbit shows that f is topologically conjugate to a restriction of B . Thus B exhibits all the known behavior of finite-dimensional dynamical systems on compact spaces. Since any compact metric space embeds in separable Hilbert space \mathcal{H} , one may consider B as acting on $l^\infty(\mathcal{H})$ to obtain a universal example. (Feldman [Fe] has recently made the observation that this example may be modified to obtain a Hilbert space example. The point x is now identified with

$$\left\{ \frac{f^n(x)}{2^n} : n = 0, 1, \dots \right\} \subset l^2(\mathcal{H}),$$

and B is replaced with $2B$.)

A further restriction is thus called for, and so we confine ourselves to *unitary* maps on separable Hilbert space \mathcal{H} . The fact that this class of maps contains the *quantizations* of finite-dimensional Hamiltonian systems (such as the real Henon map), combined with the interplay between the spectral properties and dynamics of its members, makes it a natural object of study. Let us focus on the quantizations of Hamiltonian systems on $M' = M \times \mathbf{R} \ni (x, y)$, where M is either \mathbf{T} (the 1-torus) or \mathbf{R} . Thus we have a differentiable function $H : M' \rightarrow \mathbf{R}$ (in general, $H = H_t$ is time-dependent), a Hamiltonian vector field $X = X_H := (H_y, -H_x)$, and a flow

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$\phi_t : M' \rightarrow M'$ satisfying

$$\begin{aligned} \frac{\partial}{\partial t} \phi_t &= X\phi_t \\ \phi_0 &= \text{Id.} \end{aligned}$$

We then quantize this system by replacing H with a self-adjoint operator on $L^2(M)$, which we also denote by H , and defining U_t as the unitary solution operator to the Schrodinger equation

$$\begin{aligned} i \frac{\partial}{\partial t} U_t &= H U_t \\ U_0 &= \text{Id.} \end{aligned}$$

This operator U_t then approximates the finite-time dynamics of ϕ_t in a certain sense, and is called a *quantization* of ϕ_t . See [KS] for a full discussion. One might ask if the broad features of the long-term dynamics of ϕ_t have analogues in the dynamics of U_t . If ϕ_t has an open, bounded, invariant subset, for instance, does U_t also exhibit recurrent behavior? Insofar as the recurrence properties of U_t are connected to properties of its spectrum (see [AS]), we would be looking for instance for an eigenvector of U_t . One may also fix an operator $U := U_{t_0}$ and consider the corresponding discrete dynamics.

If H is time-independent, we have the representation $U_t := e^{-itH}$, where the meaning of the exponential is given by the spectral theorem's functional calculus. If H is time-dependent, but the family U_t is *strongly continuous*, then Stone's Theorem provides a self-adjoint operator A such that, again, U_t has the representation $U_t = e^{itA}$. See for instance [RS]. Given a fixed unitary operator U , if $U = V^{-1}e^{ix}V$ is its spectral representation, then trivially $U = e^{iA}$, where $A = V^{-1}xV$. However, we will seldom have available a representation of this form, since typically the unitary operator V is not known.

There is no canonical way to perform the abovementioned replacement of H by a (possibly time-dependent) element of $\mathcal{A}(L^2(M))$, and there is a vast literature on the subject (see [KS]). The problem is essentially to reconcile the prescriptions

$$\begin{aligned} x &\mapsto M_x =: x \in \mathcal{A}(L^2(M)) \\ y &\mapsto -id/dx =: y \in \mathcal{A}(L^2(M)) \end{aligned}$$

with the non-commutativity of the latter two operators: $[x, y] := xy - yx = i$. Here M_x denotes multiplication by x . We refer the reader to the literature for a full discussion of the subtleties involved here. In our cases, however, we will be treating Hamiltonian functions that are simple enough that there is no ambiguity about what their quantizations should be.

A simple way to produce a unitary operator on $L^2(M)$ with non-trivial dynamics is to compose two non-commuting operators with known spectral properties. Any operator of the form

$$e^{ig(x)} e^{if(y)}$$

is an example. Operators of the form

$$e^{-ig(x)} e^{-iy^2/2}$$

are of particular interest to us, because they describe free motion with "kicks" to the momentum. Let

$$H = H_t = \begin{cases} y^2/2 & 0 \leq t < 1 \\ g(x) & 1 \leq t < 2 \end{cases},$$

extended by periodicity. This notation describes both a time-dependent Hamiltonian function on M' and the element of $\mathcal{A}(L^2(M))$ that quantizes it. Solving the Schrodinger equation with this operator H gives a unitary solution U_t such that

$$U_2 = e^{-ig(x)}e^{-iy^2/2} =: U.$$

Alternatively, let

$$H = H_t := y^2/2 + g(x) \sum_{n=1}^{\infty} \delta(t - n).$$

Solving the Schrodinger equation with this Hamiltonian again produces a family U_t such that $U_1 = U$. This latter Hamiltonian has the form of a traditional Schrodinger Hamiltonian:

$$H = y^2/2 + V_t(x),$$

which describes the motion of a particle in the potential V_t . In our case, we interpret the Hamiltonian as describing free motion ($V = 0$), interrupted periodically by "kicks", or instantaneous changes in the momentum. A great deal of attention has been paid to operators of this form. The so-called quantized kicked rotor, for instance, is a much-studied element of $\mathcal{U}(L^2(\mathbf{T}))$ whose dynamics and spectral properties are still largely unknown. It is defined by

$$U_\alpha := e^{-\pi i \alpha y^2} e^{-i \cos x},$$

where $\alpha \in \mathbf{R} \setminus \mathbf{Q}$. See [Be] for a discussion of this operator. Fornaess [Fo] has studied the operator

$$U_\alpha := e^{-2\pi i \alpha y} e^{-iv(x)} \in \mathcal{U}(L^2(\mathbf{T}))$$

in the case where α is the golden mean $(\sqrt{5} + 1)/2$, where $v(x)$ is a tent map on the circle, and the present author [W] has studied the same operator for general $\alpha \in \mathbf{R}$. This work is characterized by small denominator problems related to the Diophantine properties of α , recalling the similar problems encountered in the study of local holomorphic germs of self-maps of \mathbf{C} near fixed points, see [CG].

Finally, Fornaess and the author [FW] have studied

$$U_\lambda := e^{iy^2/2} e^{-i(\lambda x^3/3 - x^2 + cx/\lambda)} \in \mathcal{U}(L^2(\mathbf{R}))$$

for $\lambda \in \mathbf{R}$ and $c > 0$. It quantizes the classical Henon map on \mathbf{R}^2 . Using the fact that, up to a multiplicative constant of absolute value one, the Fourier transform may be written as

$$F = e^{-ix^2/2} e^{-iy^2/2} e^{-ix^2/2},$$

and replacing U_λ by unitary conjugates and inverses, we take instead

$$U_\lambda = e^{i\lambda x^3/3 + cx/\lambda} F.$$

Let us take $c = 0$. We are then considering the simplified operator

$$U_\lambda = e^{i\lambda x^3/3} F,$$

which, as described in [FW], quantizes the classical map

$$\begin{aligned}\phi_\lambda : \mathbf{R}^2 &\rightarrow \mathbf{R}^2 \\ (x, y) &\mapsto (y, -x + \lambda y^2),\end{aligned}$$

which is of course conjugate to ϕ_1 for all $\lambda \neq 0$. Since $\phi_\lambda \sim \phi_1$ is a small perturbation of a rotation close to the origin, the KAM theorem [L1] tells us that there is some domain containing the origin that is invariant under ϕ_1 . It is natural to ask whether an analogue holds in the quantum system; that is, is there an eigenvector for U_λ , and further, can we choose that eigenvector to be an analytic (in λ) perturbation of an eigenvector for F ? We have the following theorem, which is Theorem 4 of [FW].

Theorem. *There is no analytic family ψ_λ of perturbations of $\psi_0 := e^{-x^2/2}$ and analytic family a_λ of perturbations of $a_0 := 1$ such that*

$$U_\lambda \psi_\lambda = a_\lambda \psi_\lambda.$$

The proof shows that there is a formal obstruction to such a perturbative solution.

In this paper, we wish to consider a modification of the quantized Henon map, where the cubic term is replaced by a quadratic term. Specifically, we let

$$U_\lambda = e^{i\lambda x^2/2} F \in \mathcal{U}(L^2(\mathbf{R})).$$

Then U_λ quantizes a linear map f_λ :

$$\begin{aligned}f_\lambda : \mathbf{R}^2 &\rightarrow \mathbf{R}^2 \\ (x, y) &\mapsto (y, -x + 2\lambda y),\end{aligned}$$

which has a phase transition at $|\lambda| = 1$. For $|\lambda| < 1$, f_λ is conjugate to a rotation, while for $|\lambda| > 1$, f_λ has complementary contracting and expanding eigenspaces, and consequently no bounded invariant domain. Accordingly, we ask in this paper whether U_λ has eigenvectors for λ near 0, in contrast to the situation for the quantized Henon map. In fact we will seek to obtain a complete spectral picture for U_λ , including a description of what happens at the phase transition $|\lambda| = 1$.

1. PRELIMINARIES AND STATEMENT OF RESULTS

Let us recall the construction of the standard Hermite basis of eigenfunctions of the Fourier transform on $L^2(\mathbf{R})$. Let x and y denote the self-adjoint operators multiplication by x and $-id/dx$, respectively. Now, define inductively

$$\begin{aligned}\phi_0(x) &= e^{-x^2/2} \\ \phi_{n+1} &= (x - iy)\phi_n,\end{aligned}$$

and note that $F\phi_0 = \phi_0$. Then

$$\begin{aligned}F\phi_{n+1} &= F(x - iy)\phi_n \\ &= (-y - ix)F\phi_n \\ &= (-i)(x - iy)F\phi_n,\end{aligned}$$

which, with an easy induction, gives

$$F\phi_n = (-i)^n \phi_n.$$

The ϕ_n are also the eigenfunctions for the quantum harmonic oscillator, whose Hamiltonian is the operator

$$H := \frac{1}{2}(x^2 + y^2) = \frac{1}{2}\left(x^2 - \frac{d^2}{dx^2}\right).$$

This is easily seen as follows:

$$\begin{aligned} H\phi_0 &= \frac{1}{2}\phi_0 \\ H\phi_{n+1} &= \frac{1}{2}(x^2 + y^2)\phi_{n+1} \\ &= \frac{1}{2}(x^2 + y^2)(x - iy)\phi_n \\ &= \frac{1}{2}(x - iy)[(x^2 + y^2) + 2]\phi_n \\ &= (x - iy)(H + 1)\phi_n, \end{aligned}$$

where we have used the commutation relation $[x, y] = i$. Then a simple induction gives

$$H\phi_n = \left(n + \frac{1}{2}\right)\phi_n.$$

The corresponding unitary solution operator to the Schrodinger wave equation

$$\frac{d}{dt}(V_t\psi) = -iH(V_t\psi) \quad V_0 = \text{Id}$$

is then

$$\begin{aligned} V_t &= e^{-itH} \\ &:= (\phi_n \mapsto e^{-it(n+\frac{1}{2})}\phi_n). \end{aligned}$$

Note that $V_{\pi/2} = e^{-i\pi/4}F$. In other words, up to a multiplicative constant of absolute value one, the Fourier transform is embedded in the unitary solution operator to the quantum harmonic oscillator. It is typical to ignore this multiplicative constant, because multiplying an element $U \in \mathcal{U}(\mathcal{H})$ by $e^{i\theta}$ has no effect on expressions $(U\psi, A(U\psi))$, where $A \in \mathcal{A}(\mathcal{H})$. These are the so-called *observables* of the system. For this reason, in physical applications, one often considers oneself to be working on projective Hilbert space.

Note that the unitary operator V_t is a *quantization* of the rotation ϕ_t of \mathbf{R}^2 given by the matrix

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

in the sense that the function

$$\begin{aligned} \mathbf{R}^2 &\rightarrow \mathbf{R} \\ (x, y) &\mapsto \frac{1}{2}(x^2 + y^2) \end{aligned}$$

has ϕ_t as its Hamiltonian flow. In particular (ignoring the multiplicative constant), we say that $F = V_{\pi/2}$ is a quantization of the rotation

$$(x, y) \mapsto (y, -x).$$

Consider now the following self-map f_λ of \mathbf{R}^2 :

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 2\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \lambda \in \mathbf{R}.$$

For λ small, we may consider f_λ to be a perturbation of the rotation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then for $|\lambda| > 1$, f_λ has complementary contracting and expanding eigenspaces, whereas for $|\lambda| < 1$, f_λ is conjugate to the rotation

$$\begin{pmatrix} \lambda & \sqrt{1-\lambda^2} \\ -\sqrt{1-\lambda^2} & \lambda \end{pmatrix}.$$

We may quantize f_λ as follows: note that f_λ is the time-2 map of the Hamiltonian flow of the time-dependent function

$$(x, y) \mapsto \begin{cases} \frac{1}{2}(x^2 + y^2) & t \in [0, 1) \\ -\lambda x^2 & t \in [1, 2) \end{cases},$$

extended periodically to $t \in \mathbf{R}$. Replacing this function by the corresponding self-adjoint operator $H \in \mathcal{A}(L^2(\mathbf{R}))$, solving the Schrodinger equation, and taking the time-2 map of the resulting unitary solution operator, we obtain

$$U_\lambda := e^{i\lambda x^2} F,$$

up to the multiplicative constant $e^{i\pi/4}$, which we ignore. One immediately apparent distinction between f_λ and its quantum analogue U_λ is that, while the former may be considered a perturbation of a $-\pi/2$ rotation in the topology, say, of locally uniform convergence, the latter is clearly not a perturbation of F in the norm topology on $\mathcal{U}(L^2(\mathbf{R}))$, since

$$\|U_\lambda - F\| = 2, \quad \lambda \neq 0.$$

Thus we cannot rely on perturbative methods in searching for eigenvectors for U_λ .

Our aim is to prove the following theorems:

Theorem 1. *Let $U_\lambda = e^{i\lambda x^2} F : L^2(\mathbf{R}) \rightarrow L^2(\mathbf{R})$, $\lambda \in \mathbf{R}$. Then if $|\lambda| < 1$, $L^2(\mathbf{R})$ has a complete basis of eigenfunctions ϕ_n of U_λ , with*

$$U_\lambda(\phi_n) = (\sqrt{i\mu})\mu^n \phi_n,$$

where $\mu = \lambda - i\sqrt{1-\lambda^2}$. Here all square roots denote their principal value.

Theorem 2. *Let U_λ be as in the previous theorem, $\lambda \in \mathbf{R}$. Then if $|\lambda| \geq 1$, U_λ has a purely continuous spectrum.*

2. PROOF OF THEOREM 1

We will make use of the following lemma, which is a slight restatement of Theorem 5.1 of [FW]:

Lemma 3. *Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a C^1 function. Then*

$$\begin{aligned} e^{-if(x)}xe^{if(x)} &= x \\ e^{-if(x)}ye^{if(x)} &= y + f'(x) \\ e^{-if(y)}xe^{if(y)} &= x - f'(y) \\ e^{-if(y)}ye^{if(y)} &= y. \end{aligned}$$

Since $e^{if(x)}$ and $e^{if(y)}$ quantize the classical maps $(x, y) \mapsto (x, y + f'(x))$ and $(x, y) \mapsto (x - f'(y), y)$, respectively, this lemma may be considered an Egoroff-type result, showing that the *symbols* of the operators $U^{-1}xU$ and $U^{-1}yU$ are equal to the functions $x \circ g$ and $y \circ g$, where $U \in \mathcal{U}(L^2(\mathbf{R}))$ is a quantization of $g : \mathbf{R}^2 \rightarrow \mathbf{R}^2$, and U has the particular form above.

We now prove the following:

Theorem. *If $|\lambda| < 1$, $L^2(\mathbf{R})$ has a complete basis of eigenfunctions ϕ_n of U_λ , with*

$$U_\lambda(\phi_n) = (\sqrt{i\mu})\mu^n\phi_n,$$

and where $\mu = \lambda - i\sqrt{1 - \lambda^2}$. Here all square roots denote their principal value.

Proof. As before, let x and y denote the self-adjoint operators multiplication by x and $-id/dx$, respectively. Let $\mu = \lambda - i\sqrt{1 - \lambda^2}$. Now, define inductively

$$\begin{aligned} \phi_0(x) &= e^{i\bar{\mu}x^2/2} \\ \phi_{n+1} &= (x - \bar{\mu}y)\phi_n. \end{aligned}$$

A direct computation gives that

$$U_\lambda(\phi_0) = \sqrt{i\mu}\phi_0,$$

where the square root takes its principal value. Assume that

$$U_\lambda(\phi_j) = (\sqrt{i\mu})\mu^j\phi_j, \quad j = 1, \dots, n.$$

Then

$$\begin{aligned} U_\lambda(\phi_{n+1}) &= e^{i\lambda x^2}F(\phi_{n+1}) \\ &= e^{i\lambda x^2}F(x - \bar{\mu}y)\phi_n \\ &= e^{i\lambda x^2}(-y - \bar{\mu}x)F\phi_n \\ &= (\sqrt{i\mu})e^{i\lambda x^2}(-y - \bar{\mu}x)\mu^n e^{-i\lambda x^2}\phi_n \\ &= (\sqrt{i\mu})\mu^n(-y - \bar{\mu}x + 2\lambda x)\phi_n \\ &= (\sqrt{i\mu})\mu^{n+1}(\bar{\mu}(2\lambda - \bar{\mu})x - \bar{\mu}y)\phi_n \\ &= (\sqrt{i\mu})\mu^{n+1}(x - \bar{\mu}y)\phi_n \\ &= (\sqrt{i\mu})\mu^{n+1}\phi_{n+1}. \end{aligned}$$

where we have used

$$[y, e^{-i\lambda x^2}] = -2\lambda x e^{-i\lambda x^2}$$

in the fifth line of the above computation. This completes the proof. \square

3. PROOF OF THEOREM 2

Theorem. *Suppose that $\lambda \in \mathbf{R}$, $|\lambda| \geq 1$. If $U_\lambda = e^{i\lambda x^2} F \in \mathcal{U}(L^2(\mathbf{R}))$, then U has a purely continuous spectrum.*

Proof. We first prove the result for $\lambda = \pm 1$. Ignoring multiplicative constants of absolute value one, we have

$$\begin{aligned} Fe^{-ix^2/2}(e^{ix^2} F)e^{ix^2/2}F^{-1} &= Fe^{-ix^2/2}(e^{ix^2}e^{-ix^2/2}e^{-iy^2/2}e^{-ix^2/2})e^{ix^2/2}F^{-1} \\ &= Fe^{-iy^2/2}F^{-1} \\ &= e^{-ix^2/2}, \end{aligned}$$

and

$$\begin{aligned} Fe^{ix^2/2}(e^{-ix^2} F)e^{-ix^2/2}F^{-1} &= Fe^{ix^2/2}(e^{-ix^2}e^{-iy^2/2}e^{-ix^2/2}e^{-iy^2/2})e^{-ix^2/2}F^{-1} \\ &= Fe^{iy^2/2}(e^{-iy^2/2}e^{-ix^2/2}e^{-iy^2/2})(e^{-ix^2/2}e^{-iy^2/2})e^{-ix^2/2}F^{-1} \\ &= Fe^{iy^2/2}(F^2)F^{-1} \\ &= Fe^{iy^2/2}F^{-1}F^2 \\ &= e^{ix^2/2}F^2, \end{aligned}$$

which is a square root of e^{ix^2} . Thus U_λ has a purely absolutely continuous spectrum for $\lambda = \pm 1$.

For $|\lambda| > 1$, let $\eta = \lambda + \sqrt{\lambda^2 - 1}$. Write $U = U_\lambda$. We will begin by computing $U^k e^{-x^2/2}$. Using the fact that for $c \in \mathbf{C}$ with $\operatorname{Re}(c) \geq 0$,

$$Fe^{-cx^2/2} = c^{-1/2}e^{-x^2/2c},$$

(where the square root denotes its principal value), we have

$$\begin{aligned} Ue^{-cx^2/2} &= e^{i\lambda x^2} Fe^{-cx^2/2} \\ &= c^{-1/2}e^{-(\frac{1}{c}-2\lambda i)x^2/2} \\ &= c^{-1/2}e^{-g(c)x^2/2} \end{aligned}$$

for $\operatorname{Re}(c) \geq 0$, where g is the fractional linear map

$$g : z \mapsto \frac{1}{z} - 2i\lambda.$$

Note that g leaves the right half-plane invariant, and has fixed points at a_\pm , where

$$\begin{aligned} a_- &= -i\eta \\ a_+ &= -i/\eta. \end{aligned}$$

The derivatives at a_\pm are

$$g'(a_\pm) = -(a_\pm)^{-2} = -\eta^{\pm 2},$$

from which it follows that a_\pm are repelling and attracting, respectively. Thus g is biholomorphically equivalent to the map $z \mapsto (-\eta^{-2})z$, so that $g^n(z) \rightarrow a_-$ for all $z \in \hat{\mathbf{C}} \setminus \{a_+\}$. In particular, for $c_k := g^k(1)$,

$$|c_k - a_-| = |c_k + i\eta| = O(\eta^{-2k}),$$

from which it follows easily that

$$\left| \prod_{i=0}^{k-1} (c_i)^{-1/2} \right| = O(\eta^{-k/2}).$$

We therefore obtain

$$\begin{aligned} U^k e^{-x^2/2} &= \left(\prod_{i=0}^{k-1} (c_i)^{-1/2} \right) e^{-c_k x^2/2} \\ &= O(\eta^{-k/2}) e^{-c_k x^2/2}. \end{aligned}$$

Now, let

$$\{\phi_j; j = 0, 1, \dots\}$$

be the elements of the standard Hermite basis of $L^2(\mathbf{R})$:

$$\phi_j = (x - iy)^j e^{-x^2/2},$$

as described earlier. We will show that for each j ,

$$U^k \phi_j \rightarrow 0$$

uniformly on compact subsets of \mathbf{R} . Since this implies that the mass of an arbitrary element of $L^2(\mathbf{R})$ escapes any compact set, it follows easily that U has no point spectrum, and the theorem is proved. We induct on j . Fix a compact subset $K \subset \mathbf{R}$, and suppose that we have

$$U^k \phi_j = O(\eta^{-k/2}) p_j(x) e^{-c_k x^2/2} + O(\eta^{-k})$$

uniformly on K , where p_j is a polynomial of degree j in $x\eta^{-k}$. That is, p_j is of the form

$$p_j(x) = \sum_{i=0}^j a_{ji} (x\eta^{-k})^i$$

for some constants a_{ji} . We have shown that this holds for $j = 0$. It remains only to prove it for $j + 1$. Then since $\operatorname{Re}(c_k) > 0$ for each k —so that $e^{-c_k x^2/2}$ is rapidly-decreasing for each k —and $c_k \rightarrow -i\eta$ as $k \rightarrow \infty$, it will follow that for each j ,

$$U^k \phi_j \rightarrow 0$$

uniformly on K , as desired.

Using the result of Lemma 3 and the discussion following it, we see that

$$\begin{aligned} UxU^{-1} &= 2\lambda x - y \\ UyU^{-1} &= x, \end{aligned}$$

or, symbolically,

$$U \begin{pmatrix} x \\ y \end{pmatrix} U^{-1} = f \begin{pmatrix} x \\ y \end{pmatrix},$$

where $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is given by the matrix

$$\begin{pmatrix} 2\lambda & -1 \\ 1 & 0. \end{pmatrix}$$

Diagonalizing f gives

$$f = \phi^{-1} \begin{pmatrix} \eta & 0 \\ 0 & 1/\eta \end{pmatrix} \phi,$$

where

$$\phi = \begin{pmatrix} \eta & 1 \\ 1 & \eta. \end{pmatrix}$$

Then one easily obtains that

$$\begin{aligned}
U^{-k} \begin{pmatrix} x \\ y \end{pmatrix} U^k &= f^k \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} \eta^k \\ \eta^2 - 1 \end{pmatrix} \begin{pmatrix} (\eta^2 - \eta^{-2k})x + (-\eta + \eta^{-2k+1})y \\ (\eta - \eta^{-2k+1})x + (-1 + \eta^{-2k+2})y \end{pmatrix} \\
&= \begin{pmatrix} \eta^k \\ \eta^2 - 1 \end{pmatrix} \begin{pmatrix} \eta^2 x - \eta y \\ \eta x - y \end{pmatrix} + \begin{pmatrix} \eta^{-k} \\ \eta^2 - 1 \end{pmatrix} \begin{pmatrix} -x + \eta y \\ -\eta x + \eta^2 y \end{pmatrix}.
\end{aligned}$$

Thus

$$\begin{aligned}
U^k(x - iy)U^{-k} &= U^k x U^{-k} - i(U^k y U^{-k}) \\
&= \begin{pmatrix} \eta^k \\ \eta^2 - 1 \end{pmatrix} [(\eta^2 x - \eta y) - i(\eta x - y)] + \begin{pmatrix} \eta^{-k} \\ \eta^2 - 1 \end{pmatrix} [(-x + \eta y) - i(-\eta x + \eta^2 y)] \\
&= \begin{pmatrix} \eta^k \\ \eta^2 - 1 \end{pmatrix} V + \begin{pmatrix} \eta^{-k} \\ \eta^2 - 1 \end{pmatrix} W,
\end{aligned}$$

where

$$V := (\eta^2 - i\eta)x + (-\eta + i)y = (\eta - i)(\eta x - y),$$

and

$$W := (i\eta - 1)x + (\eta - i\eta^2)y = (i\eta - 1)(x - \eta y).$$

Thus for any rapidly-decreasing function ψ on \mathbf{R} , we clearly have

$$U^k(x - iy)U^{-k}\psi = O(\eta^k)(\eta x - y)\psi + O(\eta^{-k})$$

uniformly on K .

Now,

$$\begin{aligned}
U^k \phi_{j+1} &= U^k(x - iy)\phi_j \\
&= U^k(x - iy)U^{-k}U^k \phi_j \\
&= [O(\eta^k)(\eta x - y)][\eta^{-k/2}p_j(x)e^{-c_k x^2/2}] + O(\eta^{-k}) \\
&= O(\eta^{k/2})[p_j(x)(\eta x - y) + i(p_j)'(x)]e^{-c_k x^2/2} + O(\eta^{-k}) \\
&= O(\eta^{k/2})[p_j(x)(\eta - ic_k)x + i(p_j)'(x)]e^{-c_k x^2/2} + O(\eta^{-k}) \\
&= O(\eta^{k/2})[O(\eta^{-2k})xp_j(x) + i(p_j)'(x)]e^{-c_k x^2/2} + O(\eta^{-k}) \\
&= O(\eta^{-k/2})[O(\eta^{-k})xp_j(x) + i\eta^k(p_j)'(x)]e^{-c_k x^2/2} + O(\eta^{-k}) \\
&= O(\eta^{-k/2})p_{j+1}(x)e^{-c_k x^2/2} + O(\eta^{-k}),
\end{aligned}$$

where

$$p_{j+1} := [O(\eta^{-k})xp_j(x) + i\eta^k(p_j)'(x)]$$

is a polynomial of degree $j + 1$ in $x\eta^{-k}$, as desired. Note that we have used the commutator

$$[p_j(x), y] = i(p_j)'(x)$$

in the fourth line of the computation above. This completes the induction and proves the theorem. \square

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