AUTOMORPHISMS OF $\mathbf{C}^n$

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1. Introduction

Let $Aut(\mathbf{C}^n)$ denote the group (under composition) of injective holomorphic maps of $\mathbf{C}^n$ onto itself. While $Aut(\mathbf{C})$ consists only of the affine mappings $z \mapsto az + b$, $a \neq 0$, the group $Aut(\mathbf{C}^n)$, $n \geq 2$, is much larger and more complicated. The primary purpose of this paper is to show that there exist holomorphic automorphisms of $\mathbf{C}^n$, $n \geq 2$, with any finite number of prescribed coefficients in their Taylor expansions about the origin, subject only to the obvious condition that those coefficients be chosen so that the derivative at the origin is invertible. More precisely, we have the following:

Theorem. *Let $P = (P_1, \dots, P_n)$, $n \geq 2$, be a holomorphic polynomial mapping of $\mathbf{C}^n$ to itself, with $P'(0)$ invertible. Let $d = \max_i(deg(P_i))$. Then there exists $\phi \in Aut(\mathbf{C}^n)$ such that the d-jet of ϕ at 0 equals P.*

This is Theorem 2 of this paper.

One reason for the greater complexity of $Aut(\mathbf{C}^n)$ when $n \geq 2$ is the existence of shears and overshears in $Aut(\mathbf{C}^n)$ when $n \geq 2$. Choose linear coordinates $z = (z_1, \dots, z_n)$ on $\mathbf{C}^n$, and let $z' = (z_1, \dots, z_{n-1})$. An *overshear* is any mapping which under some linear coordinate change takes the form

$$z = (z', z_n) \mapsto (z', e^{h(z')} z_n + f(z')),$$

where f and h are holomorphic functions on $\mathbf{C}^{n-1}$. When $h \equiv 0$, such a mapping is called a *shear.* Note that shears have Jacobian determinant identically equal to one.

We have analogous definitions of overshears and shears for vector fields on $\mathbf{C}^n$. A vector field is called an *overshear* if it is conjugate under a linear change of coordinates to a vector field of the form

$$z = (z', z_n) \mapsto (0, h(z') z_n + f(z')),$$

where f and h are again holomorphic functions on $\mathbf{C}^n$. When $h \equiv 0$, the vector field is called a *shear.* In either case, the vector field is complete; i.e., it may be integrated for all real or complex time (the equivalence of completeness in real time to completeness in complex time was shown in [3]). For any $t \in \mathbf{C}$, the time-t map of the flow of a vector field which is an overshear (resp., a shear) is an overshear (resp., a shear) in $Aut(\mathbf{C}^n)$.

In [2], continuing work from [1], Andersén and Lempert proved that every biholomorphic mapping from a starshaped domain $\Omega \subset \mathbf{C}^n$ ($n \geq 2$) onto a Runge domain

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can be approximated uniformly on compact subsets of Ω by finite compositions of overshears. In [4], Forstneric and Rosay built on this result to show that if ϕ_t is of class C^2 in t, and is a biholomorphism from $\Omega \subset \mathbf{C}^n$ into $\mathbf{C}^n$, with $\phi_t(\Omega)$ Runge, for each $t \in [0,1]$, and if ϕ_0 is approximable on compact subsets of Ω by elements of $Aut(\mathbf{C}^n)$, then the same is true of each ϕ_t. They also remark that allowing the base domain Ω_t to vary with t gives the same result on compact subsets of $\bigcap_{t\in[0,1]} \Omega_t$.

In this paper, we apply essentially the same approximation technique to a family of maps indexed by a parameter varying in a closed polydisk in $\mathbf{C}^N$ for some N. In Theorem 1, we show that the maps may be approximated by automorphisms uniformly in the parameter, in such a way that the approximating automorphisms depend holomorphically on the parameter. Theorem 2 is then proven as a corollary of Theorem 1. In section 2, we develop some notation and state Theorems 1 and 2. Section 3 is devoted to lemmas, and sections 4 and 5 contain the proofs of Theorems 1 and 2, respectively.

2. Preliminaries

Throughout this paper, when X is a mapping from $\Omega \in \mathbf{C}^n$ to $\mathbf{C}^n$ and $U \subset \Omega$, $\|X\|_U$ will denote $\sup_{z\in U} |X(z)|$.

Let $P = (P_1, \ldots, P_n)$ be a polynomial mapping of $\mathbf{C}^n$ to itself, with

$$P_i(z) = a_{i,0} + \sum_{j=1}^{n} a_{ij} z_j + \sum_{2\le|\alpha|\le d} t_{i,\alpha} z^\alpha.$$

Here $d = \max_i(deg(P_i))$. Throughout this paper we will assume that $n \ge 2$. Write $a_0 = (a_{1,0}, \ldots, a_{n,0})$, $A = (a_{ij})$. Let

$$N = card\{t_{i,\alpha} : i = 1, \ldots, n,\ 2 \le |\alpha| \le d\}.$$

Reindex $\{t_{i,\alpha} : i = 1, \ldots, n,\ 2 \le |\alpha| \le d\}$ as $\{t_i\}_{i=1,\ldots,N}$. We shall use both indexings in what follows. Write P as P_t to denote its dependence on $t = (t_1, \ldots, t_N)$. Let

$$R = \max_{i=1,\ldots,N} |t_i|$$
$$D = \overline{\Delta^N(0, 3R)}$$
$$\mathcal{P} = \{P_t\}_{t\in D}.$$

Let $j_k(f)(p)$ denote the k-jet of a holomorphic mapping f at p. Recall that this is the equivalence class of germs of holomorphic mappings at p whose partial derivatives at p of order k and lower coincide with those of f. It may be identified with the polynomial mapping obtained by truncating that part of the Taylor series of f at p consisting of terms of order $k+1$ or higher.

Suppose $a_0 = P(0) = 0$, and that $A = P'(0)$ is invertible. Fix $\epsilon > 0$. We will show in Theorem 1 the existence of a neighborhood W of $0 \in \mathbf{C}^n$ so that, for any compact subset K of W, there exists a map

$$F : \mathcal{P} \to Aut(\mathbf{C}^n)$$
$$P_t \mapsto F(P_t)$$

where we write the i^{th} component of $F(P_t)$ as

$$\tilde{a}_{i,0} + \sum_{i=1}^{n} \tilde{a}_{ij} z_i + \sum_{2 \leq |\alpha| \leq d} \tilde{t}_{i,\alpha} z^\alpha + O(|z|^{d+1}),$$

with certain properties. Let

$$\tilde{a}_0 = (\tilde{a}_{1,0}, \ldots, \tilde{a}_{n,0})$$
$$\tilde{A} = (\tilde{a}_{i,j}).$$

Theorem 1. *There exists an open neighborhood W of $0 \in \mathbf{C}^n$ such that, for any compact set $K \subset W$, we can construct the map F described above to satisfy the following criteria:*

1. $\tilde{a}_0 = 0$.
2. $\tilde{A} = A$.
3. *The $\tilde{t}_{i,\alpha}$ depend holomorphically on the $t_{i,\alpha}$. That is, F induces a holomorphic mapping*

$$F^* : D \to \mathbf{C}^N$$
$$(t_1, \ldots, t_N) \mapsto (\tilde{t}_1, \ldots, \tilde{t}_N).$$

4. $\|F(P_t) - P_t\|_K < \epsilon$ *for all* $t \in D$.

As a corollary to Theorem 1 we obtain

Theorem 2. *Let $P = (P_1, \ldots, P_n)$ be a polynomial mapping of $\mathbf{C}^n$ to $\mathbf{C}^n$ with $P'(0)$ invertible. Let $d = \max_i(deg(P_i))$. Then there exists $\phi \in Aut(\mathbf{C}^n)$ such that $P = j_d(\phi)(0)$.*

3. Lemmas

We assume $P(0) = 0$ unless specified otherwise. Also, all vector fields used in this paper will depend on, and be integrated with respect to, complex time.

Lemma 1. *There exists an open neighborhood U of $0 \in \mathbf{C}^n$ such that $P_t : U \to P_t(U)$ is a biholomorphism for all $t \in D$.*

Proof. The proof is a simple generalization of the Inverse Function Theorem. □

Lemma 2. *There exists $\delta > 0$ such that*

$$B(0,\delta) \subset \bigcap_{t \in D} P_t(U)$$

for all $t \in D$. If $U_t = P_t^{-1}(B(0,\delta))$, then $W := \bigcap_{t \in D} U_t$ contains a neighborhood of 0.

Proof. For the first statement, it suffices to note that

$$dist(0, \partial P_t(U)) = dist(0, P_t(\partial U))$$

is a continuous function of t, since P_t is continuous in t, and so has compact image in $\mathbf{R}^+$. The second statement is essentially the same as the first, with P_t^{-1} in place of P_t and $B(0,\delta)$ in place of U. □

The following lemma is standard.

Lemma 3. *Let X and Y be two holomorphic time-independent vector fields on a domain $\Omega \subset \mathbf{C}^n$. Suppose that X and Y are uniformly bounded and close on Ω, and that the the time-t maps of their flows are defined for $|t| \leq R$ on some compact subset K of Ω. Then the time-t maps of the flows of X and Y are close on K. More precisely, if*

$$\|X\|_\Omega < M < \infty,$$

and

$$\|X - Y\|_\Omega < \epsilon,$$

then

$$\|\phi(\cdot, t) - \psi(\cdot, t)\|_K < \epsilon'(\epsilon),$$

where $\epsilon' \to 0$ as $\epsilon \to 0$, and $\phi(z,t)$ and $\psi(z,t)$ are the time-t maps of the flows of X and Y, respectively.

Proof. The proof is a straightforward application of Gronwall's Lemma. □

The following technical lemma, due to Andersén, will be of importance in the proof of Lemma 6.

Lemma 4. *Let $X : \mathbf{C}^n \times \overline{\Delta(0,r)} \to \mathbf{C}^n$ be a holomorphic time-dependent vector field. Let $U(z,t,\epsilon), V(z,t,\epsilon) = O_{z,t}(\epsilon^2)$. Let also*

$$z_{k+1} = z_k + \epsilon X(z_k, k\epsilon) + U(z_k, k\epsilon, \epsilon), \quad z_0 = \zeta$$
$$w_{k+1} = w_k + \epsilon X(w_k, k\epsilon) + V(w_k, k\epsilon, \epsilon), \quad w_0 = \zeta,$$

for all $\epsilon \in \mathbf{C}$ such that $k|\epsilon| \leq r$. Write $z_k = z_k(\zeta, \epsilon)$ and $w_k = w_k(\zeta, \epsilon)$ to indicate the dependence on ζ and ϵ. If u is continuous and $z_k(\zeta, t/k) \to u(\zeta, t)$ uniformly on compact sets as $k \to \infty$, then $w_k(\zeta, t/k) \to u(\zeta, t)$ uniformly on compact sets as $k \to \infty$.

Proof. The proof is essentially that of Theorem 4.1 of [1], with only minor modifications. □

The following lemma is similar to Corollary 1 to Theorem 4.1 in [1]. However, the modifications we require are significant enough that we provide a proof here.

Let $\Omega \subset \mathbf{C}^n$. Suppose that X is a holomorphic time-dependent vector field, and suppose that the time-t map of the flow of X, which we write as $\hat{\phi}(z,t)$, is defined on Ω for $t \in \overline{\Delta(0,r)} \subset \mathbf{C}$. Thus, in particular, X is defined on

$$\{(z,t) \in \mathbf{C}^n \times \mathbf{C} : t \in \overline{\Delta(0,r)},\ z \in \hat{\phi}(\cdot, t)(\Omega)\}.$$

Let K be a compact subset of Ω. For fixed t, let $|X'(\cdot,t)|$ denote the Jacobian determinant of $X(\cdot, t)$. Let

$$O_t = \hat{\phi}(\cdot, t)(\Omega)$$
$$K_t = \hat{\phi}(\cdot, t)(K)$$
$$A = \min_t dist(K_t, \partial O_t)$$
$$L = \max_t \||X'(\cdot,t)|\|_{O_t}$$
$$B = \frac{1}{L}(e^{rL} - 1)$$
$$C = \sup\{|X(z,t)| : t \in \overline{\Delta(0,r)},\ z \in O_t\}.$$

By shrinking Ω slightly, if necessary, we may assume that $L < \infty$ and $C < \infty$.

Let $k \in \mathbf{N}$, and let

$$\hat{\phi}_k(z,\epsilon) = \hat{\phi}(z, k\epsilon)$$

for all $\epsilon \in \mathbf{C}$ such that $k|\epsilon| \leq r$. Then

$$\hat{\phi}_{k+1}(z,\epsilon) = \hat{\phi}_k(z,\epsilon) + \epsilon X(\hat{\phi}_k(z,\epsilon), k\epsilon) + U(\hat{\phi}_k(z,\epsilon), k\epsilon, \epsilon),$$

where $U = O(|\epsilon|^2)$. Let $\eta_j(z,\epsilon)$ be the time-ϵ map of the flow of $X(\cdot, j\epsilon)$, defined for each $j \in \mathbf{N}$ such that $j|\epsilon| \leq r$ on some subset (possibly empty) of $\hat{\phi}_{j\epsilon}(\Omega)$. Let

$$\psi_k(z,\epsilon) = \begin{cases} \circ_{j=0}^{k-1} \eta_j(z,\epsilon) & k \geq 1 \\ z & k = 0. \end{cases}$$

Choose δ such that $0 < \delta < \frac{A}{2B}$. Choose $\epsilon_0 > 0$ so small that

$$\max_t \frac{1}{\epsilon} \|U(\cdot, t, \epsilon)\|_{O_t} < \frac{\delta}{2}$$
$$\max_t \frac{1}{\epsilon} \|V(\cdot, t, \epsilon)\|_{O_t} < \frac{\delta}{2}$$

when $|\epsilon| < \epsilon_0$.

Lemma 5. *The map ψ_k defined above is defined for all $(z,\epsilon) \in \mathbf{C}^n \times \mathbf{C}$ such that $z \in K$, $|\epsilon| < \min(\frac{A}{2C}, \epsilon_0)$ and $k|\epsilon| \leq r$. It satisfies*

$$\|\psi_k(\cdot,\epsilon) - \hat{\phi}_k(\cdot,\epsilon)\|_K \leq \frac{\delta}{L}((1 + |\epsilon|L)^k - 1). \tag{3.1}$$

Note that the right-hand-side of (3.1) is less than δB.

Proof. The proof is by induction on k. The statement is true for $k = 0$, since $\psi_0(z,\epsilon) = \hat{\phi}_0(z,\epsilon) = z$. Suppose that it holds for k. Let $(k+1)|\epsilon| \leq r$, with $|\epsilon| < \min(\frac{A}{2C}, \epsilon_0)$. Then $\psi_{k+1}(\cdot,\epsilon)$ is defined on K if and only if $\eta_k(\cdot,\epsilon)$ is defined on $\psi_k(\cdot,\epsilon)(K)$. But by (3.1),

$$dist(\psi_k(\cdot,\epsilon)(K), \partial O_{k\epsilon}) > A - \delta B > A/2.$$

Then $\eta_{k+1}(\cdot,\epsilon)$ is defined on $\psi_k(\cdot,\epsilon)(K)$ whenever $|\epsilon|C < A/2$. Furthermore, we have

$$\begin{aligned} \|\psi_{k+1}(\cdot,\epsilon) - \hat{\phi}_{k+1}(\cdot,\epsilon)\|_K &\leq \|\psi_k(\cdot,\epsilon) - \hat{\phi}_k(\cdot,\epsilon)\|_K \\ &\quad + |\epsilon| \cdot |||X'(\cdot, k\epsilon)|||_{O_{k\epsilon}} \|\psi_k(\cdot,\epsilon) - \hat{\phi}_k(\cdot,\epsilon)\|_K + \delta|\epsilon| \\ &\leq (1 + |\epsilon|L)\|\psi_k(\cdot,\epsilon) - \hat{\phi}_k(\cdot,\epsilon)\|_K + \delta|\epsilon| \\ &\leq \frac{\delta}{L}((1 + |\epsilon|L)^{k+1} - 1). \end{aligned}$$

□

The above lemma shows, in particular, that $\psi_k(z, t/k)$ is defined for $(z,t) \in K \times \overline{\Delta(0,r)}$, and

$$\|\psi_k(\cdot, t/k) - \hat{\phi}_k(\cdot, t/k)\|_K \leq \delta B,$$

when $k > r\max(\frac{1}{\epsilon_0}, \frac{2C}{A})$. Since δ was arbitrarily small, we have that

$$\psi_k(z, t/k) \to \hat{\phi}_k(z, t/k) = \hat{\phi}(z,t)$$

uniformly on K as $k \to \infty$.

Remark 1. The following lemma is stated without proof in [4]. We provide a proof here for completeness.

Lemma 6. *Let $X, Y_i, i = 1, \dots, k$ be holomorphic time-independent vector fields on $\mathbf{C}^n$, and suppose $X = Y_1 + \cdots + Y_k$. Suppose further that the time-t maps of the flows of X and the Y_i are defined for $|t| < r$ on some compact set $K \subset \mathbf{C}^n$. Let $N_2 \in \mathbf{N}$. Let $\tau_i(z,t)$ be the time-t map of the flow of kY_i, $\eta(z,t)$ the time-t map of the flow of X. Let*

$$\tau(z,t) = \tau_k(z, t/kN_2) \circ \tau_{k-1}(z, t/kN_2) \circ \cdots \circ \tau_1(z, t/kN_2).$$

Then

$$\lim_{N_2 \to \infty} \|\eta(\cdot, t) - \tau^{\circ N_2}(\cdot, t)\|_K = 0.$$

Proof. Note that $\eta(z, t/N_2)$ has the following expansion in powers of t/N_2:

$$\eta(z, t/N_2) = z + X(z)\frac{t}{N_2} + U(z, N_2),$$

where $U = O(1/N_2^2)$. Let $\eta_j(z,t) = \eta^{\circ j}(z, t/N_2), j = 1, \dots, N_2$. Then

$$\eta_{j+1}(z,t) = \eta_j(z,t) + X(\eta_j(z,t))\frac{t}{N_2} + U(\eta_j(z,t), j, N_2),$$

and $\eta_{N_2}(z,t) = \eta(z,t)$. Now,

$$\tau_i(z, t/kN_2) = z + kY_i(z)\frac{t}{kN_2} + V_i(z, N_2),$$

where $V_i = O(1/N_2^2)$. Then

$$\begin{aligned}\tau_{i+1} \circ \tau_i(z, t/kN_2) &= z + Y_i(z)\frac{t}{N_2} + V_i(z, N_2) \\ &\quad + Y_{i+1}(z + O(1/N_2))\frac{t}{N_2} + O(1/N_2^2) \\ &= z + (Y_{i+1} + Y_i)(z)\frac{t}{N_2} + O(1/N_2^2).\end{aligned}$$

Thus

$$\tau(z,t) = z + X(z)\frac{t}{N_2} + V(z, N_2),$$

where $V = O(1/N_2^2)$. Thus

$$\tau^{\circ(j+1)}(z,t) = \tau^{\circ j}(z,t) + X(\tau^{\circ j}(z,t))\frac{t}{N_2} + V(\tau^{\circ j}(z,t), j, N_2).$$

Now apply Lemma 4. □

Lemma 7. *Let $X : \Omega \to \mathbf{C}^n$ be a holomorphic time-independent vector field on $\Omega \subset \mathbf{C}^n$, and $L : \mathbf{C}^n \to \mathbf{C}^n$ an invertible linear map. If $\tilde{X} = L^{-1} \circ X \circ L$, and $f(z) = div(X)$, then $div(\tilde{X}) = f \circ L(z)$.*

Proof. Write $L = (l_{ij})$, $L^{-1} = (m_{ij})$. Then

$$\begin{aligned}
\tilde{X}(z_1, \dots, z_n) &= L^{-1} \circ X \circ L(z_1, \dots, z_n) \\
&= L^{-1} \circ X(\sum_j l_{1j} z_j, \dots, \sum_j l_{nj} z_j) \\
&= L^{-1}(X_1(\sum_j l_{1j} z_j, \dots, \sum_j l_{nj} z_j), \dots, X_n(\sum_j l_{1j} z_j, \dots, \sum_j l_{nj} z_j)) \\
&= (\sum_k m_{1k} X_k(\sum_j l_{1j} z_j, \dots, \sum_j l_{nj} z_j), \dots, \\
&\quad \sum_k m_{nk} X_k(\sum_j l_{1j} z_j, \dots, \sum_j l_{nj} z_j)).
\end{aligned}$$

Thus

$$\begin{aligned}
div(\tilde{X}) &= \sum_i \frac{\partial}{\partial z_i}(\sum_k m_{ik} X_k(\sum_j l_{1j} z_j, \dots, \sum_j l_{nj} z_j)) \\
&= \sum_i \sum_k m_{ik} \sum_r \frac{\partial X_k}{\partial z_r}(L(z)) l_{ri} \\
&= \sum_k \sum_r \delta_{kr} \frac{\partial X_k}{\partial z_r}(L(z)) \\
&= \sum_k \frac{\partial X_k}{\partial z_k}(L(z)) \\
&= f \circ L(z).
\end{aligned}$$

□

Lemma 8. *Let*

$$\begin{aligned}
X : \mathbf{C}^{n+1} &\to \mathbf{C}^n \\
(z, \lambda) &\mapsto w
\end{aligned}$$

be a holomorphic time-independent vector field on $\mathbf{C}^n$ depending holomorphically on a parameter λ. Then the time-t map of its flow,

$$\begin{aligned}
\phi : \mathbf{C}^{n+2} &\to \mathbf{C}^n \\
(z, \lambda, t) &\mapsto w,
\end{aligned}$$

also depends holomorphically on λ.

Proof. We reduce λ to a spatial coordinate as follows: extend X to a vector field on $\mathbf{C}^{n+1}$ by letting

$$\tilde{X}(z, \lambda) = (X_1(z, \lambda), \dots, X_n(z, \lambda), 0).$$

Let

$$\tilde{\phi}(z, \lambda, t) = (\phi_1(z, \lambda, t), \dots, \phi_n(z, \lambda, t), \lambda).$$

Then

$$\begin{aligned}\frac{\partial}{\partial t}(\tilde{\phi}(z,\lambda,t) &= (\frac{\partial}{\partial t}(\phi(z,\lambda,t),0)\\ &= (X(\phi(z,\lambda,t),\lambda),0)\\ &= \tilde{X}(\phi(z,\lambda,t),\lambda)\\ &= \tilde{X}(\tilde{\phi}(z,\lambda,t)).\end{aligned}$$

Also,

$$\tilde{\phi}(z,\lambda,0) = (z,\lambda).$$

Thus $\tilde{\phi}$ is the time-t map of the flow of $\tilde{X}$. In particular, $\tilde{\phi}$ depends holomorphically on its space variable $(z,\lambda) \in \mathbf{C}^{n+1}$ (see for instance [5], Theorem 2.8.2). But $\tilde{\phi} = (\phi, \lambda)$, so ϕ is holomorphic in (z,λ), and so in λ. □

Lemma 9. *Let X be a polynomial time-independent vector field on $\mathbf{C}^n$ depending holomorphically on a parameter λ. Write $X = \sum_{i=1}^n X_i \frac{\partial}{\partial z_i}$. Then we may write*

$$X = \sum_{k=1}^{N_3} Y_k,$$

where

1. *each Y_k is a complete holomorphic vector field on $\mathbf{C}^n$,*
2. *each Y_k depends holomorphically on the parameter λ, and*
3. *N_3 depends only on n and $d := \max_{i=1}^n(deg(X_i))$.*

Remark 2. Part (1) of this lemma is essentially proved in [1] and [2], though it is stated in slightly different language.

Proof. Fix $a_1,\ldots,a_{n-1} \in \mathbf{C}$ multiplicatively independent over $\mathbf{Q}$; i.e., if $\alpha \in \mathbf{Q}^{n-1}$ with

$$a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_{n-1}^{\alpha_{n-1}} = 1,$$

then $\alpha_i = 0$, $i = 1,\ldots,n-1$. Let $a_n = 1$. Write each X_i as the sum of its m^{th} homogeneous parts, $m = 0,\ldots,d$:

$$X_i = \sum_{m=0}^{d} p_{i,m}.$$

Let $M_m + 1$ be the number of multiindices α with $|\alpha| = m$; thus

$$M_m = \binom{m+n-1}{m} - 1.$$

Now we write

$$p_{i,m}(z) = \sum_{l=0}^{M_m} c_{i,l,m}(a^l \cdot z)^m,$$

where the $c_{i,l,m}$ depend holomorphically on the coefficients of $p_{i,m}$, which in turn depend holomorphically on λ. Here

$$\begin{aligned}a &= (a_1,\ldots,a_n)\\ a^l &= (a_1^l,\ldots,a_n^l),\end{aligned}$$

and

$$z \cdot w = \sum_{i=1}^{n} z_i w_i.$$

Explicitly (see [1], p. 231),

$$\begin{pmatrix} c_{i,0,m} \\ \vdots \\ c_{i,M_m,m} \end{pmatrix} = \begin{pmatrix} (a^{\alpha_0})^0 & \dots & (a^{\alpha_0})^{M_m} \\ \vdots & \ddots & \vdots \\ (a^{\alpha_{M_m}})^0 & \dots & (a^{\alpha_{M_m}})^{M_m} \end{pmatrix}^{-1} \begin{pmatrix} \binom{m}{\alpha_0}^{-1} p_{i,\alpha_0,m} \\ \vdots \\ \binom{m}{\alpha_{M_m}}^{-1} p_{i,\alpha_{M_m},m} \end{pmatrix}.$$

Here α_l denotes a multiindex (hence $a^{\alpha_l} \in \mathbf{C}$), and $p_{i,\alpha_l,m}$ the coefficient of $p_{i,m}$ corresponding to α_l. The matrix is invertible by the multiplicative independence of the a_i. Then we have

$$\begin{aligned} X_i(z) &= \sum_{m=0}^{d} \sum_{l=0}^{M_m} c_{i,l,m} (a^l \cdot z)^m \\ &= \sum_{l=0}^{M_d} q_{i,l} \circ \zeta_l(z), \end{aligned}$$

where ζ_l is the linear one-form $(a_1^l, \dots, a_n^l)$, and the $q_{i,l}$ are polynomials in one variable whose coefficients depend holomorphically on λ. Now, for $i = 1, \dots, n-1$, the vector field

$$Y_{i,l} : z \mapsto (0, \dots, 0, \overbrace{q_{i,l} \circ \zeta_l(z)}^{i^{th} \text{ component}}, 0, \dots, 0, -a_i^l q_{i,l} \circ \zeta_l(z))$$

is complete. It gives rise to the flow

$$z \mapsto z + tY_{i,l}(z),$$

as can be readily checked.

Consider the vector field

$$Z = X - \sum_{i=1}^{n-1} \sum_{l=0}^{M_d} Y_{i,l} \equiv \sum_{i=1}^{n} Z_i \frac{\partial}{\partial z_i}.$$

Note $Z_i = 0,\ i = 1, \dots, n-1$.

Let

$$f(z) = div(Z) = \frac{\partial Z_n}{\partial z_n}.$$

(Note that $f(z) \equiv 0$ implies Z is a shear.) Then f is a polynomial of degree $d-1$. As shown above for the X_i, we may write

$$f(z) = \sum_{m=0}^{d-1} \sum_{l=0}^{M_m} c_{l,m} (\zeta_l(z))^m,$$

where the ζ_l are linear forms depending only on l, and the coefficients $c_{l,m}$ depend holomorphically on λ. Write this as

$$f(z) = \sum_{l=0}^{M_{d-1}} p_l \circ \zeta_l(z),$$

where the p_l are polynomials in one variable whose coefficients depend holomorphically on λ.

Consider the vector field

$$\tilde{P}_l : z \mapsto (0, \dots, 0, p_l(z_1)z_n).$$

It is an overshear, thus complete. It gives rise to the flow

$$z \mapsto (z_1, \dots, z_{n-1}, e^{tp_l(z_1)}z_n).$$

We have $div(\tilde{P}_l) = p_l(z_1) = p_l \circ (1, 0, \dots, 0)(z)$. Let $L_l : \mathbf{C}^n \to \mathbf{C}^n$ be a linear map with

1. $(1, 0, \dots, 0) \circ L_l = \zeta_l$, and
2. L_l invertible.

Let $P_l = L_l^{-1} \circ \tilde{P}_l \circ L_l$. Then P_l is clearly complete, and by Lemma 7, $div(P_l) = p_l \circ \zeta_l(z)$. Furthermore, the coefficients of P_l depend holomorphically on those of $\tilde{P}_l$, which in turn depend holomorphically on λ.

Now, the vector field

$$W = Z - \sum_{l=0}^{M_{d-1}} P_l \equiv \sum_{i=1}^{n} W_i \frac{\partial}{\partial z_i}$$

satisfies

1. $W_i = 0,\ i = 1, \dots, n-1$, and
2. $div(W) = \dfrac{\partial W_n}{\partial z_n} = 0$.

Thus W is a shear, hence complete. Its coefficients clearly depend holomorphically on λ, since those of its summands do. The proof is complete. Note that N_3, the total number of the $Y_{i,l}$, P_l, and W, depends only on n and d, as desired. □

Remark 3. Note also that we may have one or more of the Y_i equalling zero for certain values of λ, and that, by this construction, if X fixes the origin, so do each of the Y_i.

4. Proof of Theorem 1

Fix $\epsilon_1 > 0$. Let W be as in the proof of Lemma 2. Replacing W with $int(W)$, if necessary, we may assume that W is open. Fix a compact set $K \Subset W$. Also fix a compact set K' such that $\bigcup_{t \in D} P_t(K) \subset K' \subset B(0, \delta)$. Define $F(P_0) = P_0$. Note that this is an automorphism satisfying $(1) - (4)$ of Theorem 1. Our strategy will be to extend F to subspaces of $\mathcal{P}$ of successively higher dimension. Suppose F has been defined on

$$\mathcal{P}_k := \{P_{(t_1, \dots, t_k, 0, \dots, 0)} : t_1, \dots, t_k \in \overline{\Delta(0, 3R)}\}$$

so that, in the notation of Theorem 1,

1. $\tilde{a}_0 = 0$,
2. $\tilde{A} = A$,
3. the $\tilde{t}_j,\ j = 1, \dots, N$ depend holomorphically on $t_1, \dots, t_k$, and
4. $\|F(P_{(t_1, \dots, t_k, 0, \dots, 0)}) - P_{(t_1, \dots, t_k, 0, \dots, 0)}\|_K < C_k \epsilon_1$ for all $(t_1, \dots, t_k) \in \overline{\Delta^k(0, 3R)}$ and some $C_k > 0$.

We show that we can define F on $\mathcal{P}_{k+1}$ so that conditions $(1) - (4)$ remain satisfied. Assume that t_1 through t_k are fixed, and let

$$\phi_{t_{k+1}} = P_{(t_1, \dots, t_{k+1}, 0, \dots, 0)}|_U.$$

To simplify notation, write $t' = t_{k+1}$. In the following, unless specified otherwise, t and t' will denote complex numbers.

By assumption, there exists $\tilde{\phi}_0 := F(\phi_0) \in Aut(\mathbf{C}^n)$ satisfying (1) – (4) above.

Claim 1. $\phi_{t'} \circ \phi_0^{-1}$ is the time-t' map of the time-dependent vector field X given by

$$X(z,t) = \frac{\partial}{\partial s}\phi_s \circ \phi_t^{-1}(z)|_{s=t}.$$

Proof.

$$\begin{aligned} X(\phi_t \circ \phi_0^{-1}(z), t) &= \frac{\partial}{\partial s}\phi_s \circ \phi_t^{-1} \circ \phi_t \circ \phi_0^{-1}(z)|_{s=t} \\ &= \frac{\partial}{\partial s}\phi_s \circ \phi_0^{-1}(z)|_{s=t}. \end{aligned}$$

□

Note that X depends holomorphically on t, since ϕ and ϕ^{-1} do. For each $t \in \overline{\Delta(0,3R)}$, $X(\cdot,t)$ is defined on $\phi_t(U)$. We now approximate $\phi_{t'} \circ \phi_0^{-1}$, the time-$t'$ map of the flow of X, by a composition of flows of time-independent vector fields.

Write $\phi_t \circ \phi_0(z)^{-1}$ as $\hat{\phi}(z,t)$. Let $N_1 \in \mathbf{N}$. For $j = 0, \dots, N_1 - 1$, let $\eta_j(\cdot, t'/N_1)$ be the time-t'/N_1 map of the time-independent vector field $X_{jt'/N_1} := X(\cdot, jt'/N_1)$. Let

$$\psi_{N_1}(\cdot, t'/N_1) = \circ_{j=0}^{N_1-1}\eta_j(\cdot, t'/N_1).$$

In the statement of Lemma 5, let $K = K'$, $r = 3R$. Given $\epsilon_2 > 0$, we apply Lemma 5 to find N_1 such that $\psi_{N_1}(\cdot, t'/N_1)$ is defined on K' and satisfies

$$\|\phi_{t'} \circ \phi_0^{-1} - \psi_{N_1}(\cdot, t'/N_1)\|_{K'} < \epsilon_2.$$

The proof of Lemma 5 shows that N_1 may be chosen independently of $t' \in \overline{\Delta(0,3R)}$, and the finiteness of N shows that it may be chosen independently of k. Note also that since the X_{jt'/N_1} are compositions of $X_{t'}$ with linear maps, their coefficients still depend holomorphically on t'. They are defined and holomorphic on $\phi_{jt'/N_1}(U) \supset B(0,\delta)$. We now approximate them by holomorphic vector fields on $\mathbf{C}^n$.

Given $\epsilon_3 > 0$, fix N_4 so that

$$\|\tilde{X}_{jt'/N_1} - X_{jt'/N_1}\|_{B(0,\delta)} < \epsilon_3$$

for $j = 0, \dots, N_1 - 1$, where

$$\tilde{X}_{jt'/N_1} = j_{N_4}(X_{jt'/N_1})(0).$$

Thus the $\tilde{X}_{jt'/N_1}$ are polynomial vector fields, and so defined on $\mathbf{C}^n$. Clearly they still depend holomorphically on t'. The compactness of D and the finiteness of N ensure that N_4 may be chosen independently of t' and k.

By Lemma 9, for each j we may write

$$\tilde{X}_{jt'/N_1} = Y_{j,1} + \cdots + Y_{j,N_3},$$

where the $Y_{j,i}$ are complete and depend holomorphically on t', and N_3 depends only on N_4.

Let $\tilde{\eta}_j$ be the time-t'/N_1 map of the flow of $\tilde{X}_{jt'/N_1}$. Then, if ϵ_3 was chosen sufficiently small, $\tilde{\eta}_j$ is defined on K' for all j. Now, given $\epsilon_4 > 0$, we can apply

Lemma 6 to find N_2 such that, if $\tau_{j,i}$ denotes the time-$t'/N_3N_2N_1$ map of the flow of $N_3Y_{j,i}$, and if $\tau_j = \tau_{j,N_3} \circ \cdots \circ \tau_{j,1}$, then

$$\|\tilde{\eta}_j - \tau_j^{\circ N_2}\|_{K'} < \epsilon_4.$$

By Lemmas 8 and 9, the $\tau_{j,i}$ depend holomorphically on t'. Note that they are automorphisms of $\mathbf{C}^n$, being the flows of complete time-independent vector fields on $\mathbf{C}^n$.

Let $\tilde{\psi}_{N_1} = \tilde{\eta}_{N_1-1} \circ \cdots \circ \tilde{\eta}_0$. By Lemma 3, we may choose ϵ_3 so that

$$\|\psi_{N_1} - \tilde{\psi}_{N_1}\|_{K'} < \epsilon_2.$$

We may also choose ϵ_4 so that

$$\|\tilde{\psi}_{N_1} - \circ_{j=0}^{N_1-1}\tau_j^{\circ N_2}\|_{K'} < \epsilon_2.$$

Then

$$\|\phi_{t'} \circ \phi_0^{-1} - \circ_{j=0}^{N_1-1}\tau_j^{\circ N_2}\|_{K'} < 3\epsilon_2.$$

Let

$$\tilde{\phi}_{t'} = (\circ_{j=0}^{N_1-1}\tau_j^{\circ N_2}) \circ \tilde{\phi}_0.$$

Then $\tilde{\phi}_{t'}$ is an automorphism of $\mathbf{C}^n$ depending holomorphically on $(t_1, \ldots, t_k)$ and on $t' = t_{k+1}$. By Hartogs's separate analyticity theorem, it thus depends holomorphically on $(t_1, \ldots, t_{k+1})$. For ϵ_2 sufficiently small, we have

$$\|\tilde{\phi}_{t'} - \phi_{t'}\|_K < \tilde{C}_{k+1}\epsilon_1,$$

for all $(t_1, \ldots, t_{k+1}) \in \overline{\Delta^{k+1}(0, 3R)}$, and some $\tilde{C}_{k+1} > 0$. By construction, $\tilde{\phi}_{t'}(0) = 0$. Let

$$\tilde{A} = (\phi_{t'})'(0) \circ ((\tilde{\phi}_{t'})'(0))^{-1} = A \circ ((\tilde{\phi}_{t'})'(0))^{-1}.$$

Then $\tilde{A}$ depends holomorphically on $(t_1, \ldots, t_{k+1})$ and is close to the identity. If we let

$$F(P_{(t_1, \ldots, t_{k+1}, 0, \ldots, 0)}) = \tilde{A} \circ \tilde{\phi}_{t'},$$

then

$$\|F(P_{(t_1, \ldots, t_{k+1}, 0, \ldots, 0)}) - P_{(t_1, \ldots, t_{k+1}, 0, \ldots, 0)})\|_K < C_{k+1}\epsilon_1$$

for some $C_{k+1} > 0$, and $F(P_{(t_1, \ldots, t_{k+1}, 0, \ldots, 0)})$ satisfies properties $(1) - (4)$ above.

We now continue until F has been extended to $\mathcal{P}_N = D$. Then F^* is holomorphic in D, and, if ϵ_1 was chosen sufficiently small relative to ϵ, Theorem 1 is proven. □

5. Proof of Theorem 2

By post-composing with a translation if necessary, we may assume that $P(0) = 0$. Take

$$F^* : D \to \mathbf{C}^N$$
$$t \mapsto \tilde{t}$$

as in Theorem 1.

Claim 2. $\overline{\Delta^N(0, R)} \subset F^*(D)$.

Proof. In Theorem 1, choose K to contain an open neighborhood of 0. By the Cauchy estimates and the compactness of D, $\|F^* - \mathrm{id}\|_D < \epsilon'(\epsilon)$, where $\epsilon'(\epsilon) \to 0$ as $\epsilon \to 0$. Thus $F^* - \mathrm{id}$ maps D into $B(0,\epsilon') \subset \mathbf{C}^n$. By the Cauchy estimates, therefore, the eigenvalues of $(F^* - \mathrm{id})'(t)$ are less than or equal to $\epsilon'/R < \epsilon'$ for all $t \in (2/3)D = \overline{\Delta^N(0,2R)}$. Thus, if λ is an eigenvalue of $(F^*)'(t), t \in (2/3)D$, then $|\lambda - 1| \leq \epsilon'$; so in particular $|\lambda| \geq 1 - \epsilon'$. Thus

$$|\det(F^*)'(t)| \geq (1-\epsilon')^N \neq 0 \qquad \forall t \in (2/3)D.$$

Then, by the inverse function theorem, $F^*|_{(2/3)D}$ is an open mapping, and since $\|F^*(t) - t\|_D < \epsilon'$, we must have $\overline{\Delta^N(0,R)} \subset F^*(D)$, as desired, for ϵ' sufficiently small. □

Thus, in particular, $t \in F^*(D)$, and we are done. □

6. Final Remark

Theorem 2 may be used to produce examples of automorphisms of $\mathbf{C}^2$ tangent to the identity, and having an open set of points attracted to the origin and biholomorphic to $\mathbf{C}^2$ on which the map is biholomorphically conjugate to

$$(z,w) \mapsto (z+1,w).$$

These will be given in a future paper.

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