ATTRACTING BASINS FOR AUTOMORPHISMS OF $\mathbf{C}^2$

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1. Introduction

It is a well-known result of iteration theory of one complex variable that any germ of holomorphic function at zero of the form

$$f(z) = \lambda z + O(|z|^2),$$

with $f'(0) = \lambda$ a root of unity, is either linear or has a basin of attraction to the origin; that is, an open set U, with $0 \in \partial U$, such that $f^n(z) \to 0$ locally uniformly as $n \to \infty$ for $z \in U$. This phenomenon is discussed in detail in, for example, [1] and [5].

In $\mathbf{C}^n$, $n \geq 2$, the situation is somewhat different. For simplicity, we will consider mappings of the form

$$F(z) = z + O(|z|^2).$$

Assume $F \neq \mathrm{Id}$. Then F may or may not have a domain of attraction to the origin. Easy examples of the former situation are furnished by product mappings (f, f), f as above, and of the latter by shears $(z, w) \mapsto (z, w + g(z))$, where $g(0) = 0$ and $g'(0) = 0$. Further examples are provided by Fatou in his brief treatment of this question in [2].

An interesting question, however, is whether an automorphism of $\mathbf{C}^n$ tangent to the identity can have such a domain, and if so, whether that domain is biholomorphic to $\mathbf{C}^n$. (We note that Fatou's examples in [2] are not realizable as automorphisms.) This clearly cannot happen when $n = 1$, since the automorphism group of $\mathbf{C}$ is the group of affine mappings $z \mapsto az + b$, $a \neq 0$. In this paper, we provide the following answer.

Theorem 1. *There exist automorphisms of $\mathbf{C}^2$ tangent to the identity with an invariant domain of attraction to the origin, biholomorphic to $\mathbf{C}^2$, on which the automorphism is biholomorphically conjugate to the map*

$$(x, y) \mapsto (x - 1, y).$$

The plan of this paper is as follows. In Section 2, we make use of the technique of directional blow-up of the origin, which has been employed extensively in the study of vector fields and diffeomorphisms on $\mathbf{R}^2$, to find domains of attraction to the origin for a particular class of germs of holomorphic mappings of $\mathbf{C}^2$ tangent to the identity. We then use the following theorem from [8] to show that this class of germs can be realized as automorphisms of $\mathbf{C}^2$.

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Theorem 2 ([8])**.** *Let $P = (P_1, \dots, P_n)$ be any polynomial mapping from $\mathbf{C}^n$ to itself with $P'(0)$ invertible. Let $d \geq \max_i(deg(P_i))$. Then there exists an automorphism ϕ of $\mathbf{C}^n$ such that $\phi(z) - P(z) = O(|z|^{d+1})$.*

Sections 3 through 10 are devoted to proving that the domain of attraction found in Section 2 is biholomorphic to $\mathbf{C}^2$, and the automorphism conjugate there to translation. In them we adapt some techniques used by Ueda in his proof of the following theorem from [6] (we state the theorem in a more general form proven by Hakim in [3]).

Theorem 3 ([6]),([3])**.** *Let $F = (f_1, f_2)$ be an automorphism of $\mathbf{C}^2$ with $F(0) = 0$ and*

$$F'(0) = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix},$$

$0 < |b| < 1$. Suppose further that F has no curve of fixed points through the origin. Then F has a domain of attraction to the origin biholomorphic to $\mathbf{C}^2$ and on which F is biholomorphically conjugate to the mapping

$$(x, y) \mapsto (x - 1, y).$$

Note that there exist Hénon mappings satisfying the hypotheses of Theorem 3. In constrast, the automorphisms of Theorem 1 clearly cannot be polynomial.

In the interest of sparing the reader some technicalities, the argument of this paper has not been given in the greatest possible generality. The final section, therefore, contains remarks about extending the above construction to much larger classes of automorphism. The details of this more general argument may be found in [8].

2. Construction of an invariant domain of attraction

By Theorem 2, there exist infinitely many automorphisms F of $\mathbf{C}^2$ of the form $F = (f_1, f_2)$, where

$$f_1(x, y) = x + x^2 + O(|z|^3) \tag{2.1}$$

$$f_2(x, y) = y + 2xy + O(|z|^3). \tag{2.2}$$

Let

$$\begin{aligned} \phi : \mathbf{C}^2 &\to \mathbf{C}^2 \\ (x, s) &\mapsto (x, xs) =: (x, y). \end{aligned}$$

It is easily checked that ϕ is a biholomorphism away from the s-axis.

Lemma 1. *There exists a solution $\tilde{F}$ to the functional equation*

$$\phi \circ \tilde{F} = F \circ \phi \tag{2.3}$$

on a set W of the form $\mathbf{C}^2 \setminus X$, X a one-dimensional analytic variety. In particular, $\tilde{F}$ is defined on a neighborhood of the s-axis.

Proof. $\tilde{f}_1(x,s) := f_1(x,xs) = x(g(x,s))$, where $g(x,s) = 1+x+O_s(|x|^2)$. It follows that g is nonzero in a set of the form

$$N = \{(x,s) : |x| < \epsilon(s)\},$$

where ϵ is a continuous, strictly positive function of s. Now, define $\tilde{F} = (\tilde{f}_1, \tilde{f}_2)$ in $\mathbf{C}^2 \setminus \{(x,s) : f_1(x,xs) = 0\}$ by

$$\tilde{f}_1(x,s) = f_1(x,xs) \tag{2.4}$$

$$\tilde{f}_2(x,s) = \frac{f_2(x,xs)}{f_1(x,xs)}. \tag{2.5}$$

Since F is holomorphic on $\mathbf{C}^2$, the expansions (2.1) and (2.2) converge in all of $\mathbf{C}^2$. Since

$$f_2(x,xs) = xs + 2x^2s + O_s(|x|^3) = x(s + 2xs + O_s(|x|^2))$$

vanishes to first order on the s-axis, the domain of definition of $\tilde{F}$ may thus be extended to

$$W := \mathbf{C}^2 \setminus \{(x,s) : g(x,s) = 0\}.$$

In particular, $N \subset W$.

Now, for all $(x,s) \in W$, $\tilde{f}_2$ can be written

$$\tilde{f}_2(x,s) = \frac{xs + 2x^2s + O_s(|x|^3)}{x + x^2 + O_s(|x|^3)} \tag{2.6}$$

$$= \frac{s + 2xs + O_s(|x|^2)}{1 + x + O_s(|x|^2)}, \tag{2.7}$$

since the expansions in the numerator and denominator of (2.7) converge in all of $\mathbf{C}^2$, and the denominator is nonzero in W. By shrinking, if necessary, the function ϵ in Lemma 1, in N the equality (2.7) takes the form

$$\tilde{f}_2(x,s) = (s + 2xs + O_s(|x|^2)(1 - x + O_s(|x|^2)) \tag{2.8}$$

$$= s + sx + O_s(|x|^2). \tag{2.9}$$

This expansion converges in N. Thus we see, in particular, that $\tilde{F}$ fixes the s-axis. It is trivial to verify that $\tilde{F}$ satisfies the functional equation (2.3) in W. □

Proposition 2. *$\tilde{F}$ has an invariant domain $D_{(x,s)}$ of uniform attraction to the origin.*

Proof. We make the coordinate change $(x,s) \mapsto (1/x, s) \equiv (u,s)$ in $\mathbf{C}^2 \setminus \{(x,s) : x = 0\}$. Let $u_n = u \circ \tilde{F}^n$, $s_n = s \circ \tilde{F}^n$. Then

$$u_1 = u - 1 + O_s(\frac{1}{|u|}) \tag{2.10}$$

$$s_1 = s(1 + \frac{1}{u}) + O_s(\frac{1}{|u|^2}), \tag{2.11}$$

where these series converge on $\{(u,s) : (1/u, s) \in N\}$. Fix $\delta > 0$. Then, in particular, there exists $R > 0$ such that (2.10) and (2.11) converge on

$$\{(u,s) : |u| > R, |s| < \delta\}.$$

In the course of this article we will often have occasion to increase R.

Let

$$D_1 = U' \times \{s : |s| < \delta\},$$

where

$$U' = \{u : \mathrm{Re}(u) < -R\}.$$

By Taylor's Theorem, there exists $k > 0$ such that

$$|u_1 - u + 1| < \frac{k}{|u|} < \frac{k}{R} \tag{2.12}$$

$$|s_1 - s(1 + 1/u)| < \frac{k}{|u|^2} < \frac{k}{R^2} \tag{2.13}$$

in D_1. Note that increasing R decreases D_1, so that k need not be increased. Choose R large enough that $k/R < \min(\delta/2, 1/2)$.

Now, for $a < -1$, let

$$b(a) = \sqrt{3a^2 + 3a + 9/16}.$$

Note that b then solves the equation

$$\sqrt{a^2 + b^2} - \sqrt{(a+1)^2 + b^2} = 1/2$$

for $a < -1$. Let

$$U = \{u \in U' : |\mathrm{Im}(u)| < b(\mathrm{Re}(u))\}.$$

Then

$$|u| - |u + 1| > 1/2$$

in U, so that

$$\frac{k}{|u|} < \frac{\delta}{2} < \delta(|u| - |u + 1|)$$

in U, or

$$\frac{k}{|u|^2} < \delta(1 - |1 + 1/u|),$$

or

$$\delta|1 + 1/u| + \frac{k}{|u|^2} < \delta$$

in U.

Let

$$D = U \times \{s : |s| < \delta\} \subset D_1.$$

We claim that $(u_1, s_1)(D) \subset D$. Let $(u, s) \in D$. Then

$$\begin{aligned} |s_1| &< |s(1 + 1/u)| + \frac{k}{|u|^2} \\ &< \delta|1 + 1/u| + \frac{k}{|u|^2} \\ &< \delta. \end{aligned}$$

Also, when a is large, $\frac{\partial b}{\partial a} \sim -\sqrt{3}$. So for R sufficiently large,

$$\begin{aligned} u \in U &\implies \Delta(u - 1, 1/2) \subset U \\ &\implies u_1 \in U. \end{aligned}$$

Thus $(u_1, s_1) \in D$, and the claim is proved.

Now, for $(u, s) \in D$, we have

$$|u_1 - u + 1| = |u_1 - (u - 1)| < \frac{k}{R} < \frac{1}{2}, \tag{2.14}$$

so

$$\begin{aligned}|u_n - u + n| &\le \sum_{i=1}^{n} |u_i - u_{i-1} + 1| \\ &\le n/2\end{aligned}$$

so that

$$n/2 < |u_n - u| < 3n/2. \tag{2.15}$$

Thus we obtain the following lemma, which will be useful in the chapters to come.

Lemma 3. *Let* $(u, s) \equiv (u_0, s_0) \in D$, $k > 0$. *Then* $|u_n| \sim n$. *In particular,*

$$\sum_{i=0}^{\infty} \frac{1}{|u_i|^k}$$

converges uniformly in D *if* $k > 1$, *and diverges if* $k \le 1$.

It remains only to show that $s_n \to 0$.

Lemma 4. *Let*

$$P_n = \prod_{i=0}^{n} \left| 1 + \frac{1}{u_i} \right|.$$

Then $P_n \to 0$ *uniformly for* $(u, s) \equiv (u_0, s_0) \in D$. *More precisely,* $P_n = O(1/|u_n|) = O(1/n)$.

Proof. Since

$$P_n = \left| \frac{1 + u_0}{u_n} \right| \prod_{i=0}^{n-1} \left| \frac{u_{i+1} + 1}{u_i} \right|,$$

it suffices to show that

$$S_n := \prod_{i=0}^{n} \left| \frac{u_{i+1} + 1}{u_i} \right|$$

converges as $n \to \infty$ for $(u, s) \equiv (u_0, s_0) \in D$. But

$$\big| |u_{i+1} + 1| - |u_i| \big| \le |u_{i+1} - u_i + 1| < \frac{k}{|u_i|},$$

so

$$-\frac{k}{|u_i|} + |u_i| < |u_{i+1} + 1| < \frac{k}{|u_i|} + |u_i|,$$

or

$$1 - \frac{k}{|u_i|^2} < \left| \frac{u_{i+1} + 1}{u_i} \right| < 1 + \frac{k}{|u_i|^2}.$$

Now, S_n converges if and only if $\log(S_n)$ converges. But

$$\log(S_n) = \sum_{i=0}^{n} \log \left| \frac{u_{i+1} + 1}{u_i} \right|,$$

and

$$\log(1 - \frac{k}{|u_i|^2}) < \log \left| \frac{u_{i+1} + 1}{u_i} \right| < \log(1 + \frac{k}{|u_i|^2}).$$

Now, $\log(1+x) = x + O(|x|^2)$ for x close to 0, so $c_1|x| < |\log(1+x)| < c_2|x|$ for all x in a fixed compact neighborhood of 0 and some $c_1, c_2 > 0$. Thus

$$\frac{c_1 k}{|u_i|^2} < \left|\log\left|\frac{u_{i+1}+1}{u_i}\right|\right| < \frac{c_2 k}{|u_i|^2}$$

for $u_i \in U$. So S_n converges if $\sum_{i=0}^{\infty} 1/|u_i|^2$ converges. But this follows from Lemma 3. □

Lemma 5. *If* $(u, s) \in D$*, then*

$$|s_n| < |s| \prod_{i=0}^{n-1} \left|1 + \frac{1}{u_i}\right| + \sum_{j=0}^{n-2} \frac{k}{|u_j|^2} \prod_{i=j+1}^{n-1} \left|1 + \frac{1}{u_i}\right| + \frac{k}{|u_{n-1}|^2}. \tag{2.16}$$

Proof. The proof is by induction on n. For $n = 1$, we have

$$\begin{aligned} |s_1| - |s|\,|1 + 1/u| &\le |s_1 - s(1 + 1/u)| \\ &< \frac{k}{|u|^2} \end{aligned}$$

by (2.13). Now, assume that (2.16) holds for $n \le N$. Then

$$|s_{N+1}| < |s_N| \left|1 + \frac{1}{u_N}\right| + \frac{k}{|u_N|^2},$$

by (2.13). The statement follows immediately for $n = N + 1$. □

Now, by Lemmas 3 and 4, we have that each summand in (2.16) goes to zero as $n \to \infty$. Fix $\epsilon > 0$. Fix N_1 so large that

$$\sum_{i=N_1}^{\infty} \frac{k}{|u_i|^2} < \epsilon/2.$$

Choose $n >> N_1$ so large that each of the first $N_1 + 1$ summands in the estimate (2.16) for $|s_n|$ is less than $\frac{\epsilon}{2(N_1+1)}$. Then

$$\begin{aligned} |s_n| &< (N_1 + 1)\left(\frac{\epsilon}{2(N_1 + 1)}\right) + \sum_{j=N_1}^{n-2} \frac{k}{|u_j|^2} \prod_{i=j+1}^{n-1} \left|1 + \frac{1}{u_i}\right| + \frac{k}{|u_{n-1}|^2} \\ &< \epsilon/2 + \sum_{j=N_1}^{\infty} \frac{k}{|u_j|^2} \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Thus $s_n \to 0$ as $n \to \infty$.

For more precise information on the rate at which $|s_n|$ converges to zero, note that the first term in (2.16) goes to zero at the same rate as $1/n$, by Lemma 4, and the last at the same rate as $1/n^2$, by Lemma 3. The remaining term,

$$\sum_{j=0}^{n-2} \frac{k}{|u_j|^2} \prod_{i=j+1}^{n-1} \left|1 + \frac{1}{u_i}\right|,$$

goes to zero at the same rate as

$$\begin{aligned}\sum_{j=0}^{n-2} \frac{k}{|u_j|^2}\frac{j}{n} &\sim \frac{1}{n}\sum_{j=0}^{n-2}\frac{1}{j^2}j \\ &\sim \frac{1}{n}\sum_{j=0}^{n-2}\frac{1}{j} \\ &\sim \frac{\log n}{n}.\end{aligned}$$

Thus we have the following lemma.

Lemma 6. *Let* $(u,s) \equiv (u_0, s_0) \in D$. *Then*

$$|s_n| = o\left(\frac{1}{n^q}\right)$$

for any $0 < q < 1$.

Write $D = D_{(u,s)}$. Let

$$D_{(x,s)} := \{(x,s) : (1/x, s) \in D_{(u,s)}\}.$$

Then $D_{(x,s)}$ is a bounded open set which is forward invariant under $\tilde{F}$ and satisfies $\tilde{F}^n \to 0$ uniformly on $D_{(x,s)}$, as desired. This concludes the proof of Proposition 2. □

Let $D_{(x,y)} = \phi(D_{(x,s)})$, and let

$$\Omega = \bigcup_{n=0}^{\infty} F^{-n}(D_{(x,y)}). \tag{2.17}$$

3. Semi-conjugacy to translation

We will now prove that F is semi-conjugate to translation on $D_{(x,y)}$; that is, we show that there exists $\psi : D_{(x,y)} \to \mathbf{C}$ such that

$$\psi \circ F(p) = \psi(p) - 1 \tag{3.1}$$

for all $p \in D_{(x,y)}$.

The function ψ is often referred to in the literature as the Abel-Fatou function. Write

$$u_1 = u - 1 + \frac{c(s)}{u} + O_s(\frac{1}{|u|^2}), \tag{3.2}$$

where c is entire. Thus c satisfies a Lipschutz condition

$$|c(s) - c(s')| < K|s - s'|$$

for all $s, s' \in \{|s| < \delta\}$.

Choose and fix a branch of the logarithm on U. Let

$$\mu_n = u_n + n + c(s_n)\log u_n.$$

Lemma 7. μ_n *converges as* $n \to \infty$ *to a holomorphic function* μ *on* $D_{(u,s)}$, *where*

$$\mu(u,s) = u + c(s)\log u + \eta(u,s), \tag{3.3}$$

η *a holomorphic bounded function on* $D_{(u,s)}$ *satisfying* $\eta \to 0$ *uniformly in* $D_{(u,s)}$ *as* $u \to \infty$.

Proof. We have

$$\begin{aligned}|\mu_{n+1}-\mu_n| &= |u_{n+1}-u_n+1+c(s_{n+1})\log u_{n+1}-c(s_n)\log u_n| \\ &= |u_{n+1}-u_n+1+c(s_{n+1})\frac{\log u_{n+1}}{\log u_n}+(c(s_{n+1})-c(s_n))\log u_n| \\ &\le |u_{n+1}-u_n+1+c(s_{n+1})\frac{\log u_{n+1}}{\log u_n}|+|(c(s_{n+1})-c(s_n))\log u_n| \\ &=: A+B.\end{aligned}$$

Also, using (2.11), we obtain

$$s_{n+1}-s_n=\frac{s_n}{u_n}+O(\frac{1}{|u_n|^2})$$

in $D_{(u,s)}$, where we have omitted the subscript s from the big O expression, since s_n is uniformly bounded in $D_{(u,s)}$.

Now, from (3.2), we have

$$\begin{aligned}\log\left(\frac{u_{n+1}}{u_n}\right) &= \log\left(\frac{u_n-1+O(1/|u_n|)}{u_n}\right) \\ &= \log\left(1-\frac{1}{u_n}+O(\frac{1}{|u_n|^2})\right) \\ &= -\frac{1}{u_n}+O(\frac{1}{|u_n|^2}),\end{aligned}$$

so that

$$\begin{aligned}A &= |(u_n-1+\frac{c(s_n)}{u_n}+O(\frac{1}{|u_n|^2}))-u_n+1-\frac{c(s_{n+1})}{u_n}+O(\frac{1}{|u_n|^2})| \\ &= |\frac{c(s_n)-c(s_{n+1})}{u_n}+O(\frac{1}{|u_n|^2})| \\ &\le K|\frac{s_{n+1}-s_n}{u_n}|+O(\frac{1}{|u_n|^2}) \\ &= K|\frac{s_n}{u_n^2}|+O(\frac{1}{|u_n|^2}) \\ &= O(\frac{1}{|u_n|^2}).\end{aligned}$$

Also,

$$\begin{aligned}B &\le K|s_{n+1}-s_n||\log u_n| \\ &= K|\frac{s_n}{u_n}+O(\frac{1}{|u_n|^2})||\log u_n| \\ &= o(\frac{1}{n^q})\cdot|\frac{\log u_n}{u_n}|+O(|\frac{\log u_n}{|u_n|^2}|)\end{aligned}$$

for any $0<q<1$, where we have used Lemma 6 in the last line.

This expression depends uniformly on u_n, so that it goes to zero uniformly in $D_{(u,s)}$ as $u_n\to\infty$. If $0<q'<2$, we have also, by Lemma 3, that

$$A+B=o(\frac{1}{n^{q'}}).$$

Thus

$$|\mu_{n+1} - \mu_n| = o(\frac{1}{n^{q'}}),$$

so that

$$\mu_n - \mu_0 = \sum_{i=0}^{\infty} (\mu_{i+1} - \mu_i) \tag{3.4}$$

converges absolutely uniformly on $D_{(u,s)}$ to a holomorphic limit η. Our estimates above show that $\eta \to 0$ uniformly in $D_{(u,s)}$ as $u \to \infty$.

Let $\mu = \lim \mu_n = \mu_0 + \eta = u + c(s) \log u + \eta$. □

Let $\psi(x,y) = \mu(u,s) = \mu(1/x, y/x)$. Then

$$\begin{aligned}
\psi \circ F(p) &= \lim_{n\to\infty} [u_{n+1}(p) + n + c(s_{n+1}(p)) \log(u_{n+1}(p))] \\
&= \lim_{n\to\infty} [u_{n+1}(p) + (n+1) + c(s_{n+1}(p)) \log(u_{n+1}(p))] - 1 \\
&= \psi(p) - 1,
\end{aligned}$$

so that ψ satisfies the functional equation (3.1) in $D_{(x,y)}$.

4. New coordinates on $D_{(x,y)}$

Consider the mapping from $D_{(x,y)}$ to $\mathbf{C}^2$ given by

$$(x,y) \mapsto (\psi(x,y), s) = (\psi(x,y), y/x) =: (t,v).$$

Lemma 8. *(t,v) is a biholomorphism from $D_{(x,y)}$ onto its image.*

Proof. It suffices to show that (t,v) is injective in $D_{(x,y)}$. Suppose, then, that there exist points (x_1,y_1) and (x_2,y_2) in $D_{(x,y)}$ such that

$$(\psi(x_1,y_1), s_1) = (\psi(x_2,y_2), s_2),$$

where $s_i = y_i/x_i$, $i = 1,2$. Then $s_1 = s_2 =: s$. Thus $\psi(x_1,y_1) = \psi(x_2,y_2)$, or, putting $u_1 = 1/x_1$ and $u_2 = 1/x_2$, $\mu(u_1,s) = \mu(u_2,s)$ So we need only show that for each s with $|s| < \delta$, the mapping

$$\begin{aligned}
\mu_s : U &\to \mathbf{C} \\
u &\mapsto \mu(u,s)
\end{aligned}$$

is injective in U. By shrinking U slightly if necessary, we may assume that μ_s is holomorphic across ∂U for all $|s| < 1$. For $r >> R$, let $U_r = \{u \in U : |u| < r\}$. Note that

$$\frac{\partial \mu_s(u)}{\partial u} = 1 + \frac{c(s)}{u} + \frac{\partial \eta(u,s)}{\partial u}.$$

Recall that $\eta(u,s) \to 0$ uniformly in $D_{(u,s)}$ as $u \to \infty$. The Cauchy estimates then give that $\dfrac{\partial \eta(u,s)}{\partial u} \to 0$ uniformly in $D_{(u,s)}$ as $u \to \infty$. Thus we may assume that R was chosen large enough that $\dfrac{\partial \mu_s(u)}{\partial u}$ is uniformly close to 1 for $(u,s) \in D_{(u,s)}$. Thus μ_s disturbs the tangents to ∂U_r by uniformly small amounts. As u traces out ∂U_r, therefore, $\mu_s(u)$ must trace out a simple closed curve. The argument principle then gives that μ_s is injective in each U_r. Since this is true for each r as $r \to \infty$, each μ_s is injective in U. □

5. Modification of $D_{(x,y)}$

For reasons which will become apparent in the sections to follow, we wish to modify our domain $D_{(x,y)}$ so that its image under the coordinate map (t, v) is a simple product. Let

$$T = \bigcap_{|s|<\delta} \mu_s(U).$$

Recall that $\eta(u, s) \to 0$ uniformly in $D_{(u,s)}$ as $u \to \infty$, so that for any $\epsilon > 0$, we may assume that R was chosen so that

$$|\mu_s(u) - (u + c(s)\log(u))| < \epsilon$$

for $(u, s) \in D_{(u,s)}$. Thus T contains $\{u \in U + c(s)\log(U) : dist(u, \partial(U + c(s)\log(U))) > \epsilon\}$, and therefore has the same asymptotic boundary behavior as U. Replacing T by $int(T)$ if necessary, we may assume T is open and satisfies instead

$$T \subset \bigcap_{|s|<\delta} \mu_s(U).$$

Let

$$D'_{(u,s)} = \{(u, s) \in D_{(u,s)} : \mu(u, s) \in T\}$$
$$D'_{(x,y)} = \{(x, y) : (1/x, y/x) \in D'_{(u,s)}\}.$$

Then we have the following.

Lemma 9.

$$(t, v)(D'_{(x,y)}) = T \times \{v : |v| < \delta\}. \tag{5.1}$$

Proof. It is clear that the left-hand-side of (5.1) is contained in the right-hand-side. We need only show the reverse containment. Given $t_0 \in T$, $|v_0| < \delta$, choose $u_0 \in U$ such that $\mu_{v_0}(u_0) = t_0$ (this is possible by the definition of T). Let $x_0 = 1/u_0$ and $y_0 = v_0/u_0$. Then $p := (x_0, y_0) \in D'_{(x,y)}$, with $t(p) = t_0$, $v(p) = v_0$. □

6. Extension of the Abel-Fatou function to Ω

Until now, we have made no use of the fact that $F \in Aut(\mathbf{C}^n)$. All of the results obtained thus far apply to general germs of holomorphic mappings F of the form

$$f_1(x, y) = x + x^2 + O(|z|^3)$$
$$f_2(x, y) = y + 2xy + O(|z|^3).$$

Now, however, we will begin to make use of the global properties of F.

We extend ψ to Ω via (3.1) as follows: for $p \in \Omega$, choose an n such that $F^n(p) \in D'_{(x,y)}$. Define

$$\psi(p) = \psi \circ \tilde{F}^n(p) + n. \tag{6.1}$$

Then ψ is holomorphic in $\tilde{\Omega}$ and satisfies (3.1) there. It is straightforward to check that it is well- defined; i.e., that the definition (6.1) is independent of n.

Because of the asymptotic boundary behavior of T, this extended ψ maps Ω onto $\mathbf{C}$, since its image contains $T + n$ for all $n \in \mathbf{N}$. We wish to make ψ a global coordinate on Ω. In the sections to follow, we will put a new coordinate on the fibers of ψ.

7. Construction of an invariant one-form on $D_{(x,y)}$

The Jacobian matrix of the coordinate change

$$(u,s) \mapsto (t,v)$$

is

$$\begin{pmatrix} \partial t/\partial u & \partial t/\partial s \\ 0 & 1 \end{pmatrix}. \tag{7.1}$$

We have seen that η is uniformly bounded on $D'_{(u,s)}$, and that $\eta \to 0$ uniformly in $D'_{(u,s)}$ as $u \to \infty$. By the Cauchy estimates, therefore, $\dfrac{\partial \eta}{\partial u}$ and $\dfrac{\partial \eta}{\partial s} \to 0$ uniformly in $D'_{(u,s)}$ as $u \to \infty$. Now, differentiating (3.3) with respect to u and s, we obtain

$$\frac{\partial t}{\partial u} \to 1, \quad \frac{\partial t}{\partial s} \to 0$$

uniformly in $D'_{(u,s)}$ as $u \to \infty$. Now, the Jacobian matrix of the inverse coordinate transformation

$$(t,v) \mapsto (u,s)$$

is

$$\begin{pmatrix} \frac{1}{\partial t/\partial u} & \frac{-\partial t/\partial s}{\partial t/\partial u} \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} \partial u/\partial t & \partial u/\partial v \\ \partial s/\partial t & \partial s/\partial v \end{pmatrix}. \tag{7.2}$$

Thus $\dfrac{\partial u}{\partial v}$ is bounded in $D'_{(t,v)} := T \times \{v : |v| < \delta\}$, and approaches 0 uniformly in $D'_{(t,v)}$ as $t \to \infty$. (Note that $t \to \infty$ if and only if $u \to \infty$.)

Now, in terms of the coordinates (t,v), F takes the form

$$t_1 = t - 1$$
$$v_1 = g(t,v),$$

where

$$g(t,v) = s_1(u(t,v), s(t,v)).$$

We wish to estimate $\dfrac{\partial v_1}{\partial v}$. We have

$$\frac{\partial v_1}{\partial v} = \frac{\partial s_1}{\partial v} = \frac{\partial s_1}{\partial u}\frac{\partial u}{\partial v} + \frac{\partial s_1}{\partial s}\frac{\partial s}{\partial v}.$$

From (2.11), we obtain

$$\frac{\partial s_1}{\partial u} = -\frac{s}{u^2} + O_s(\frac{1}{|u|^2}) = O_s(\frac{1}{|u|^2})$$
$$\frac{\partial s_1}{\partial s} = 1 + \frac{1}{u} + O_s(\frac{1}{|u|^2}),$$

and from (7.2), we have

$$\frac{\partial u}{\partial v} = O_s(1)$$
$$\frac{\partial s}{\partial v} = 1.$$

Since we are restricting our attention to domains on which s is bounded, we may drop the subscript s from the above expressions. They combine to give

$$\frac{\partial v_1}{\partial v} = 1 + \frac{1}{u} + O(\frac{1}{|u|^2}). \tag{7.3}$$

For $(t, v) \in D'_{(t,v)}$, consider

$$\xi(t, v) := \prod_{n=0}^{\infty} \left(1 - \frac{1}{t_n}\right) \left(\frac{\partial v_{n+1}}{\partial v_n}\right). \tag{7.4}$$

Then

$$\log(\xi(t, v)) = \sum_{n=0}^{\infty} \log \left[\left(1 - \frac{1}{t_n}\right) \left(\frac{\partial v_{n+1}}{\partial v_n}\right)\right] \tag{7.5}$$

But

$$\begin{aligned} \log \left[\left(1 - \frac{1}{t}\right) \left(\frac{\partial v_1}{\partial v}\right)\right] &= \frac{1}{u} - \frac{1}{t} + O(\frac{1}{|u|^2}) + O(\frac{1}{|t|^2}) \\ &= \frac{t-u}{ut} + O(\frac{1}{|u|^2}) \\ &= O(|\log u|)O(\frac{1}{|u|^2}) + O(\frac{1}{|u|^2}) \\ &= o(\frac{1}{|u|^k}) \end{aligned}$$

for any $k < 2$. With Lemma 3, this shows the convergence of the sum in (7.5). Thus ξ is a well-defined holomorphic function on $D'_{(t,v)}$. Our estimate above also shows that $\log(\xi) \to 0$ uniformly in $D'_{(t,v)}$ as $t \to \infty$. Thus $\xi \to 1$ uniformly in $D'_{(t,v)}$ as $t \to \infty$. So given $\epsilon > 0$, we may assume, by choosing R sufficiently large in Section 2, that

$$|\xi(t, v) - 1| < \epsilon \tag{7.6}$$

in $D'_{(t,v)}$.

Now, let

$$\hat{\xi}(t, v) = t\xi(t, v). \tag{7.7}$$

Then

$$\begin{aligned} \hat{\xi}(t_1, v_1) &= (t-1) \prod_{n=1}^{\infty} \left(1 - \frac{1}{t_n}\right) \left(\frac{\partial v_{n+1}}{\partial v_n}\right) \\ &= t\left(1 - \frac{1}{t}\right) \prod_{n=1}^{\infty} \left(1 - \frac{1}{t_n}\right) \left(\frac{\partial v_{n+1}}{\partial v_n}\right) \\ &= t \prod_{n=0}^{\infty} \left(1 - \frac{1}{t_n}\right) \prod_{n=1}^{\infty} \left(\frac{\partial v_{n+1}}{\partial v_n}\right), \end{aligned}$$

so that

$$\hat{\xi}(t_1, v_1)\frac{\partial v_1}{\partial v} = \hat{\xi}(t, v). \tag{7.8}$$

Define the one-form ω on $D'_{(x,y)}$ by

$$\omega(p) = \hat{\xi}(t(p), v(p))dv.$$

Then ω is invariant under F; i.e., it satisfies

$$F^*(\omega) = \omega. \tag{7.9}$$

8. New coordinate on the fibers of ψ

We wish to integrate ω on fibers of ψ. As a preparatory step, we show that these fibers are connected and simply connected.

Lemma 10. *For each $t \in \mathbf{C}$, $\psi^{-1}(t)$ is connected and simply connected.*

Proof. For each $t \in \mathbf{C}$, $\psi^{-1}(t)$ is exhausted by the sequence

$$\Delta_n := \psi^{-1}(t) \cap F^{-n}(D'_{(x,y)}), \quad n \in \mathbf{N},$$

each element of which is biholomorphic under F^n to

$$\psi^{-1}(t-n) \cap D'_{(x,y)},$$

which in turn, for n sufficiently large, is biholomorphic under the coordinate map (t, v) to the disk $\{|v| < \delta\}$. □

Now define a function χ on $D'_{(x,y)}$ as follows. Given $p = (x, y) \in D'_{(x,y)}$, let C be a path joining $(x, 0)$ to p on $\psi^{-1}(t(p))$. Then let

$$\begin{aligned}
\chi(p) &= \int_C \omega \\
&= \int_0^{v(p)} \hat{\xi}(t(p), \nu) d\nu \\
&= t(p) \int_0^{v(p)} \xi(t(p), \nu) d\nu.
\end{aligned}$$

The mapping χ_t of

$$\psi^{-1}(t) \cap D'_{(x,y)} \cong \{|v| < \delta\}$$

into $\mathbf{C}$ defined by $\chi_t(v(p)) = \chi(x(p), y(p))$ is then injective, in view of (7.6), and covers the disk $\{|w| < |t|(1-\epsilon)\delta\}$.

Notice that χ satisfies

$$\begin{aligned}
\chi(F(p)) &= \int_0^{v(F(p))} \hat{\xi}(t(F(p)), \nu) d\nu \\
&= \int_{v(F(x,0))}^{v(F(p))} \hat{\xi}(t(p) - 1, \nu) d\nu + \int_0^{v(F(x,0))} \hat{\xi}(t(p) - 1, \nu) d\nu \\
&= \chi(p) + \kappa(t(p)),
\end{aligned}$$

where in the last equality we have made use of (7.9), and where

$$\kappa(t(p)) := \int_0^{v(F(x,0))} \hat{\xi}(t(p) - 1, \nu) d\nu. \tag{8.1}$$

Now we extend χ to $\{p \in \Omega : t(p) \in T\} = \psi^{-1}(T)$ by defining

$$\chi(p) = \chi(F^n(p)) - [\kappa(t(p)) + \kappa(t(p) - 1) + \cdots + \kappa(t(p) - (n-1))], \tag{8.2}$$

where n is chosen so that $F^n(p) \in D'_{(x,y)}$. It is clear that this definition is independent of n, and valid when $t(p) \in T$ (since κ is defined there). It is also clear from this definition that for $t \in T$, the mapping χ_t from $\psi^{-1}(t)$ to $\mathbf{C}$ defined by $\chi_t(v) = \chi((x(p), y(p))$ is injective.

Lemma 11. *The mapping χ_t defined above is a biholomorphism of $\psi^{-1}(t)$ onto* $\mathbf{C}$.

Proof. It remains only to show that the mapping is onto $\mathbf{C}$. Fix $t \in T$. Consider again the sets Δ_n, each of which maps biholomorphically under $\chi_{t-n} \circ F^n$ onto a set containing the disk $\{|w| < |t-n|(1-\epsilon)\delta\}$. In view of (8.2), therefore, Δ_n maps biholomorphically under χ_t onto a set containing the disk

$$\{|w + \kappa(t) + \kappa(t-1) + \cdots + \kappa(t-n)| < |t-n|(1-\epsilon)\delta\}, \tag{8.3}$$

which has center

$$c_n = -(\kappa(t) + \kappa(t-1) + \cdots + \kappa(t-n))$$

and radius

$$r_n = |t-n|(1-\epsilon)\delta.$$

It is clear that $r_n \to \infty$ as $n \to \infty$.

We now estimate $\kappa(t)$. From (2.11), we have that

$$\begin{aligned} v(F(x,0)) = s_1(1/x,0) &= s_1(u,0) \\ &= 0 + O(\frac{1}{|u|^2}) \\ &= O(\frac{1}{|t|^2}). \end{aligned}$$

From (7.6) and (7.7), we have that $\hat{\xi} = O(|t|)$ in $D'_{(x,y)}$. Choosing our path of integration in (8.1) to be a radius from 0 to $v(F(x,0))$ in $\mathbf{C}$, we see that

$$\begin{aligned} \kappa(t) &= \left[O(\frac{1}{|t|^2})\right][O(|t|)] \\ &= O(\frac{1}{|t|}). \end{aligned}$$

Thus

$$c_n \sim \sum_{i=1}^{n} \frac{1}{i} \sim \log n$$

as $n \to \infty$, while

$$r_n \sim n$$

as $n \to \infty$. Thus

$$|c_n/r_n| \to 0$$

as $n \to \infty$. □

9. Biholomorphism of Ω onto $\mathbf{C}^2$

We have now that the mapping (ψ, χ) is a biholomorphism of $\psi^{-1}(T)$ onto $T \times \mathbf{C}$. Furthermore, for $n \in \mathbf{N}$, $(\psi, \chi \circ F^n)$ is a biholomorphism of $\psi^{-1}(T+n)$ onto $T + n \times \mathbf{C}$.

Now, the function $\psi : \Omega \to \mathbf{C}$, the open cover $\{\psi^{-1}(T+n)\}_{n\in\mathbf{N}}$ of Ω, and the coordinates $(\psi, \chi \circ F^n)$ on each $\psi^{-1}(T+n)$ define on Ω a structure of locally trivial fiber bundle with base $\mathbf{C}$ and fiber $\mathbf{C}$. From (8.2), we have for $n > m$ that

$$\begin{aligned} f_{n,m}(t) &:= \chi \circ F^n(p) - \chi \circ F^m(p) \\ &= \kappa(t(p) - m) + \kappa(t(p) - (m+1)) + \cdots + \kappa(t(p) - (n-1)), \end{aligned}$$

on $(T+n)\cap(T+m) = T+m$. Thus the $\{f_{n,m}\}$ are holomorphic transition functions subordinate to the cover $\{T+n\}$ of $\mathbf{C}$, satisfying

$$\begin{aligned} f_{n,m} &= -f_{m,n} \\ f_{n,m} + f_{m,l} + f_{l,n} &= 0. \end{aligned}$$

Since the additive Cousin problem can be solved on $\mathbf{C}$, for each $n \in \mathbf{N}$ there exists on each a holomorphic function f_n, defined on $T+n$, such that

$$f_{n,m} = f_n - f_m \quad \forall\, n, m \in \mathbf{N}.$$

Now, for $p \in \Omega$, choose n such that $t \circ F^n(p) \in T$. Define

$$\tilde{\chi}(p) = \chi \circ F^n(p) - f_n(t(p)).$$

This definition is independent of n, and its restriction to each fiber $\psi^{-1}(t)$ is again a biholomorphism of $\psi^{-1}(t)$ onto $\mathbf{C}$, since this restriction differs from the restriction to $\psi^{-1}(t)$ of the old coordinate $\chi \circ F^n$ by the constant $f_n(t)$. Thus $(\psi, \tilde{\chi})$ are global coordinates on Ω, and they map Ω onto $\mathbf{C}^2$.

10. Invariant coordinate on the fibers of ψ

We construct in this section a coordinate on the fibers of ψ which has the advantage of being invariant under F.

Consider the following equivalence relation on Ω. Let $p \sim q$ if $p = F^n(q)$ for some $n \in \mathbf{Z}$. Let π_1 be the projection of Ω onto $\Omega/\sim$. Let $\pi_2(t) = e^{2\pi i t}$. Since

$$\begin{aligned} \pi_2 \circ \psi \circ F(z) &= e^{2\pi i(\psi(z)-1)} \\ &= \pi_2 \circ \psi(z), \end{aligned}$$

there exists $\mu : \Omega \to \mathbf{C} \setminus \{0\}$ such that the following diagram commutes:

$$\begin{array}{ccc} \psi^{-1}(T) & \xrightarrow{\psi} & T \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \Omega/\sim & \xrightarrow[\mu]{} & \mathbf{C} \setminus \{0\} \end{array} \tag{10.1}$$

Let

$$\begin{aligned} S_a &= \psi^{-1}(T \setminus \overline{(T-1)}) \\ S_b &= \psi^{-1}\left((T - \frac{1}{2}) \setminus \overline{(T - \frac{3}{2})}\right) \\ X_a &= \pi_1(S_a) \\ X_b &= \pi_1(S_b) \\ Y_a &= \mu(X_a) \\ Y_b &= \mu(X_b) \\ \pi_a &= \pi_1|_{S_a} \\ \pi_b &= \pi_1|_{S_b} \end{aligned}$$

Then $X_a \cup X_b$ covers $\Omega/\sim$, and π_a, π_b are injective and hence biholomorphisms. Consider the functions

$$\begin{aligned} g_a : X_a &\to \mathbf{C} \\ p &\mapsto \chi \circ \pi_a^{-1}(p) \\ g_b : X_b &\to \mathbf{C} \\ p &\mapsto \chi \circ \pi_b^{-1}(p) \end{aligned}$$

Then g_a and g_b map fibers of μ biholomorphically onto $\mathbf{C}$. Now,

$$\begin{aligned} X_a \cap X_b &= \{p \in \Omega/\sim : \operatorname{Re}(\mu(p)) \neq 0, -1/2\} \\ &= A \cup B, \end{aligned}$$

where

$$\begin{aligned} A &= \{p \in \Omega/\sim : -1/2 < \operatorname{Re}(\mu(p)) < 0\} \\ B &= \{p \in \Omega/\sim : -1 < \operatorname{Re}(\mu(p)) < -1/2\}. \end{aligned}$$

Note also that

$$\begin{aligned} Y_a \cap Y_b &= \mu(X_a) \cap \mu(X_b) \\ &= \mu(X_a \cap X_b). \end{aligned}$$

On $X_a \cap X_b$, consider the difference

$$f_{a,b} := \chi \circ \pi_b^{-1} - \chi \circ \pi_a^{-1}.$$

Note that if $p \in A$, $\pi_a^{-1}(p) = \pi_b^{-1}(p)$, while if $p \in B$,

$$\psi \circ \pi_b^{-1}(p) = \psi \circ \pi_a^{-1}(p) - 1. \tag{10.2}$$

Thus

$$f_{a,b}(p) = \begin{cases} 0 & p \in A \\ \kappa(\psi \circ \pi_a^{-1}(p)) & p \in B \end{cases} \tag{10.3}$$

Note that $f_{a,b}(p)$ depends only on $\mu(p)$, and may thus be considered a holomorphic function on $Y_a \cap Y_b \subset \mathbf{C}^* \equiv \mathbf{C} \setminus \{0\}$. Since $Y_a \cup Y_b$ covers $\mathbf{C}^*$, and the additive Cousin problem is solvable on $\mathbf{C}^*$, there exist holomorphic functions

$$\begin{aligned} g_a &: Y_a \to \mathbf{C} \\ g_b &: Y_b \to \mathbf{C} \end{aligned}$$

such that $f_{a,b} = g_b - g_a$ on $Y_a \cap Y_b$. Then

$$\begin{aligned} f_{a,b} &= \chi \circ \pi_b^{-1} - \chi \circ \pi_a^{-1} \\ &= (g_b - g_a) \circ \mu. \end{aligned}$$

Thus

$$\hat{\chi} := \chi \circ \pi_a^{-1} - g_a \circ \mu \tag{10.4}$$

is a holomorphic function on $\Omega/\sim$ mapping fibers of μ biholomorphically onto $\mathbf{C}$. $\hat{\chi} \circ \pi_1$ is then a holomorphic function on Ω mapping fibers of ψ biholomorphically onto $\mathbf{C}$. It is furthermore invariant under F. Thus the coordinates

$$(\psi, \hat{\chi} \circ \pi_1) =: (t, q)$$

map Ω biholomorphically onto $\mathbf{C}^2$, and satisfy

$$(t_1(p), q_1(p)) := (t \circ F(p), q \circ F(p)) = (t(p) - 1, q(p)) \tag{10.5}$$

for $p \in \Omega$.

11. Many Fatou-Bieberbach petals

It is easy now to construct automorphisms of $\mathbf{C}^2$ with arbitrarily many Fatou-Bieberbach petals, each of which is a basin of attraction to the origin, and on each of which the automorphism is conjugate to translation.

Fix $n \in \mathbf{N}$, and, using Theorem 2, choose $F \in Aut(\mathbf{C}^2)$ of the form $F = (f_1, f_2)$, where

$$\begin{aligned} f_1(x, y) &= x + x^{n+1} + O(|z|^{2n+1}) \\ f_2(x, y) &= y + (n+1)x^n y + O(|z|^{2n+1}). \end{aligned}$$

As in Section 2, we blow up F by $\phi : (x, s) \to (x, xs) = (x, y)$ to $\tilde{F} : \mathbf{C}^2 \setminus X \to \mathbf{C}^2$, X an analytic hypersurface which does not meet the s-axis, and find that near the s-axis $\tilde{F}$ has the expansion

$$\begin{aligned} x_1 &= x + x^{n+1} + O_s(|x|^{2n+1}) \\ s_1 &= s + nsx^n + O_s(|x|^{2n}). \end{aligned}$$

Let $g(x, s) = (nx^n, s)$. Let $D_{(x,s)} \subset \mathbf{C}^2$ be defined as in Section 2. Since it is simply connected and does not meet the s-axis, we may define on $D_{(x,s)}$ n distinct branches of g^{-1}. Call these $h_1, \ldots, h_n$. For any i between 1 and n, consider

$$F_i := g \circ \tilde{F} \circ h_i,$$

defined on $D_{(x,s)}$. Then F_i has the form

$$\begin{aligned} x_1 &= x + x^2 + O_s(|x|^3) \\ s_1 &= s + sx + O_s(|x|^2). \end{aligned}$$

Let

$$D_i = \phi \circ h_i(D_{(x,s)}).$$

Note that the D_i are disjoint, since ϕ is injective away from the s-axis. Then F is conjugate on D_i to F_i. If we let

$$\Omega_i = \bigcup_{n=0}^{\infty} F^{-n}(D_i),$$

the argument of the previous sections shows that each Ω_i is a Fatou-Bieberbach domain on which F is conjugate to translation.

12. A generic class of automorphisms of $\mathbf{C}^2$

In Section 2, we chose to work with a rather specific class of automorphism to simplify the exposition. However, the argument may be extended to much more general classes of automorphism.

The family of quadratic polynomial mappings tangent to the identity has been classified up to linear conjugacy by Ueda ([7]). We list the conjugacy classes here, along with a normal form for each class:

$$N_1(\alpha_1,\alpha_2): \quad \begin{aligned} x_1 &= x + \alpha_1 x^2 + (\alpha_2+1)xy \\ y_1 &= y + (\alpha_1+1)xy + \alpha_2 y^2 \end{aligned}$$

$$N_{2,1}(\alpha): \quad \begin{aligned} x_1 &= x + \alpha x^2 + xy \\ y_1 &= y + (\alpha+1)xy + y^2 \end{aligned}$$

$$N_{2,2}(\alpha): \quad \begin{aligned} x_1 &= x + \alpha x^2 \\ y_1 &= y + (\alpha+1)xy \end{aligned}$$

$$N_{3,1}: \quad \begin{aligned} x_1 &= x + xy \\ y_1 &= y + x^2 + y^2 \end{aligned}$$

$$N_{3,2}: \quad \begin{aligned} x_1 &= x + x^2 \\ y_1 &= y + x^2 + xy \end{aligned}$$

$$N_{3,3}: \quad \begin{aligned} x_1 &= x \\ y_1 &= y + x^2 \end{aligned}$$

$$N_4: \quad \begin{aligned} x_1 &= x + x^2 \\ y_1 &= y + xy \end{aligned}$$

$$N_0: \quad \begin{aligned} x_1 &= x \\ y_1 &= y. \end{aligned}$$

Here $N_1(\alpha_1,\alpha_2)$, indexed by two complex parameters, consists of those mappings which leave invariant exactly three lines; $N_{2,1}(\alpha)$ and $N_{2,2}(\alpha)$, each indexed by one complex parameter, consist of those mappings which leave invariant exactly two lines; $N_{3,1}$, $N_{3,2}$, and $N_{3,3}$ consist of those which leave invariant exactly one line. The remaining two classes leave invariant every line.

In [8], with the help of a coordinate change suggested by Hakim in [4], we prove the following.

Theorem 12. *Let F be an automorphism of $\mathbf{C}^2$ whose quadratic part F_2 satisfies one of the following conditions.*

1. *$F_2 \in N_1(\alpha_1,\alpha_2)$, and one of α_1, α_2, $-1-\alpha_1-\alpha_2$ has strictly positive real part and is not in $\mathbf{N}$.*
2. *$F_2 \in N_{2,1}(\alpha)$, $\mathrm{Re}(\alpha) > 0$, and $\alpha \notin \mathbf{N}$.*
3. *$F_2 \in N_{2,2}(\alpha)$, and $\mathrm{Re}(\alpha) > 0$.*

Then F has a basin of attraction Ω to the origin, with Ω biholomorphic to $\mathbf{C}^2$, and $F|_\Omega$ biholomorphically conjugate to the translation $(x,y) \mapsto (x-1,y)$

Remark. There is still a basin of attraction when we allow the above numbers to belong to $\mathbf{N}$, but I have been unable to prove the global statements about that basin in this case.

We refer the reader to [8] for the proof of this theorem. Since each of Ueda's classes listed above is invariant under the mapping taking a germ to its inverse, under the hypotheses of Theorem 12 it is in fact the case that both F and F^{-1} have basins of attraction satisfying the above properties.

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