A QUANTIZED HENON MAP

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ABSTRACT. We quantize the classical Henon map on $\mathbb{R}^2$, obtaining a unitary map on $L^2(\mathbb{R})$ whose dynamics we study, developing analogies to the classical dynamics.

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1. THE CLASSICAL AND QUANTIZED HENON MAPS

We wish to study a quantization of the following classical Henon map on $\mathbb{R}^2$:

$$(x, y) \mapsto (x^2 - y + c, x),$$

$c \in \mathbf{R}$. A quantization of *functions* on $\mathbb{R}^2$ is a map σ from the space of functions to the space of self-adjoint linear operators on $L^2(\mathbb{R})$ which extends the prescriptions

$$x \quad \rightarrow \quad \text{multiplication by } x \tag{1}$$

$$y \quad \rightarrow \quad -i\hbar\partial_x. \tag{2}$$

Here $\hbar$ is a positive constant, and we define the Fourier transform F on $L^2(\mathbb{R})$ by

$$F\psi(x) = 1/\sqrt{2\pi\hbar}\int_{\mathbf{R}} \psi(t)e^{-ixt/\hbar}.$$

This integral formula makes sense on a dense subspace of $L^2(\mathbb{R})$, for instance on the Schwarzian space $\mathcal{S}$ of smooth rapidly-decreasing functions, and the definition may be extended to the whole space by continuity. In fact it may be extended by duality to the space $\mathcal{S}'$ of so-called tempered distributions; it is sometimes convenient to work in this larger space.

In this paper, we will often use x and y to refer also to their corresponding quantizations, without explicit reference to σ. Note that $y = F^{-1}xF$. Also, for measurable functions $f : \mathbf{R} \rightarrow \mathbf{C}$, we will let $f(x)$ denote multiplication by $f(x)$,

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and, in accordance with the functional calculus of the spectral theorem [RS], $f(y)$ will denote the operator $F^{-1}f(x)F$. In other words, $f(y)$ is the *pseudodifferential* operator with symbol f (see [T]).

Given a Hamiltonian flow ϕ_t on $\mathbb{R}^2$, let H be the Hamiltonian function which gives rise to it; then solving the Schrodinger equation

$$i\hbar\partial_t U_t\psi = \sigma(H)U_t\psi$$
$$U_0 = \mathrm{Id}$$

gives a unitary solution operator U_t on $L^2(\mathbb{R})$, which we call the quantization of ϕ_t.

It is known that the Henon map cannot be embedded in the flow generated by a time-independent Hamiltonian function. It can, however, be embedded in the flow generated by a piecewise constant Hamiltonian, and this is the approach we will take. Of course, there is no canonical way to do this. Furthermore, there is no canonical definition of the map σ: one would like σ to convert the Poisson bracket of functions into the Lie bracket of operators; however, this desire is incompatible with the requirements (1) and (2), and the various attempts to achieve it in an approximate sense have given rise to various notions of quantization. All of these notions, however, share the feature of approximating, in some asymptotic sense (as $\hbar \to 0$) and for states localized near a point of phase space, the action of the corresponding classical map on points–see for instance the expository article [KS]. Our point of view is to choose *some* quantization of the Henon map and study its dynamics, with a view to deciding whether the main features of the classical dynamics have analogues in the quantum version, or are rather artifacts of the classical limit.

In studying elements of $\mathcal{H} := L^2(\mathbb{R})$ or of $\mathcal{U}(\mathcal{H})$, the unitary operators on $\mathcal{H}$, we will always ignore multiplicative constants of absolute value one, the reason being that they have no effect on expressions of the form

$$(U^n\psi, A(U^n\psi)),$$

the so-called observables of the system, where here A is any self-adjoint operator.

To simplify the situation, we will fix $\hbar = 1$, and instead allow the Henon map to vary under conjugation by a dilation λ. Thus we will consider the family of Henon maps

$$g_\lambda : (x, y) \mapsto (\lambda x^2 - y + c/\lambda, x).$$

Let us write g_λ as the composition of the rotation $(x, y) \mapsto (-y, x)$ and the shear $(x, y) \mapsto (x + \lambda y^2 + c/\lambda, y)$. The first of these is the time-$(\pi/2)$ map of the flow generated by the Hamiltonian $-1/2(x^2 + y^2)$, whose quantization is the self-adjoint operator $1/2(\partial_x^2 - x^2)$. This is the Hamiltonian of the reverse harmonic oscillator, and it is well-known that solving its Schrodinger equation gives the unitary solution operator

$$\psi \mapsto \sum_{n=0}^{\infty} e^{i(n+1/2)t}(\psi, \phi_n)\phi_n,$$

whose normalized eigenfunctions ϕ_n are defined in terms of the Hermite polynomials:

$$\phi_n = \pi^{-1/4}(2^n n!)^{-1/2}(-1)^n e^{x^2/2}\frac{d^n}{dx^n}(e^{-x^2}).$$

The time-$(\pi/2)$ map of this operator is

$$\psi \mapsto e^{\pi i/4} \sum_{n=0}^{\infty} i^n (\psi, \phi_n)\phi_n,$$

which is just the operator $e^{\pi i/4}F^{-1}$ in its spectral representation (see [Ti]). Thus, ignoring the multiplicative constant, we quantize the rotation by F^{-1}, the inverse Fourier transform.

The shear $(x, y) \mapsto (x + \lambda y^2 + c/\lambda, y)$ is in turn the time-one map of the flow generated by the Hamiltonian $\lambda y^3/3 + cy/\lambda$. Quantizing, solving the Schrodinger equation, and taking the time-one map of its solution operator gives the unitary operator $e^{-(\lambda y^3/3+cy/\lambda)} := F^{-1}e^{-(\lambda x^3/3+cx/\lambda)}F$. Thus we will take for our quantization of the Henon map the unitary operator

$$\begin{aligned} U &:= e^{-i(\lambda y^3/3+cy/\lambda)}F^{-1} \\ &= F^{-1}e^{-i(\lambda x^3/3+cx/\lambda)}. \end{aligned}$$

It is easy to check (by direct calculation) that the operator $e^{ix^2/2}e^{iy^2/2}e^{ix^2/2}$ differs from F^{-1} by a multiplicative constant of absolute value one. Thus U is equivalent to the operator

$$e^{ix^2/2}e^{iy^2/2}e^{-i(\lambda x^3/3-x^2/2+cx/\lambda)},$$

so we find that

$$e^{-ix^2/2}Ue^{ix^2/2} \sim \tilde{U} := e^{iy^2/2}e^{-i(\lambda x^3/3-x^2+cx/\lambda)},$$

where "$\sim$" denotes equality up to a multiplicative constant of absolute value one. We will make use of this form of the quantized Henon map–a quantization "by shears"–in section 4.

2. Statement of results

Let $U = F^{-1}e^{-i(\lambda x^3/3+cx/\lambda)}$, $\lambda > 0$ and $c \in \mathbb{R}$. We prove the following (all terms will be defined in the succeeding sections).

Theorem 1: Suppose that $0 < 2a_0 < b_0$ with $((1+\lambda)b_0)^{1.7} < (\lambda a_0/2)^2$, that $c_0 < d_0$ are real numbers satisfying $|c_0|, |d_0| < 10a_0$, and that $\psi \in L^2(\mathbb{R})$ has unit norm and is ϵ, ϵ supported in $[a_0, b_0] \times [c_0, d_0]$, where

$$\epsilon = \frac{1}{(\ln\ \ln(a_0))^2}.$$

Then if a_0 and b_0 are sufficiently large, for each $n = 1, 2, \ldots$ we have that $U^n\psi$ is $1/100, 1/100$ supported in $[a_n, b_n]\times[a_{n-1}, b_{n-1}]$, with $a_n, b_n \to \infty$ super-exponentially.

Theorem 2: If $\lambda > 0$ is fixed and $c > 0$ is sufficiently large, or if $c > 1$ is fixed and $\lambda > 0$ is sufficiently small, then given any $\psi \in L^2(\mathbb{R})$ with $\|\psi\| = 1$, any $\epsilon > 0$ and any $R > 0$, there exists $n \in \mathbb{N}$ such that $U^n\psi$ is ϵ, ϵ supported in $[R, \infty) \times [R, \infty)$. In particular, U has no discrete spectrum.

Theorem 3: Let y be the operator $-i\partial_x$; then

$$\begin{aligned} U^{-1}xU &= -y + \lambda x^2 + c/\lambda \\ U^{-1}yU &= x \end{aligned}$$

on a dense subspace of $L^2(\mathbb{R})$. As a consequence, if $c > 1$ or $\lambda > \sqrt{2(1-c)}$, then U has no eigenvector in the Sobolev space $H^1 := \{\psi \in L^2(\mathbb{R}) : x\psi \in L^2(\mathbb{R})\}$.

Theorem 4: Assume that $c = 0$. Write $U_\lambda = F^{-1}e^{-i\lambda x^3/3}$. Then there is no analytic family ψ_λ of perturbations of $e^{-x^2/2}$ and analytic family a_λ of perturbations of $1 \in \mathbb{C}$ such that

$$U_\lambda \psi_\lambda = a_\lambda \psi_\lambda.$$

(In contrast, if we replaced the three in the exponent by a two, the theorem is false.)

3. A basin of attraction to ∞: proof of Theorem 1

One of the principles of quantization is that there should be a correspondence between the behavior of the classical map on points and its unitary quantization on states which are "pointlike"; that is, localized around a point in phase space. Let us make the following definition: given $\epsilon > 0$ and $\delta > 0$, say that $\psi \in L^2(\mathbb{R})$ is ϵ, δ supported in a rectangle $[a,b] \times [c,d] \in \mathbb{R}^2$ if

$$\begin{aligned} \int_{[a,b]^c} |\psi|^2 &\leq \epsilon^2 \\ \int_{[c,d]^c} |F\psi|^2 &\leq \delta^2. \end{aligned}$$

We would like estimates which describe how $U(\psi)$ is localized in phase space, in say the above sense, given information on how ψ is localized in phase space. Then we will draw conclusions about the dynamics of U from those estimates. The following theorem is classical.

THEOREM 3.1. *Let $\ell(x)$ be a real-valued C^2 function on $[a,b]$. Then*

$$\left| \int_a^b e^{i\ell(x)} dx \right| \leq \frac{2\pi^2 (b-a) \|\ell''\|_\infty}{(\min |\ell'|)^2} + \frac{2\pi}{\min |\ell'|}.$$

Proof: We can assume that $\ell' \neq 0$ on $[a,b]$. Hence we may assume that $\ell' > 0$.

Fix a $\gamma \in [a,b]$ and let $\delta = \frac{2\pi}{\ell'(\gamma)}$. Assume that $\gamma + \delta \leq b$.

Then

$$
\begin{aligned}
\int_\gamma^{\gamma+\delta} e^{i\ell(x)}dx &= \int_\gamma^{\gamma+\delta} e^{i(\ell(\gamma)+\ell'(\gamma)(x-\gamma)+\frac{\ell''(g(x))}{2}(x-\gamma)^2)}dx,\ \gamma \le g(x) \le \gamma+\delta,\\
&= \int_\gamma^{\gamma+\delta} e^{i(\ell(\gamma)-\ell'(\gamma)\gamma)}\\
&* \ (1+L(x,\gamma))e^{i\ell'(\gamma)x}dx\\
0 &= \int_\gamma^{\gamma+\delta} e^{i(\ell(\gamma)-\ell'(\gamma)\gamma)}e^{i(\ell'(\gamma))x}dx\\
1+L(x,\gamma) &= e^{i\frac{\ell''(g(x))}{2}(x-\gamma)^2}\\
|L(x,\gamma)| &\le \frac{sup_{[\gamma,\gamma+\delta]}|\ell''|}{2}(x-\gamma)^2\\
|\int_\gamma^{\gamma+\delta} e^{i\ell(x)}dx| &\le \int_\gamma^{\gamma+\delta} |L(x,\gamma)|\\
&\le \int_\gamma^{\gamma+\delta} \frac{sup_{[\gamma,\gamma+\delta]}|\ell''|}{2}\frac{4\pi^2}{\min_{[\gamma,\gamma+\delta]}(\ell')^2}
\end{aligned}
$$

Next we divide the interval $[a,b]$ into a finite number of intervals $[a_1,b_1],\dots,[a_n,b_n]$ where $a_1=a, b_n=b, b_k=a_{k+1}$, and $b_j=a_j+\frac{2\pi}{\ell'(a_j)}, j<n$ and $b_n \le a_n+\frac{2\pi}{\ell'(a_n)}$.

Hence

$$
\begin{aligned}
\left|\int_a^b e^{i\ell(x)}dx\right| &= \left|\sum_{j=1}^n \int_{a_j}^{b_j} e^{i\ell(x)}dx\right|\\
&\le \sum_{j=1}^{n-1}\int_{a_j}^{b_j} \frac{sup_{[a_j,b_j]}|\ell''|}{2}\frac{4\pi^2}{\min_{[a_j,b_j]}(\ell')^2}\\
&+ \int_{a_n}^{b_n} dx\\
&\le \sum_{j=1}^{n-1}\int_{a_j}^{b_j} \frac{sup_{[a,b]}|\ell''|}{2}\frac{4\pi^2}{\min_{[a,b]}(\ell')^2}\\
&+ \int_{a_n}^{b_n} dx\\
&\le \frac{2\pi^2(b-a)\|\ell''\|_\infty}{(\min|\ell'|)^2}+\frac{2\pi}{\min|\ell'|}
\end{aligned}
$$

∎

Suppose now that $[a,b],[c,d]$ are two given intervals, and ℓ a $\mathcal{C}^2$ function on $[a,b]$. We denote by $[e,f]$ the range of ℓ' and $C=c+e, D=d+f$. Suppose that ψ is ϵ,δ supported in $[a,b]\times[c,d]$, $\|\psi\|=1$. Let $\tilde\psi=\chi_{[a,b]}F^{-1}\chi_{[c,d]}F\psi$. Then we have the following basic estimate:

THEOREM 3.2. *With ψ, $\tilde{\psi}$ as above, we have $\|\tilde{\psi}-\psi\| \leq \epsilon+\delta$ and if $x \notin [C,D]$, $r := dist(x,[C,D])$,*

$$|Fe^{i\ell}\tilde{\psi}(x)| \leq \sqrt{\frac{d-c}{2\pi}}\left(\frac{2\pi^2(b-a)\|\ell''\|_\infty}{r^2}+\frac{2\pi}{r}\right)$$

Proof:

$$\begin{aligned}
\|\tilde{\psi}-\psi\| &= \|\tilde{\psi}-\chi_{[a,b]}\psi\| + \|\chi_{[a,b]}\psi-\psi\| \\
&= \|\chi_{[a,b]}[F^{-1}\chi_{[c,d]}F\psi-\psi]\| + \|\chi_{[a,b]}\psi-\psi\| \\
&\leq \|F^{-1}\chi_{[c,d]}F\psi-\psi\| + \|\chi_{[a,b]}\psi-\psi\| \\
&= \|\chi_{[c,d]}F\psi-F\psi\| + \|\chi_{[a,b]}\psi-\psi\| \\
&\leq \delta+\epsilon.
\end{aligned}$$

Next we estimate $Fe^{i\ell}\tilde{\psi}(x),\ x \notin [C,D]$:

$$\begin{aligned}
|Fe^{i\ell}\tilde{\psi}(x)| &= |Fe^{i\ell}\chi_{[a,b]}F^{-1}\chi_{[c,d]}F\psi(x)| \\
&= |\left(Fe^{i\ell}\chi_{[a,b]}\right)*\left(\chi_{[c,d]}F\psi\right)\left(x\right)| \\
&= \left|\int_{\mathbb{R}}(\chi_{[c,d]}F\psi)(t)(Fe^{i\ell}\chi_{[a,b]})(x-t)\,dt\right| \\
&= \left|\int_c^d (F\psi)(t)(Fe^{i\ell}\chi_{[a,b]})(x-t)\,dt\right| \\
&\leq \left(\int_c^d |F\psi|\right)\sup\{|Fe^{i\ell}\chi_{[a,b]})(x-t)|; t\in[c,d]\} \\
&\leq \left(\int_c^d |F\psi|^2\right)^{1/2}\left(\int_c^d 1\right)^{1/2}\sup\{|Fe^{i\ell}\chi_{[a,b]})(x-t)|; -t\in[-d,-c]\} \\
&\leq \sqrt{d-c}\,\sup\{|Fe^{i\ell}\chi_{[a,b]})(t)|; t\in[x-d,x-c]\} \\
&= \sqrt{\frac{d-c}{2\pi}}\sup\{|\int_{\mathbb{R}} e^{i\ell(u)}\chi_{[a,b]}(u)e^{-iut}\,du|;\ t\in[x-d,x-c]\} \\
&= \sqrt{\frac{d-c}{2\pi}}\sup\{|\int_a^b e^{i[\ell(u)-tu]}\,du|;\ t\in[x-d,x-c]\} \\
&\leq \sqrt{\frac{d-c}{2\pi}}\sup\left\{\frac{2\pi^2(b-a)\|(\ell-tu)''\|_\infty}{(\min|(\ell-tu)'|)^2}+\frac{2\pi}{\min|(\ell-tu)'|};\ t\in[x-d,x-c]\right\} \\
&= \sqrt{\frac{d-c}{2\pi}}\sup\left\{\frac{2\pi^2(b-a)\|\ell''\|_\infty}{(\min|\ell'-t|)^2}+\frac{2\pi}{\min|\ell'-t|};\ t\in[x-d,x-c]\right\} \\
&\leq \sqrt{\frac{d-c}{2\pi}}\left(\frac{2\pi^2(b-a)\|\ell''\|_\infty}{r^2}+\frac{2\pi}{r}\right),
\end{aligned}$$

since the inequalities $x-d \leq t \leq x-c$ and $e \leq \ell' \leq f$ imply that $x-(d+f) \leq t-\ell' \leq x-(c+e)$, from which we obtain $|\ell'-t| > r$.

■

THEOREM 3.3. *Let* $R := \max\{\frac{16\pi(d-c)}{(\epsilon+\delta)^2}, \left(\frac{16\pi^3(d-c)(b-a)^2\|\ell''\|_\infty^2}{3(\epsilon+\delta)^2}\right)^{1/3}\}$*. Then* $e^{i\ell}\psi$ *is* $\epsilon, 2(\epsilon+\delta)$ *supported on* $[a,b]\times[C-R, D+R]$.

Proof: Let $I=[C-R,D+R]$. We have

$$\begin{aligned}
\int_{I^c} |Fe^{i\ell}\tilde{\psi}(t)|^2\,dt &\leq 2\int_R^\infty \left(\sqrt{\frac{d-c}{2\pi}}\left(\frac{2\pi^2(b-a)\|\ell''\|_\infty}{r^2}+\frac{2\pi}{r}\right)\right)^2 dr \\
&\leq \frac{4(d-c)}{2\pi}\left[\int_R^\infty \left(\frac{2\pi^2(b-a)\|\ell''\|_\infty}{r^2}\right)^2 dr + \int_R^\infty \left(\frac{2\pi}{r}\right)^2 dr\right] \\
&\leq \frac{2(d-c)}{\pi}\left[\frac{4\pi^4(b-a)^2\|\ell''\|_\infty^2}{3R^3}+\frac{4\pi^2}{R}\right] \\
&\leq (\epsilon+\delta)^2.
\end{aligned}$$

Thus

$$\begin{aligned}
\|Fe^{i\ell}\psi\|_{I^c} &\leq \|Fe^{i\ell}\psi - Fe^{i\ell}\tilde{\psi}\|_{I^c} + \|Fe^{i\ell}\tilde{\psi}\|_{I^c} \\
&\leq \|Fe^{i\ell}\psi - Fe^{i\ell}\tilde{\psi}\| + \|Fe^{i\ell}\tilde{\psi}\|_{I^c} \\
&= \|\psi-\tilde{\psi}\| + \|Fe^{i\ell}\tilde{\psi}\|_{I^c} \\
&\leq (\epsilon+\delta)+(\epsilon+\delta).
\end{aligned}$$

■

THEOREM 3.4. *Suppose that* ψ *is* ϵ,δ *supported on* $[a,b]\times[c,d]$*, with* $0<a<b$*,* $d-c<10b$*. Let* $\ell(x)=-\lambda x^3/3 - c'x/\lambda$*. Then* $e^{i\ell}\psi$ *is* $\epsilon, 2(\epsilon+\delta)$ *supported on* $[a,b]\times[c-\lambda b^2-c'/\lambda-R, d-\lambda a^2-c'/\lambda+R]$*, where*

$$R=\max\left\{\frac{160\pi b}{(\epsilon+\delta)^2}, \left(\frac{640\lambda^2\pi^3 b^5}{3(\epsilon+\delta)^2}\right)^{1/3}\right\}.$$

Proof: This follows directly from the previous estimate, with $\|\ell\|_\infty^2=(2\lambda b)^2$ and $[e,f]=[-\lambda b^2-c'/\lambda, -\lambda a^2-c'/\lambda]$. ■

Now, F^{-1}, the inverse Fourier transform, maps any pair $(\psi, F\psi)$ to the pair $(F^{-1}\psi,\psi)=(F^2(F\psi),\psi)$. Recall that F^2 is the involution $\psi(x)\mapsto\psi(-x)$. Thus if ψ is ϵ,δ supported in a rectangle $I\times J$, $F^{-1}\psi$ is δ,ϵ supported in $-J\times I$. Thus we obtain immediately the following:

THEOREM 3.5. *Let* U *be the unitary map* $F^{-1}e^{-i(\lambda x^3/3+c'/\lambda)}$ *on* $L^2(\mathbb{R})$*. Suppose that* ψ *is* ϵ,δ *supported on* $[a,b]\times[c,d]$*, with* $0<a<b$*,* $d-c<10b$*. Then* $U\psi$ *is* $2(\epsilon+\delta),\epsilon$ *supported on*

$$[\lambda a^2+c'/\lambda-d-R, \lambda b^2+c'/\lambda-c+R]\times[a,b],$$

where

$$R=\max\left\{\frac{160\pi b}{(\epsilon+\delta)^2}, \left(\frac{640\lambda^2\pi^3 b^5}{3(\epsilon+\delta)^2}\right)^{1/3}\right\}.$$

It now becomes necessary to reduce the error in the above estimate. We can achieve a more accurate localization of a function in a rectangle by perturbing the function appropriately, while allowing the rectangle to expand. This is the point of the following theorem.

THEOREM 3.6. *Suppose that* ψ *is* ϵ, ϵ *supported in* $[a,b]\times[c,d]$, *with* $\|\psi\|=1$. *Let*

$$\tilde{\psi} = F^{-1}e^{-\frac{(x-c)^2\epsilon}{(d-c)^2}}F\chi_{[a,b]}\psi.$$

Then $\|\psi-\tilde{\psi}\|\leq 3\epsilon$. *Let* $[\tilde{c},\tilde{d}]\supset[c,d]$ *be the interval consisting of those* x *for which*

$$|x-c|\leq \frac{\sqrt{\ln 72}(d-c)}{\sqrt{\epsilon}}.$$

Let $[\tilde{a},\tilde{b}]\supset[a,b]$ *be the interval consisting of those* x *for which*

$$dist(x,[a,b])\leq \frac{2\sqrt{\epsilon}}{d-c}\sqrt{\ln\frac{72\sqrt{(d-c)(b-a)}}{\epsilon^{5/4}}}.$$

Then $\tilde{\psi}$ *is* $\epsilon/36, \epsilon/36$ *supported on* $[\tilde{a},\tilde{b}]\times[\tilde{c},\tilde{d}]$.

Note that $\tilde{\psi}$ is not necessarily of norm 1; however,

$$1-3\epsilon\leq\|\tilde{\psi}\|\leq 1+3\epsilon.$$

Proof:

$$\begin{aligned}
\|\psi-\tilde{\psi}\| &= \|F\psi - e^{-\frac{(x-c)^2\epsilon}{(d-c)^2}}F\chi_{[a,b]}\psi\| \\
&\leq \|e^{-\frac{(x-c)^2\epsilon}{(d-c)^2}}F\chi_{[a,b]}\psi - e^{-\frac{(x-c)^2\epsilon}{(d-c)^2}}F\psi\| + \|(e^{-\frac{(x-c)^2\epsilon}{(d-c)^2}}-1)F\psi\| \\
&\leq \|F\chi_{[a,b]}\psi - F\psi\| + \|\chi_{[c,d]}(e^{-\frac{(x-c)^2\epsilon}{(d-c)^2}}-1)F\psi\| \\
&+ \|\chi_{\mathbb{R}\setminus[c,d]}(e^{-\frac{(x-c)^2\epsilon}{(d-c)^2}}-1)F\psi\| \\
&\leq \|\chi_{[a,b]}\psi-\psi\| + \epsilon\|\chi_{[c,d]}F\psi\| + \|\chi_{\mathbb{R}\setminus[c,d]}F\psi\| \\
&\leq \epsilon+\epsilon+\epsilon.
\end{aligned}$$

Next,

$$\begin{aligned}
\|\chi_{\mathbb{R}\setminus[\tilde{c},\tilde{d}]}F\tilde{\psi}\| &= \|\chi_{\mathbb{R}\setminus[\tilde{c},\tilde{d}]}e^{-\frac{(x-c)^2\epsilon}{(d-c)^2}}F\chi_{[a,b]}\psi\| \\
&\leq \frac{1}{72}\|\chi_{\mathbb{R}\setminus[\tilde{c},\tilde{d}]}F\chi_{[a,b]}\psi\| \\
&\leq \frac{\epsilon}{72}+\frac{1}{72}\|\chi_{\mathbb{R}\setminus[\tilde{c},\tilde{d}]}F\psi\| \\
&\leq \frac{\epsilon}{36}.
\end{aligned}$$

Finally,

$$
\begin{aligned}
\tilde{\psi} &= F^{-1}e^{-\frac{(x-c)^2\epsilon}{(d-c)^2}}F\chi_{[a,b]}\psi \\
&= (F^{-1}e^{-\frac{(x-c)^2\epsilon}{(d-c)^2}}) * (\chi_{[a,b]}\psi) \\
&= \left(\frac{d-c}{\sqrt{2\epsilon}}e^{icx}e^{-\frac{x^2(d-c)^2}{4\epsilon}}\right) * (\chi_{[a,b]}\psi).
\end{aligned}
$$

Thus

$$
\begin{aligned}
|\tilde{\psi}(x)| &= \frac{d-c}{\sqrt{2\epsilon}}\left|\int_a^b \psi(t)e^{ic(x-t)}e^{-\frac{(x-t)^2(d-c)^2}{4\epsilon}}\,dt\right| \\
&\le \frac{d-c}{\sqrt{2\epsilon}}\int_a^b |\psi(t)|e^{-\frac{(x-t)^2(d-c)^2}{4\epsilon}}\,dt
\end{aligned}
$$

If $\mathrm{dist}(x,[a,b]) = r > 0$, then

$$
|\tilde{\psi}(x)| \le \frac{(d-c)\sqrt{b-a}}{\sqrt{2\epsilon}}e^{-\frac{r^2(d-c)^2}{4\epsilon}}.
$$

Thus

$$
\begin{aligned}
\left(\int_{\mathrm{dist}(x,[a,b])\ge\rho}|\tilde{\psi}(x)|^2\,dx\right)^{1/2} &\le 2\frac{\sqrt{b-a}(d-c)}{\sqrt{2\epsilon}}\sqrt{\int_\rho^\infty e^{-\frac{r^2(d-c)^2}{2\epsilon}}dr} \\
&\le 2\frac{\sqrt{b-a}(d-c)}{\sqrt{2\epsilon}}\sqrt{\frac{\sqrt{2\epsilon}}{d-c}\frac{\sqrt{\pi}}{2}e^{-\frac{\rho^2(d-c)^2}{2\epsilon}}} \\
&= \left(\frac{2\pi}{\epsilon}\right)^{1/4}\sqrt{(b-a)(d-c)}e^{-\frac{\rho^2(d-c)^2}{4\epsilon}} \\
&\le \epsilon/36,
\end{aligned}
$$

where we have used the following inequality, valid for $a > 0$:

$$
\begin{aligned}
\int_\rho^\infty e^{-ax^2}\,dx &= \int_0^\infty e^{-a(x+\rho)^2}\,dx \\
&= e^{-a\rho^2}\int_0^\infty e^{-ax^2-2ax\rho}\,dx \\
&\le e^{-a\rho^2}\int_0^\infty e^{-ax^2}\,dx \\
&= \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-a\rho^2}.
\end{aligned}
$$

■

Define

$$
\epsilon(x) = \frac{1}{(\ln\ \ln(x))^2}
$$

for $x > 1$, and

$$
R(a,b) = \max\left\{\frac{40\pi b}{\epsilon(a)^2}, \left(\frac{160\lambda^2\pi^3 b^5}{3\epsilon(a)^2}\right)^{1/3}\right\}.
$$

Let us assume that $1 << a_0 < 2a_0 < b_0$, with b_0 sufficiently large that

$$R(a_0,b_0)+10a_0-\frac{c'}{\lambda}+1<R(b_0,b_0)+10b_0-\frac{c'}{\lambda}+1<b_0^{1.7}.$$

Define

$$\rho_1(a,b)=\frac{2\sqrt{\epsilon(a)}}{b-a}\sqrt{\ln\frac{72\sqrt{(b-a)(\lambda(b^2-a^2)+10b+2R(a,b))}}{\epsilon(a)^{5/4}}}$$

and

$$\rho_2(a,b)=\frac{\sqrt{\ln 72}(b-a)}{\sqrt{\epsilon(a)}}.$$

Assume that a_0 is sufficiently large that $\rho_1(a_0,b_0)<1$, and that b_0 is sufficiently large that $\rho_2(a_0,b_0)<b_0^{1.1}$.

Assume that $c_0<d_0$ are chosen such that $|c_0|,|d_0|<10a_0$. Given $\psi\in L^2(\mathbb{R})$, $\|\psi\|=1$, such that ψ is $\epsilon(a_0),\epsilon(a_0)$ supported in $[a_0,b_0]\times[c_0,d_0]$, we obtain from the previous two theorems that there exists $\psi_1\in L^2(\mathbb{R})$, $\|\psi_1-U\psi\|<3\epsilon(a_0)$, such that ψ_1 is $\epsilon(a_0)/9,\epsilon(a_0)/9$ supported in $[\tilde{a},\tilde{b}]\times[\tilde{c},\tilde{d}]$, where

$$\begin{aligned}
\tilde{a} &= \lambda a_0^2+\frac{c'}{\lambda}-d_0-R(a_0,b_0)-\rho_1(a_0,b_0)>\lambda a_0^2-b_0^{1.7}\\
\tilde{b} &= \lambda b_0^2+\frac{c'}{\lambda}-c_0+R(a_0,b_0)+\rho_1(a_0,b_0)<\lambda b_0^2+b_0^{1.7}\\
\tilde{c} &= a_0-\rho_2(a_0,b_0)>a_0-b_0^{1.1}\\
\tilde{d} &= b_0+\rho_2(a_0,b_0)>b_0+b_0^{1.1}
\end{aligned}$$

Suppose that a_0,b_0 are chosen so that $((1+\lambda)b_0)^{1.7}<(\lambda a_0/2)^2$. For $n\geq 1$, define

$$a_n=\frac{2}{\lambda}\left(\frac{\lambda a_0}{2}\right)^{2^n},$$

$$b_n=\frac{1}{1+\lambda}((1+\lambda)b_0)^{2^n}.$$

Note that then

$$\begin{aligned}
((1+\lambda)b_n)^{1.7} &= [((1+\lambda)b_0)^{1.7}]^{2^n}\\
&< \left(\frac{\lambda a_0}{2}\right)^{2^{n+1}}\\
&= \left(\frac{\lambda a_n}{2}\right)^2
\end{aligned}$$

for each $n\geq 1$. Let also $c_n=a_{n-1}-b_{n-1}^{1.1}$, $d_n=b_{n-1}+b_{n-1}^{1.1}$ for each $n\geq 1$. Now, a simple induction shows that $2a_n<b_n$ for all n, and we have also that

$$|c_n|<d_n<2b_{n-1}^{1.1},$$

while

$$\begin{aligned} 10a_n &= 10\left(\frac{2}{\lambda}\right)\left(\frac{\lambda a_{n-1}}{2}\right)^2 \\ &> \frac{20}{\lambda}((1+\lambda)b_{n-1})^{1.7} \\ &> 20b_{n-1}^{1.7} \\ &> 2b_{n-1}^{1.1}. \end{aligned}$$

Thus

$$|c_n|, |d_n| < 10a_n$$

for all n. Furthermore,

$$\lambda b_n^2 + b_n^{1.7} < (1+\lambda)b_n^2 = b_{n+1},$$

and

$$\begin{aligned} \lambda a_n^2 - b_n^{1.7} &= 2a_{n+1} - \frac{1}{(1+\lambda)^{1.7}}[((1+\lambda)b_0)^{1.7}]^{2^n} \\ &> 2a_{n+1} - \frac{1}{(1+\lambda)^{1.7}}\left(\frac{\lambda a_0}{2}\right)^{2^{n+1}} \\ &= 2a_{n+1} - \frac{\lambda}{2(1+\lambda)^{1.7}}a_{n+1} \\ &> (2-1/2)a_{n+1} \\ &> a_{n+1} \end{aligned}$$

for each n.

Now, let us assume that a_0 is chosen large enough that

$$a_0 < \frac{\lambda a_0^2}{2}$$

Then $a_n \to \infty$ super-exponentially. Also,

$$\epsilon(a_n) := \frac{1}{(\ln\ \ln(a_n))^2} \le \frac{C(a_0)}{(n+1)^2},$$

where $C(a_0) \to 0$ as $a_0 \to \infty$. If we suppose also that a_0 is chosen large enough that

$$(\ln a_n)^3 > \ln\left(\frac{\lambda}{2}\right) + 2\ln a_n$$

for all n, then we obtain that $\epsilon(a_{n+1}) > \epsilon(a_n)/9$ for all n. Thus we have the following theorem:

THEOREM 3.7. *Suppose that* $0 < 2a_0 < b_0$ *with* $((1+\lambda)b_0)^{1.7} < (\lambda a_0/2)^2$, *that* $c_0 < d_0$ *are real numbers satisfying* $|c_0|, |d_0| < 10a_0$, *and that* $\psi \in L^2(\mathbb{R})$ *has unit norm and is* ϵ, ϵ *supported in* $[a_0, b_0] \times [c_0, d_0]$, *where*

$$\epsilon = \frac{1}{(\ln\ \ln(a_0))^2}.$$

Then if a_0 *and* b_0 *are sufficiently large, for each* $n \in \mathbb{N}$ *we have that* $U^n\psi$ *is* $1/100$ *supported in* $[a_n, b_n]$, *with* $a_n, b_n \to \infty$ *super-exponentially.*

Proof: Let $\psi_0 = \psi$. Define the sequences a_n, b_n, c_n, d_n as above. For each $n = 0, 1, \ldots$, we find a ψ_{n+1} which is $\epsilon(a_{n+1}), \epsilon(a_{n+1})$ supported on $[a_{n+1}, b_{n+1}] \times [c_{n+1}, d_{n+1}]$, and such that $\|\psi_{n+1} - U\psi_n\| \leq 3\epsilon(a_n)$. Then

$$
\begin{aligned}
\|\psi_n - U^n\psi_0\| &\leq 3\sum_{i=0}^{n-1} \epsilon(a_n) \\
&\leq 3C(a_0)\sum_{i=0}^{\infty} \frac{1}{(n+1)^2} \\
&\leq 6C(a_0) \\
&\leq 1/200
\end{aligned}
$$

for a_0 sufficiently large. Since ψ_n is $1/200, 1/200$ supported on $[a_n, b_n] \times [c_n, d_n]$ (again assuming that a_0 is sufficiently large), the result follows.

■

COROLLARY 3.8. *Under the same hypotheses, for each n, $U^n\psi$ is $1/100, 1/100$ supported in $[a_n, b_n]\times[c_n, d_n]$, where c_n and d_n also go to infinity super-exponentially.*

Proof: We have already pointed out that if any $\phi \in L^2(\mathbb{R})$ is ϵ, ϵ supported in $[a, b] \times [c, d]$, then $U\phi$ is $4\epsilon, \epsilon$ supported in $[a', b'] \times [c', d']$, where $c' = a$ and $d' = b$. Thus we may take $c_n = a_{n-1}$ and $d_n = b_{n-1}$ and apply the theorem.

■

4. Dynamics for large c, small λ: proof of Theorem 2

In this section we will begin by considering the operator

$$\tilde{U} := e^{iy^2/2}e^{-i(\lambda x^3/3 - x^2 + cx/\lambda)},$$

which, as we have described in the introduction, is unitarily equivalent to the operator U. We wish to show that if certain conditions are put on c and λ– specifically, if $\lambda > 0$ is fixed and $c > 0$ is sufficiently large, or if $c > 1$ is fixed and $\lambda > 0$ is sufficiently small– then all states are sent to infinity by the iterates of $\tilde{U}$ in the sense that for any $\psi \in L^2(\mathbb{R})$ and any interval $I_R := (-\infty, R] \subset \mathbb{R}$,

$$\|\tilde{U}^n\psi\|_{I_R} \to 0$$

and

$$\|F\tilde{U}^n\psi\|_{-I_R} \to 0$$

as $n \to \infty$. Then we will be able to draw similar conclusions with U in place of $\tilde{U}$. In particular, if these conditions are satisfied, then U has no discrete spectrum. Note that for the classical map $(x, y) \mapsto (x^2 + c - y, x)$, $c > 1$ implies that there is no domain of points with bounded orbit.

We begin by defining the so-called Airy function: let

$$Ai(x) = Fe^{-ix^3/3}.$$

(Here of course we must use the natural extension of F to the space $\mathcal{S}'$ of tempered distributions, dual to the space $\mathcal{S}$ of smooth, rapidly-decreasing functions on $\mathbb{R}$.) It is known that Ai is real-valued on $\mathbb{R}$, bounded, real-analytic, and, most importantly for our purposes, that it decays exponentially at $+\infty$: precisely, there exist $A, B > 0$ such that

$$|Ai(x)| < Bx^{-1/4}e^{-(2/3)x^{3/2}}$$

for $x > A$. Increasing A if necessary, we will replace this by the estimate

$$|Ai(x)| < e^{-2x}$$

for $x > A$. See [T] for a description of these and other properties of the Airy function.

Let

$$\begin{aligned} \mu(x) &= Fe^{-i(\lambda x^3/3 - x^2 + cx/\lambda)} \\ &= Fe^{-i[\frac{\lambda}{3}(x-\frac{1}{\lambda})^3 + \frac{(c-1)x}{\lambda} + \frac{1}{3\lambda^2}]}. \end{aligned}$$

Using elementary properties of the Fourier transform, it follows that

$$\mu(x) = (e^{-i/3\lambda^2})(e^{-i[\frac{1}{\lambda}(x+\frac{c-1}{\lambda})]})\lambda^{-1/3}Ai\left(\lambda^{-1/3}(x+\frac{c-1}{\lambda})\right).$$

Thus we have the following estimate:

LEMMA 4.1. *If*

$$x > \lambda^{1/3}A - \frac{c-1}{\lambda},$$

then

$$|\mu(x)| \leq \lambda^{-1/3}e^{-2(\lambda^{-1/3}(x+\frac{c-1}{\lambda}))}.$$

Proof: This follows immediately from the estimate on Ai. ■

Now, for $C \in \mathbb{R}$, define

$$\mathcal{C}_C = \{\psi \in L^2(\mathbb{R}) : |\mathrm{F}\psi(\mathrm{x})| < \lambda^{-1/6}\mathrm{e}^{[\lambda^{-1/3}(\mathrm{C}-\mathrm{x})]} \text{ for } \mathrm{x} > \mathrm{C}\}.$$

Note that if $\psi \in \mathcal{C}_C$, then

$$\int_C^\infty |F\psi|^2 < 1/2.$$

THEOREM 4.2. *Let* $T = \sup|Ai(x)|$, *and suppose that*

$$\delta := \frac{c-1}{\lambda} - \lambda^{1/3}(A + \ln(T+1)) > 0.$$

Then

$$\tilde{U} : \mathcal{C}_C \to \mathcal{C}_{C-\delta}.$$

Proof: Note that $\sup|\mu(x)| = \lambda^{-1/3}T$. Let $\psi \in \mathcal{C}_C$ and suppose that $x > C - \delta$. Then

$$\begin{aligned} x &> C - \frac{c-1}{\lambda} + \lambda^{1/3}(A + \ln(T+1)) \\ &> C - \frac{c-1}{\lambda} + \lambda^{1/3}A. \end{aligned}$$

We have

$$\begin{aligned}
|F\tilde{U}\psi(x)| &= |Fe^{iy^2/2}e^{-i(\lambda x^3/3-x^2+cx/\lambda)}\psi(x)| \\
&= |e^{ix^2/2}Fe^{-i(\lambda x^3/3-x^2+cx/\lambda)}\psi(x)| \\
&= |Fe^{-i(\lambda x^3/3-x^2+cx/\lambda)}\psi(x)| \\
&= |\mu * F\psi(x)| \\
&= \left|\int_{\mathbb{R}} \mu(x-t)F\psi(t)\ dt\right| \\
&\leq D_1 + D_2 + D_3,
\end{aligned}$$

where

$$\begin{aligned}
D_1 &= \left|\int_{-\infty}^{C} \mu(x-t)F\psi(t)\ dt\right| \\
D_2 &= \left|\int_{C}^{x-\lambda^{1/3}A+(c-1)/\lambda} \mu(x-t)F\psi(t)\ dt\right| \\
D_3 &= \left|\int_{x-\lambda^{1/3}A+(c-1)/\lambda}^{\infty} \mu(x-t)F\psi(t)\ dt\right|.
\end{aligned}$$

Then

$$\begin{aligned}
D_1 &\leq \left(\int_{-\infty}^{C} |F\psi(t)|^2\ dt\right)^{1/2}\left(\int_{-\infty}^{C} |\mu(x-t)|^2\ dt\right)^{1/2} \\
&\leq 1\cdot\left(\int_{-\infty}^{C} |\mu(x-t)|^2\ dt\right)^{1/2} \\
&\leq \lambda^{-1/3}\left(\int_{-\infty}^{C} e^{-4(\lambda^{-1/3}(x-t+\frac{c-1}{\lambda}))}\ dt\right)^{1/2} \\
&= \frac{1}{2}\lambda^{-1/3}\lambda^{1/6}(e^{-4\lambda^{-1/3}(x-C+\frac{c-1}{\lambda})})^{1/2} \\
&< \lambda^{-1/3}\lambda^{1/6}e^{2\lambda^{-1/3}(C-\frac{c-1}{\lambda}-x)},
\end{aligned}$$

$$\begin{aligned}
D_2 &\leq \int_{C}^{x-\lambda^{1/3}A+(c-1)/\lambda} (\lambda^{-1/6}e^{\lambda^{-1/3}(C-t)})(\lambda^{-1/3}e^{-2\lambda^{-1/3}(x-t+\frac{c-1}{\lambda})})\ dt \\
&= \lambda^{-1/2}\int_{C}^{x-\lambda^{1/3}A+(c-1)/\lambda} e^{-\lambda^{-1/3}(2x-t-C+\frac{2(c-1)}{\lambda})}\ dt \\
&= \lambda^{-1/2}\lambda^{1/3}[e^{-\lambda^{-1/3}(x+\lambda^{1/3}A-C+\frac{c-1}{\lambda})} - e^{-\lambda^{-1/3}(2x-2C+\frac{2(c-1)}{\lambda})}],
\end{aligned}$$

and

$$\begin{aligned}
D_3 &\leq \int_{x-\lambda^{1/3}A+(c-1)/\lambda}^{\infty} (\lambda^{-1/6}e^{\lambda^{-1/3}(C-t)})(\lambda^{-1/3}T)\ dt \\
&= \lambda^{-1/2}T\int_{x-\lambda^{1/3}A+(c-1)/\lambda}^{\infty} e^{\lambda^{-1/3}(C-t)}\ dt \\
&= \lambda^{-1/2}T\lambda^{1/3}e^{\lambda^{-1/3}(C-x+\lambda^{1/3}A-\frac{c-1}{\lambda})}.
\end{aligned}$$

Combining these estimates, we have

$$\begin{aligned} D_1+D_2+D_3 &< \lambda^{-1/6}e^{2\lambda^{-1/3}(C-\frac{c-1}{\lambda}-x)} \\ &+ \lambda^{-1/6}[e^{-\lambda^{-1/3}(x+\lambda^{1/3}A-C+\frac{c-1}{\lambda})}-e^{-\lambda^{-1/3}(2x-2C+\frac{2(c-1)}{\lambda})}] \\ &+ \lambda^{-1/6}Te^{\lambda^{-1/3}(C-x+\lambda^{1/3}A-\frac{c-1}{\lambda})} \\ &= \lambda^{-1/6}e^{\lambda^{-1/3}(C-\frac{c-1}{\lambda}-x)}[e^{-A}+Te^{A}] \\ &< \lambda^{-1/6}e^{\lambda^{-1/3}(C-\frac{c-1}{\lambda}-x)}(T+1)e^{A} \\ &= \lambda^{-1/6}e^{\lambda^{-1/3}(C-\frac{c-1}{\lambda}+\lambda^{1/3}(A+\ln(T+1))-x)} \\ &= \lambda^{-1/6}e^{\lambda^{-1/3}(C-\delta-x)}. \end{aligned}$$

■

Note that, given $\psi\in L^2(\mathbb{R})$, an arbitrarily small perturbation of ψ lies in some $\mathcal{C}_C$. Thus the L^2 mass of $F\tilde{U}^n\psi$ moves to $-\infty$ for every $\psi\in L^2(\mathbb{R})$.

COROLLARY 4.3. *For c,λ satisfying the above condition, the operator $\tilde{U}$ has no discrete spectrum, and neither therefore does U.*

We would like to show now that, under the same conditions on c and λ, the L^2 mass of $\tilde{U}^n\psi$ moves to $+\infty$ for every $\psi\in L^2(\mathbb{R})$.

THEOREM 4.4. *Suppose again that the condition*

$$\delta := \frac{c-1}{\lambda}-\lambda^{1/3}(A+\ln(T+1))>0$$

is satisfied, and that $\psi\in\mathcal{C}_C$ for some $C\in\mathbb{R}$. Then, given $R>0$,

$$\|\tilde{U}^n\psi\|_{(-\infty,R]}\to 0$$

as $n\to\infty$.

Proof: By the previous theorem, $\tilde{U}^n\psi\in\mathcal{C}_{C(n)}$ for $n=1,2,\ldots$, where $C(n)=C-n\delta$. Then

$$\begin{aligned} |\tilde{U}^{n+1}\psi(x)| &= |e^{iy^2/2}e^{-i(\lambda x^3/3-x^2+cx/\lambda)}\tilde{U}^n\psi(x)| \\ &= |e^{-ix^2/2}(e^{ix^2/2}e^{iy^2/2}e^{ix^2/2})e^{-ix^2/2}e^{-i(\lambda x^3/3-x^2+cx/\lambda)}\tilde{U}^n\psi(x)| \\ &= |e^{-ix^2/2}F^{-1}e^{-ix^2/2}e^{-i(\lambda x^3/3-x^2+cx/\lambda)}\tilde{U}^n\psi(x)| \\ &= |F^{-1}e^{-i(\lambda x^3/3-x^2/2+cx/\lambda)}\tilde{U}^n\psi(x)| \\ &= |(F^{-1}e^{-i(\lambda x^3/3-x^2/2+cx/\lambda)})*(F^{-1}\tilde{U}^n\psi)(x)| \\ &= |\nu*(F^{-1}\tilde{U}^n\psi)(x)|, \end{aligned}$$

where

$$\begin{aligned} \nu &= F^{-1}e^{-i(\lambda x^3/3-x^2/2+cx/\lambda)} \\ &= F^2(Fe^{-i[\frac{\lambda}{3}(x-\frac{1}{2\lambda})^3+\frac{(c-1/4)x}{\lambda}+\frac{1}{24\lambda^2}]}) \\ &= F^2\left(e^{-i\frac{1}{24\lambda^2}}e^{-i[\frac{1}{2\lambda}(x+\frac{c-1/4}{\lambda})]}\lambda^{-1/3}Ai\left(\lambda^{-1/3}(x+\frac{c-1/4}{\lambda})\right)\right) \\ &= (e^{-i\frac{1}{24\lambda^2}})(e^{-i[\frac{1}{2\lambda}(-x+\frac{c-1/4}{\lambda})]})\lambda^{-1/3}Ai\left(\lambda^{-1/3}(-x+\frac{c-1/4}{\lambda})\right). \end{aligned}$$

Thus

$$|\nu(x)| = \lambda^{-1/3} \left| Ai \left(\lambda^{-1/3}(-x + \frac{c-1/4}{\lambda}) \right) \right| .$$

Recalling the estimate

$$|Ai(x)| < e^{-2x}, \quad x > A,$$

we have

$$|\nu(x)| < \lambda^{-1/3} e^{2\lambda^{-1/3}(x - \frac{c-1/4}{\lambda})}$$

whenever

$$x < \frac{c-1/4}{\lambda} - \lambda^{1/3} A,$$

or

$$|\nu(x-t)| < \lambda^{-1/3} e^{2\lambda^{-1/3}(x-t-\frac{c-1/4}{\lambda})}$$

whenever

$$t > x - \frac{c-1/4}{\lambda} + \lambda^{1/3} A.$$

Furthermore, since $\tilde{U}^n \psi \in \mathcal{C}_{C(n)}$, we have

$$\begin{aligned} |F^{-1}\tilde{U}^n \psi(t)| &= |F^2(F\tilde{U}^n\psi)(t)| \\ &= |F\tilde{U}^n \psi(-t)| \\ &< \lambda^{-1/6} e^{\lambda^{-1/3}(C(n)+t)} \end{aligned}$$

whenever $t < -C(n)$.

Suppose now that

$$x < -C(n) + \frac{c-1/4}{\lambda} - \lambda^{1/3} A.$$

Then

$$\begin{aligned} |\tilde{U}^{n+1}\psi(x)| &= |\nu * (F^{-1}\tilde{U}^n\psi)(x)| \\ &= \left| \int_{\mathbb{R}} \nu(x-t) F^{-1}\tilde{U}^n \psi(t)\, dt \right| \\ &\leq D_1 + D_2 + D_3, \end{aligned}$$

where

$$\begin{aligned} D_1 &= \left| \int_{-\infty}^{x - \frac{c-1/4}{\lambda} + \lambda^{1/3}A} \nu(x-t) F^{-1}\tilde{U}^n\psi(t)\, dt \right| \\ D_2 &= \left| \int_{x - \frac{c-1/4}{\lambda} + \lambda^{1/3}A}^{-C(n)} \nu(x-t) F^{-1}\tilde{U}^n\psi(t)\, dt \right| \\ D_3 &= \left| \int_{-C(n)}^{\infty} \nu(x-t) F^{-1}\tilde{U}^n\psi(t)\, dt \right| . \end{aligned}$$

Then we have

$$\begin{aligned} D_1 &< \lambda^{-1/3} T \int_{-\infty}^{x - \frac{c-1/4}{\lambda} + \lambda^{1/3}A} \lambda^{-1/6} e^{\lambda^{-1/3}(C(n)+t)}\, dt \\ &= \lambda^{-1/6} T e^{\lambda^{-1/3}(C(n) + x - \frac{c-1/4}{\lambda} + \lambda^{1/3}A)}, \end{aligned}$$

$$\begin{aligned} D_2 &< \int_{x-\frac{c-1/4}{\lambda}+\lambda^{1/3}A}^{-C(n)} \left(\lambda^{-1/3}e^{2\lambda^{-1/3}(x-t-\frac{c-1/4}{\lambda})}\right)\left(\lambda^{-1/6}e^{\lambda^{-1/3}(C(n)+t)}\right)\,dt \\ &= \lambda^{-1/2}\int_{x-\frac{c-1/4}{\lambda}+\lambda^{1/3}A}^{-C(n)} e^{\lambda^{-1/3}(C(n)+2x-t-\frac{2(c-1/4)}{\lambda})}\,dt \\ &= \lambda^{-1/6}[e^{\lambda^{-1/3}(C(n)+x-\lambda^{1/3}A-\frac{c-1/4}{\lambda})} - e^{2\lambda^{-1/3}(C(n)+x-\frac{c-1/4}{\lambda})}], \end{aligned}$$

and

$$\begin{aligned} D_3 &\leq \left(\int_{-C(n)}^{\infty} |F^{-1}\tilde{U}^n\psi(t)|^2\,dt\right)^{1/2}\left(\int_{-C(n)}^{\infty} |\nu(x-t|^2\,dt\right)^{1/2} \\ &\leq 1\cdot\left(\int_{-C(n)}^{\infty} |\nu(x-t|^2\,dt\right)^{1/2} \\ &< \lambda^{-1/3}\left(\int_{-C(n)}^{\infty} e^{4\lambda^{-1/3}(x-t-\frac{c-1/4}{\lambda})}\right)^{1/2} \\ &= \lambda^{-1/3}\left(\frac{\lambda^{1/3}}{4}e^{4\lambda^{-1/3}(C(n)+x-\frac{c-1/4}{\lambda})}\right)^{1/2} \\ &= \frac{1}{2}\lambda^{-1/6}e^{2\lambda^{-1/3}(C(n)+x-\frac{c-1/4}{\lambda})} \\ &< \lambda^{-1/6}e^{2\lambda^{-1/3}(C(n)+x-\frac{c-1/4}{\lambda})}. \end{aligned}$$

Combining these estimates, we get

$$\begin{aligned} D_1+D_2+D_3 &< \lambda^{-1/6}[Te^{\lambda^{-1/3}(C(n)+x+\lambda^{1/3}A-\frac{c-1/4}{\lambda})} + e^{\lambda^{-1/3}(C(n)+x-\lambda^{1/3}A-\frac{c-1/4}{\lambda})}] \\ &= \lambda^{-1/6}e^{\lambda^{-1/3}(C(n)+x-\frac{c-1/4}{\lambda})}[Te^{A}+e^{-A}] \\ &< \lambda^{-1/6}e^{\lambda^{-1/3}(C(n)+x-\frac{c-1/4}{\lambda})}[(T+1)e^{A}] \\ &= \lambda^{-1/6}e^{\lambda^{-1/3}(C(n)+x-\frac{c-1/4}{\lambda}+\lambda^{1/3}(A+\ln(T+1)))}. \end{aligned}$$

To summarize, for each $n \in \mathbb{N}$, we have

$$|\tilde{U}^{n+1}\psi(x)| < \lambda^{-1/6}e^{\lambda^{-1/3}(C(n)+x-\frac{c-1/4}{\lambda}+\lambda^{1/3}(A+\ln(T+1)))}$$

for all

$$x < -C(n) + \frac{c-1/4}{\lambda} - \lambda^{1/3}A.$$

Since $C(n) = C - n\delta$, letting $n \to \infty$ gives the desired result. ∎

COROLLARY 4.5. *Let $\psi \in L^2(\mathbb{R})$, and let $U = F^{-1}e^{-i(\lambda x^3/3+cx/\lambda)}$. Suppose that c and λ satisfy the condition of the theorem. Then for any $R > 0$, we have*

$$\|U^n\psi\|_{I_R} \to 0$$

and

$$\|FU^n\psi\|_{I_R} \to 0$$

as $n \to \infty$.

Proof: Since $\cup\{\mathcal{C}_C : C \in \mathbb{R}\}$ is dense in $L^2(\mathbb{R})$, it suffices to prove the corollary under the assumption that $e^{-ix^2/2}\psi \in \mathcal{C}_C$ for some C. We have

$$\begin{aligned} |U^n\psi(x)| &= |e^{ix^2/2}\tilde{U}^n e^{-ix^2/2}\psi(x)| \\ &= |\tilde{U}^n e^{-ix^2/2}\psi(x)|. \end{aligned}$$

The first statement therefore follows from the theorem. The second then follows immediately from the first, since

$$\begin{aligned} |FU^{n+1}\psi(x)| &= |F(F^{-1}e^{-i(\lambda x^3/3+cx/\lambda)})U^n\psi(x)| \\ &= |e^{-i(\lambda x^3/3+cx/\lambda)})U^n\psi(x)| \\ &= |U^n\psi(x)|. \end{aligned}$$

■

5. Eigenvalue questions: proofs of Theorems 3 and 4

We first prove an Egorov-type theorem for unitary operators which are quantizations of shears; that is, of the form $e^{if(x)}$ or $e^{if(y)}$. The time-one maps of the Hamiltonian functions $f(x)$ and $f(y)$ on $\mathbb{R}^2$ are

$$(x, y) \mapsto (x, y - f'(x))$$

and

$$(x, y) \mapsto (x + f'(y), y),$$

respectively, and their corresponding quantizations are $e^{-if(x)}$ and $e^{-if(y)}$. (As usual, for a function $g : \mathbb{R} \to \mathbb{C}$, we use the notation $g(x)$ to denote multiplication by g, and $g(y)$ to denote $F^{-1}g(x)F$.)

THEOREM 5.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$ function. Then*

$$\begin{aligned} e^{if(x)}xe^{-if(x)} &= x \\ e^{if(x)}ye^{-if(x)} &= y - f'(x) \\ e^{if(y)}xe^{-if(y)} &= x + f'(y) \\ e^{if(y)}ye^{-if(y)} &= y. \end{aligned}$$

Proof: The first and last of these equalities are trivial. For the second, suppose that $\psi \in \mathcal{S} \subset L^2(\mathbb{R})$. Then

$$\begin{aligned} e^{if(x)}xe^{-if(x)}\psi &= e^{if(x)}(-i\partial_x)e^{-if(x)}\psi \\ &= e^{if(x)}[-i(-if'(x)e^{-if(x)}\psi) + e^{-if(x)}(-i\partial_x\psi)] \\ &= (y - f'(x))\psi. \end{aligned}$$

Now the third follows from the second:

$$\begin{aligned} e^{if(y)}xe^{-if(y)} &= F^{-1}e^{if(x)}FxF^{-1}e^{-if(x)}F \\ &= F^{-1}e^{if(x)}(-y)e^{-if(x)}F \\ &= F^{-1}(-y + f'(x))F \\ &= x + f'(y). \end{aligned}$$

The equalities are therefore valid on a dense subspace of $L^2(\mathbb{R})$. ■

COROLLARY 5.2. *If*

$$U = F^{-1}e^{-i(\lambda x^3/3+cx/\lambda)},$$

then

$$\begin{aligned} U^{-1}xU &= -y + \lambda x^2 + c/\lambda \\ U^{-1}yU &= x. \end{aligned}$$

Proof: This follows immediately from the theorem and the identities $FxF^{-1} = -y$ and $FyF^{-1} = x$. ■

Recall the following consequence of the Cauchy-Schwarz inequality, known as the Heisenberg uncertainty principle: let $\psi \in L^2(\mathbb{R})$ satisfy $\|\psi\| = 1$, $x\psi \in L^2(\mathbb{R})$, and $y\psi \in L^2(\mathbb{R})$. Let

$$\Delta_x\psi = \sqrt{(\psi, x^2\psi) - (\psi, x\psi)^2}$$

and

$$\Delta_y\psi = \sqrt{(\psi, y^2\psi) - (\psi, y\psi)^2}.$$

Then

$$\begin{aligned} \Delta_x\psi \cdot \Delta_y\psi &= \|(x - (\psi, x\psi))\psi\| \cdot \|(y - (\psi, y\psi))\psi\| \\ &\geq |((x - (\psi, x\psi))\psi, (y - (\psi, y\psi))\psi)| \\ &\geq \mathrm{Im}((x - (\psi, x\psi))\psi, (y - (\psi, y\psi))\psi) \\ &= \frac{1}{2i}(\psi, [(x - (\psi, x\psi)), (y - (\psi, y\psi))]\psi) \\ &= \frac{1}{2i}(\psi, [x, y]\psi) \\ &= \frac{1}{2i}(\psi, i\psi) \\ &= 1/2. \end{aligned}$$

THEOREM 5.3. *Let*

$$U = F^{-1}e^{-i(\lambda x^3/3+cx/\lambda)}.$$

If $c > 1$ or $\lambda > \sqrt{2(1-c)}$, then U has no eigenvectors in the Sobolev space $H^1 := \{\psi \in L^2(\mathbb{R}) : x\psi \in L^2(\mathbb{R})\}$.

Proof: Suppose that ψ is such an eigenvector. Then $(U\psi, AU\psi) = (\psi, A\psi)$ for all self-adjoint operators A such that ψ is in the domain of A. Also note that

$$\begin{aligned} |F\psi(x)| &= |FU\psi(x)| \\ &= |e^{-i(\lambda x^3/3+cx/\lambda)}\psi(x)| \\ &= |\psi(x)|. \end{aligned}$$

Thus if $f : \mathbb{R} \to \mathbb{R}$ is any function such that

$$\int_{\mathbb{R}} f|\psi|^2 < \infty,$$

then

$$\begin{aligned}(\psi, f(y)\psi) &= (\psi, F^{-1}f(x)F\psi)\\ &= (F\psi, f(x)F\psi)\\ &= \int_{\mathbb{R}} f|F\psi|^2\\ &= \int_{\mathbb{R}} f|\psi|^2\\ &= (\psi, f(x)\psi).\end{aligned}$$

In particular, $x\psi \in L^2(\mathbb{R})$ implies that $y\psi \in L^2(\mathbb{R})$. Also, $\Delta_x\psi = \Delta_y\psi \geq 1/\sqrt{2}$. Then

$$\begin{aligned}(\psi, x\psi) &= (U\psi, xU\psi)\\ &= (\psi, U^{-1}xU\psi)\\ &= (\psi, (-y + \lambda x^2 + c/\lambda)\psi)\\ &= -(\psi, y\psi) + \lambda(\psi, x^2\psi) + c/\lambda\\ &= -(\psi, x\psi) + \lambda[(\psi, x\psi)^2 + (\Delta_x\psi)^2] + c/\lambda.\end{aligned}$$

Thus

$$(\psi, x\psi) = 1 \pm \sqrt{1 - (\lambda^2(\Delta_x\psi)^2 + c)}.$$

Therefore we require

$$0 \leq 1 - (\lambda^2(\Delta_x\psi)^2 + c) \leq 1 - (\lambda^2/2 + c),$$

from which we conclude that $c < 1$ and $\lambda < \sqrt{2(1-c)}$. ■

We wish to remark on the following curiosity: consider the operator $e^{i\lambda x^2}F$ for λ in a neighborhood of $0 \in \mathbb{R}$. For $-1 < \lambda < 1$, it has the eigenvector

$$e^{i\lambda x^2/2-(\sqrt{1-\lambda^2})x^2/2},$$

while for $|\lambda| = 1$, it has a purely continuous spectrum, since, ignoring multiplicative constants of absolute value one, we have

$$\begin{aligned}Fe^{-ix^2/2}(e^{ix^2}F)e^{ix^2/2}F^{-1} &= Fe^{-ix^2/2}(e^{ix^2}e^{-ix^2/2}e^{-iy^2/2}e^{-ix^2/2})e^{ix^2/2}F^{-1}\\ &= Fe^{-iy^2/2}F^{-1}\\ &= e^{-ix^2/2},\end{aligned}$$

and

$$\begin{aligned}Fe^{ix^2/2}(e^{-ix^2}F)e^{-ix^2/2}F^{-1} &= Fe^{ix^2/2}(e^{-ix^2}e^{-iy^2/2}e^{-ix^2/2}e^{-iy^2/2})e^{-ix^2/2}F^{-1}\\ &= Fe^{iy^2/2}(e^{-iy^2/2}e^{-ix^2/2}e^{-iy^2/2})(e^{-ix^2/2}e^{-iy^2/2})e^{-ix^2/2})F^{-1}\\ &= Fe^{iy^2/2}(F^2)F^{-1}\\ &= Fe^{iy^2/2}F^{-1}F^2\\ &= e^{ix^2/2}F^2,\end{aligned}$$

which is a square root of e^{ix^2}. The phase change at $|\lambda| = 1$ occurs exactly where the eigenvalues of the underlying classical map, which is linear, become real, so that the map changes from being conjugate to a rotation for $|\lambda| < 1$ to having a saddle for $|\lambda| > 1$.

We wish to investigate whether the situation is similar if we replace the exponent two by a three in the definition of this operator.

THEOREM 5.4. *There is no neighborhood I of $0 \in \mathbb{R}$, analytic family $\{\psi_\lambda\}$: $\psi_\lambda \in L^2(\mathbb{R})$ of perturbations of $\psi_0 := e^{-x^2/2}$, and analytic family $\{a_\lambda\}$: $a_\lambda \in \{|z| = 1\} \subset \mathbb{C}$ of perturbations of $a_0 := 1$, such that*

$$e^{i\lambda x^3} F\psi_\lambda = a_\lambda \psi_\lambda$$

for each $\lambda \in I$.

Proof: Assume the contrary. Write

$$\psi_\lambda = e^{-x^2/2} + \lambda\psi_1 + \lambda^2\psi_2 + \dots$$

and

$$a_\lambda = 1 + \lambda a_1 + \lambda^2 a_2 + \dots.$$

Then, expanding the equation

$$F\psi_\lambda = e^{-i\lambda x^3} a_\lambda \psi_\lambda$$

in powers of λ, we obtain

$$\begin{aligned} (F - \mathrm{Id})e^{-x^2/2} &= 0 \\ (F - \mathrm{Id})\psi_1 &= (a_1 - ix^3)e^{-x^2/2} \\ (F - \mathrm{Id})\psi_2 &= (a_1 - ix^3)\psi_1 + (a_2 - x^6/2)e^{-x^2/2} \\ &\vdots \end{aligned}$$

Consider the eigenspaces T_1, T_{-i}, T_{-1}, T_i of the Fourier transform. In order to solve the above equations, a necessary condition is that their right-hand sides be perpendicular to T_1. In particular, taking the inner product of the second with $e^{-x^2/2}$ and setting the result equal to zero, we find that $a_1 = 0$. Let V be the inverse of the restriction of $F - \mathrm{Id}$ to $T_1^\perp$. Then

$$\begin{aligned} V|_{T_{-i}} &= (-1 + i)/2 \\ V|_{T_i} &= (-1 - i)/2. \end{aligned}$$

Let

$$\begin{aligned} \phi_1 &= xe^{-x^2/2} \\ \phi_2 &= (1 - 2x^2)e^{-x^2/2} \\ \phi_3 &= (3x - 2x^3)e^{-x^2/2} \\ \phi_4 &= (3 - 12x^2 + 4x^4)e^{-x^2/2} \\ \phi_5 &= (15x - 20x^3 + 8x^5)e^{-x^2/2} \\ \phi_6 &= (15 - 90x^2 + 80x^4 - 16x^6)e^{-x^2/2}. \end{aligned}$$

Then $F\phi_n = (-i)^n\phi_n$ for each n. Solving the second equation above, we may write

$$\psi_1 = V(-ix^3e^{-x^2/2}) + t_1$$

for some $t_1 \in T_1$. Substituting into the third equation, we obtain

$$\begin{aligned}
(F-\mathrm{Id})\psi_2 &= -x^3V(x^3e^{-x^2/2}) - ix^3t_1 + (a_2 - x^6/2)e^{-x^2/2} \\
&= -x^3V((3/2)\phi_1 - (1/2)\phi_3) - ix^3t_1 + (a_2 - x^6/2)e^{-x^2/2} \\
&= -x^3\left[\frac{-1+i}{2}(3/2)\phi_1 - \frac{-1-i}{2}(1/2)\phi_3\right] - ix^3t_1 + (a_2 - x^6/2)e^{-x^2/2} \\
&= -x^3\left(\left[\frac{3(-1+i)}{4}(x) + \frac{1+i}{4}(3x-2x^3)\right] + (a_2 - x^6/2)\right)e^{-x^2/2} - ix^3t_1 \\
&= (a_2 - (3i/2)x^4 + (i/2)x^6)e^{-x^2/2} - ix^3t_1.
\end{aligned}$$

Again we require that the right-hand side lie in $T_1^{\perp}$. Since $-ix^3t_1$ is an odd function, it is in $T_1^{\perp}$. And since $(a_2 - (3i/2)x^4 + (i/2)x^6)e^{-x^2/2}$ is an even function, we have

$$\begin{aligned}
\mathrm{Proj}((a_2 - (3i/2)x^4 + (i/2)x^6)e^{-x^2/2}, T_1) &= (1/2)(F+\mathrm{Id})(a_2 - (3i/2)x^4 + (i/2)x^6)e^{-x^2/2} \\
&= [(2a_2+3i) - (27i/2)x^2 + (9i/2)x^4]e^{-x^2/2} \\
&\neq 0,
\end{aligned}$$

as may be easily checked using the fact that $Fx = -yF = i\partial_x F$. ■

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