

INFINITE-DIMENSIONAL COMPLEX DYNAMICS: A QUANTUM RANDOM WALK

BRENDAN WEICKERT

ABSTRACT. We describe a unitary operator $U(\alpha)$ on $L^2(\mathbf{T})$, depending on a real parameter α , that is a quantization of a simple piecewise holomorphic dynamical system on the cylinder $\mathbb{C}^* \cong \mathbf{T} \times \mathbb{R}$. We give results describing the spectrum of $U(\alpha)$ in terms of the diophantine properties of α , and use these results to compare the quantum to classical dynamics. In particular, we prove that for almost all α , the quantum dynamics localizes, whereas the classical dynamics does not. We also give a condition implying that the quantum dynamics does not localize.

1. **Introduction.** In the physical world, we often draw conclusions about infinite-dimensional dynamical systems by considering finite-dimensional approximations. A particularly simple class of infinite-dimensional dynamical systems is given by the class of unitary self-maps of a separable Hilbert space; moreover, certain such unitary maps are closely connected to Hamiltonian self-maps of finite-dimensional complex manifolds through the process of quantization. The latter are considered to approximate the former, in a certain sense (see [4], for instance). Of course, since the study of dynamics considers arbitrarily high-order iterates of maps, there is no obvious reason why dynamical conclusions about a system should be related to dynamical conclusions about an approximation of that system; nevertheless, in the quantum dynamics literature it seems that one often expects there to be a link. The point of this paper is to provide a simple class of counterexamples to this intuition. We will describe a situation in which, roughly speaking, the quantum orbits of a system are bound while their classical counterparts are unbound. This system has been previously studied by Fornæss [3]; we will describe his results below.

The two dynamical systems are obtained as follows. Consider the following time-dependent Hamiltonian function on the cylinder $\mathbf{T} \times \mathbb{R}$:

$$H(x, y, t) = \begin{cases} -4\pi\alpha y, & \text{if } t \bmod 1 \in [0, 1/2) \\ 2v(x), & \text{if } t \bmod 1 \in [1/2, 1), \end{cases}$$

where

$$v(x) = \begin{cases} x - \pi/2 & \text{if } 0 \leq x \leq \pi \\ -x + 3\pi/2 & \text{if } \pi \leq x \leq 2\pi. \end{cases}$$

The time-1 map of the flow defined by this Hamiltonian is defined everywhere on the cylinder except for the lines $\{x - 2\pi\alpha = 0\}$ and $\{x - 2\pi\alpha = \pi\}$, and is equal

Date: May 2, 2000.

1991 Mathematics Subject Classification. Primary: 37N20.

The author is partly supported by an NSF postdoctoral fellowship.

where defined to the map

$$g_\alpha(x, y) := (x - 2\pi\alpha, y - v'(x - 2\pi\alpha)).$$

For almost every $(x, y) \in \mathbb{T} \times \mathbb{R}$, all forward iterates $g_\alpha^n(x, y)$ are defined.

To obtain the quantization of g , we replace H by the following self-adjoint operator on $L^2(\mathbb{T})$:

$$H(x, y, t) = \begin{cases} 4\pi i\alpha \partial/\partial x, & \text{if } t \bmod 1 \in [0, 1/2) \\ 2v(x), & \text{if } t \bmod 1 \in [1/2, 1). \end{cases}$$

Solving the Schrodinger equation

$$i \frac{\partial}{\partial t} \psi_t = H \psi_t$$

yields a unitary solution operator U_t whose time-1 map is the operator

$$U = U(\alpha) : \psi(x) \mapsto e^{-iv(x)} \psi(x + 2\pi\alpha).$$

One can think of g_α and its quantum analogue $U(\alpha)$ as modeling a sort of random walk (for irrational values of α). Since the projection of the iterates of any point in the x -direction are uniformly distributed on the circle (in a sense made precise by the ergodic theorem), one may think of the rotation as supplying a random parameter according to which the y -coordinate of a point is shifted by plus or minus one. Fornaess [3] has studied the operator $U(\alpha)$ in the case where α is the golden mean. He proves

THEOREM 1.1 ([3]). *Let $U = U(\alpha)$ be as above, with $\alpha = (\sqrt{5} + 1)/2$. For any $\psi \in L^2(\mathbb{T})$ with $\|\psi\| = 1$ and any $\epsilon > 0$, there exists an integer N such that for any n , writing*

$$U^n \psi = \sum_m c_m^n e^{imx},$$

we have

$$\sum_{|m| < N} |c_m^n|^2 > 1 - \epsilon.$$

The aim of this paper is to strengthen this result by giving a more general diophantine condition which, if satisfied by α , implies that $U(\alpha)$ has a pure point spectrum. This implies the localization result of Fornaess's theorem. Furthermore, this condition is satisfied by almost all real α . Specifically, we prove the following:

THEOREM 1.2. *Let $\alpha \in \mathbb{R}$ satisfy*

$$|\alpha - p/q| \geq \frac{c}{q^\beta}$$

for some $c > 0$, some $2 \leq \beta < 5/2$, and all $p/q \in \mathbb{Q}$. Then the operator $U(\alpha)$ has a pure point spectrum. In particular, $U(\alpha)$ has a pure point spectrum for almost every $\alpha \in \mathbb{R}$.

Remark 1.3. *In fact, the hypothesis of this theorem may be weakened to require that the diophantine condition be satisfied only for odd denominators.*

We also prove that if α is too close to a certain type of resonance—that is, α fails a certain diophantine condition—then the spectrum of $U(\alpha)$ is purely continuous. This implies a type of delocalization of orbits in the Fourier transform direction, as will be described in section three below. We prove

THEOREM 1.4. *Suppose that there exists a sequence p_k/q_k of rational numbers with odd denominators q_k and $(p_k, q_k) = 1$, such that*

$$q_k^8 |\alpha - p_k/q_k| \rightarrow 0$$

as $k \rightarrow \infty$. Then $U(\alpha)$ has purely continuous spectrum. In particular, there exists a subsequence n_k such that U^{n_k} goes weakly to zero.

We also treat the case of resonance. There are two qualitatively different types of behavior, depending on whether the denominator is odd or even.

THEOREM 1.5. *Suppose that $\alpha = p/q$ is rational. Then if q is odd, $U(\alpha)$ has purely absolutely continuous spectrum. If q is even and $(p, q) = 1$, then $U(\alpha)^q = \text{Id}$.*

Finally, we prove the following result, which shows that the classical dynamics is diffusive. I do not think that the result is original, but have been unable to find it in the literature, and so have provided a proof here for the reader's convenience.

THEOREM 1.6. *Suppose that $\alpha \in \mathbb{R}$ is irrational. Then for almost every $x \in \mathbb{T}$, all orbits $\{g_\alpha^n(x, y)\}_{n \geq 0}$ are defined for all n and unbounded.*

Acknowledgment. *Many thanks to Alex Furman for a stimulating discussion on the subject.*

2. Quantum localization: case where α is diophantine with exponent less than $5/2$. Let v be periodic with period 2π , $v(x) = x - \pi/2$ on $[0, \pi]$ and $v(x) = -x + 3\pi/2$ on $[\pi, 2\pi]$. Let $\alpha \in \mathbb{R}$. Consider the following operator on $L^2(\mathbb{T})$:

$$U(\alpha) : \psi(x) \rightarrow e^{-iv(x)} \psi(x + 2\pi\alpha).$$

THEOREM 2.1. *Let $\alpha \in \mathbb{R}$ satisfy*

$$|\alpha - p/q| \geq \frac{c}{q^\beta}$$

for some $c > 0$, some $2 \leq \beta < 5/2$, and all $p/q \in \mathbb{Q}$. Then the operator $U(\alpha)$ has a pure point spectrum. In particular, $U(\alpha)$ has a pure point spectrum for almost every $\alpha \in \mathbb{R}$.

Proof. First, note that the Diophantine condition is satisfied if and only if

$$|e^{2\pi i n \alpha} - 1| \geq cn^{1-\beta}$$

for some $c > 0$, or

$$\left| \frac{1}{e^{2\pi i n \alpha} - 1} \right| \leq (1/c)n^{\beta-1}.$$

Let

$$\sum v_n e^{in x}$$

be the Fourier expansion of v . Then it is an easy exercise to show that $|v_n| \sim 1/n^2$. For each n , define

$$g_n = \frac{v_n}{e^{2\pi i n \alpha} - 1}.$$

Then $g_n \sim n^{\beta-3}$; thus $\sum |g_n|^2 < \infty$, and therefore

$$g(x) := \sum g_n e^{in x} \in L^2(\mathbb{T}).$$

For $m \in \mathbf{Z}$, let $\psi_m(x) = e^{i(g(x)-mx)}$. Then

$$\begin{aligned} U\psi_m(x) &= e^{-iv(x)}\psi_m(x+2\pi\alpha) \\ &= e^{-iv(x)}e^{i[g(x+2\pi\alpha)-m(x+2\pi\alpha)]} \\ &= e^{-2\pi i\alpha m}e^{i[g(x+2\pi\alpha)-v(x)-mx]} \\ &= e^{-2\pi i\alpha m}\psi_m(x), \end{aligned}$$

since

$$g(x+2\pi\alpha) - v(x) = g(x),$$

as may be shown by taking the Fourier transforms of both sides:

$$\begin{aligned} \sum g_n e^{2\pi i\alpha n} \delta_n - \sum v_n \delta_n &= \sum v_n \left(\frac{e^{2\pi i\alpha n}}{e^{2\pi i\alpha n} - 1} - 1 \right) \delta_n \\ &= \sum \frac{v_n}{e^{2\pi i\alpha n} - 1} \delta_n \\ &= \sum g_n \delta_n, \end{aligned}$$

as desired.

Thus $\{e^{i(g(x)-mx)} : m \in \mathbf{Z}\}$ is a complete set of eigenvectors of U . (This is clear, since it is a unitary image of $\{e^{inx} : n \in \mathbf{Z}\}$.) \square

Remark 2.2. *It is easy to check that in fact $v_n = 0$ for n even. Therefore we may weaken the hypothesis in the theorem to require only that there exist $2 \leq \beta < 5/2$ such that $|\alpha - p/q| \geq c/q^\beta$ whenever q is odd.*

3. Quantum delocalization: case where α is close to resonance. We would now like to investigate the behavior of the operator $U(\alpha)$, described above, in the case where the rotation number α fails some weaker Diophantine condition. Let α be an irrational number in $(0, 1)$, and let $\{p_k/q_k\}$ be the rational approximations given by the continued fraction expansion of α . That is, if we write $\alpha = 1/(a_1 + 1/(a_2 + 1/(a_3 + \dots)))$, then we take $p_0 = 0$, $p_1 = 1$, and $p_{n+1} = p_{n-1} + a_{n+1}p_n$, and $q_0 = 1$, $q_1 = a_1$, and $q_{n+1} = q_{n-1} + a_{n+1}q_n$. It is clear that no two consecutive denominators q_k, q_{k+1} can be even. In particular, throwing away the elements of the sequence with even denominator, we have an infinite sequence $\{p_k/q_k\}$ of rational numbers with odd denominator approximating α . For $\beta \geq 2$, we say that α satisfies condition $(*)_\beta$ if $\liminf q_k^\beta |\alpha - p_k/q_k| = 0$.

THEOREM 3.1. *If α is an irrational number satisfying $(*)_8$, then the spectrum of $U(\alpha)$ is purely continuous.*

We first prove a lemma.

For $\theta \in \mathbf{R}$, define

$$\begin{aligned} r_{n,m}(\theta) &= (e^{imx}, U(\theta)^n e^{imx}) \\ &= e^{2\pi im\theta} \frac{1}{2\pi} \int_0^{2\pi} e^{-iv_n(x,\theta)} dx, \end{aligned}$$

where

$$v_n(x, \theta) := \sum_{k=0}^{n-1} v(x + 2\pi k\theta).$$

LEMMA 3.2. *Suppose that q is an odd positive integer. Then*

$$|r_{n,m}(p/q)| \leq \max\left(\frac{4q^2}{\pi(n-q^2+1)}, 1\right).$$

Proof. Write $n = n'q + s$, where $s < q$. Note that

$$v_n(\cdot, p/q) = v_{n'q+s}(\cdot, p/q) = n'v_q(\cdot, p/q) + v_s(\cdot, p/q).$$

Clearly $v_\ell(\cdot, p/q)$ is a continuous, piecewise linear function for any ℓ : there exist intervals I_j , with $\cup I_j = \mathbb{T}$ such that $v_\ell(x, p/q) = m_j x + c_j$ for $x \in I_j$. Since v' is discontinuous at two points in the circle, and since translating creates at most two more possible points of discontinuity, we see that v'_ℓ has at most 2ℓ points of discontinuity in the circle. Since $v_n = n'v_q + v_s$, we have that v'_n has at most $2q + 2s \leq 4q - 2$ points of discontinuity. The number of intervals I_j such that $v_n|_{I_j}$ is linear for each j may therefore be chosen to be less than $4q$ for any n . Since $v' = \pm 1$ and there is an odd number of terms in the sum defining v_q , it follows that $|v'_q| \geq 1$ in \mathbb{T} . Also, $|v'_s| \leq s \leq q - 1$ in \mathbb{T} . Thus $|v'_n| = |(n'v_q + v_s)'| \geq n' - q + 1$. Now, if $n' \leq q - 1$ (i.e., $n \leq q^2 - 1$), we use the unit bound for $|r_{n,m}|$. Otherwise, writing $I_j = (a_j, b_j)$, we have

$$\begin{aligned} |r_{n,m}(p/q)| &\leq \frac{1}{2\pi} \sum_j \left| \int_{I_j} e^{-iv_n(x)} dx \right| \\ &= \frac{1}{2\pi} \sum_j \left| \frac{i}{v'_n} (e^{-in(m_j b_j + c_j)} - e^{-in(m_j a_j + c_j)}) \right| \\ &\leq \frac{1}{2\pi} \left(\frac{1}{n' - q + 1} \right) \sum_{j=1}^{4q} (2) \\ &= \frac{8q}{2\pi} \left(\frac{1}{n' - q + 1} \right) \\ &= \frac{4q}{\pi} \left(\frac{q}{n - q^2 + q - s} \right) \\ &\leq \frac{4q}{\pi} \left(\frac{q}{n - q^2 + 1} \right). \end{aligned}$$

□

For $\theta \in \mathbb{R}$, define

$$s_{N,m}(\theta) := \frac{1}{N} \sum_{n=0}^{N-1} |r_{n,m}(\theta)|^2.$$

If q is odd, then we have by the above lemma that $s_{N,m}(p/q) \rightarrow 0$ as $N \rightarrow \infty$. By the Wiener criterium ([5], p. 340), therefore, $U(p/q)$ has purely continuous spectrum. Furthermore, since

$$\limsup N s_{N,m} < \infty$$

for each m , we have that the singular continuous spectrum of $U(p/q)$ is empty [1]. Therefore we have the following corollary to the lemma:

COROLLARY 3.3. *If q is odd, then $U(p/q)$ has purely absolutely continuous spectrum.*

Remark 3.4. *This corollary is also clear from a simple computation of $U(p/q)^q$, which is multiplication by a piecewise linear function that is not constant on any interval in the case where q is odd.*

Proof of theorem 3.1: For any $\theta_1, \theta_2 \in \mathbb{R}$, we have

$$\begin{aligned}
|r_{n,m}(\theta_1) - r_{n,m}(\theta_2)| &= \left| \frac{1}{2\pi} \int_0^{2\pi} \left(e^{-iv_n(x,\theta_1)+2\pi im\theta_1} - e^{-iv_n(x,\theta_2)+2\pi im\theta_2} \right) dx \right| \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} (|v_n(x,\theta_1) - v_n(x,\theta_2)| + 2\pi|m||\theta_1 - \theta_2|) dx \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k=0}^{n-1} (|v(x+2\pi k\theta_1) - v(x+2\pi k\theta_2)|) + 2\pi|m||\theta_1 - \theta_2| \right) dx \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k=0}^{n-1} (2\pi k|\theta_1 - \theta_2|) + 2\pi|m||\theta_1 - \theta_2| \right) dx \\
&= 2\pi|\theta_1 - \theta_2| \left(|m| + \sum_{k=0}^{n-1} k \right) \\
&\leq 2\pi|\theta_1 - \theta_2| (|m| + n^2),
\end{aligned}$$

where we have used the fact that H is Lipschitz with constant one.

For each n and m , we have

$$\begin{aligned}
|r_{n,m}(\alpha)|^2 &\leq (|r_{n,m}(p/q)| + |r_{n,m}(\alpha) - r_{n,m}(p/q)|)^2 \\
&\leq 2|r_{n,m}(p/q)|^2 + 2|r_{n,m}(\alpha) - r_{n,m}(p/q)|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
s_{N,m}(\alpha) &\leq 2s_{N,m}(p/q) + \frac{2}{N} \sum_{n=0}^{N-1} [2\pi|\alpha - p/q|(|m| + n^2)]^2 \\
&\leq \frac{2}{N} \sum_{n=0}^{q^2-1} (1) + \frac{2}{N} \sum_{n=q^2}^{N-1} \frac{16q^4}{\pi^2(n - q^2 + 1)^2} + \frac{2}{N} \sum_{n=0}^{N-1} (2\pi|\alpha - p/q|(|m| + n^2))^2 \\
&\leq \frac{2}{N} [q^2 + \left(\frac{16q^4}{\pi^2}\right) (2) + 4\pi^2|\alpha - p/q|^2 \sum_{n=0}^{N-1} (|m| + n^2)^2] \\
&\leq \frac{Cq^4}{N} + K(m)N^4|\alpha - p/q|^2
\end{aligned}$$

where C is a constant and K is a function of m alone.

Since α satisfies $(*)_8$, there exists a sequence $\{p_k/q_k\}$ of rational numbers, with q_k odd, satisfying

$$\ell_k := q_k^8 \left| \alpha - \frac{p_k}{q_k} \right| \rightarrow 0$$

as $k \rightarrow \infty$. Let N_k be a sequence of positive integers satisfying

$$(1/c)\ell_k^{-1/4}q_k^4 \leq N_k \leq c\ell_k^{-1/4}q_k^4$$

for some $c > 0$. Then

$$\begin{aligned} s_{N_k, m}(\alpha) &\leq cC\ell_k^{1/4} + c^4K(m)\ell_k^{-1}q_k^{1/6} \left| \alpha - \frac{pk}{qk} \right|^2 \\ &= cC\ell_k^{1/4} + c^4K(m)\ell_k. \end{aligned}$$

Thus $s_{N_k, m}(\alpha) \rightarrow 0$ as $k \rightarrow \infty$, which implies, by the Wiener criterium [5], that $U(\alpha)$ has purely continuous spectrum. \blacksquare

We wish finally to point out that a certain type of resonance implies trivially a pure point spectrum.

THEOREM 3.5. *If q is even and $(p, q) = 1$, then $U(p/q)^q = \text{Id}$.*

Proof. It suffices to prove that $v_q \equiv 0$. Given any point $x \in L^2(\mathbb{T})$ outside of a finite exceptional set, it is clear that under the given hypothesis, $\{x + 2\pi jp/q\}_{j=0}^{q-1}$ has the same number of points in $(0, \pi)$ as in $(\pi, 2\pi)$. Thus $\sum_{j=0}^{q-1} v'(x + 2\pi jp/q) = 0$ where defined. Thus v_q has zero derivative almost everywhere, and since it is a continuous function, it is constant. That it is zero follows from $\int_0^{2\pi} v = 0$. \square

4. Failure of localization in the classical case.

THEOREM 4.1. *Suppose that $\alpha \in \mathbb{R}$ is irrational. Then for almost every $x \in \mathbb{T}$, all orbits $\{g_\alpha^n(x, y)\}_{n \geq 0}$ are defined for all n and unbounded.*

Proof. For $x \in \mathbb{T}$, let $T : x \rightarrow x - 2\pi\alpha$. Define

$$\begin{aligned} f(x) &= \sup_{n \geq 0} \sum_{k=0}^{n-1} v'(T^k x) \\ g(x) &= \inf_{n \geq 0} \sum_{k=0}^{n-1} v'(T^k x), \end{aligned}$$

where $v' \in L^\infty(\mathbb{T})$ is the step function

$$v'(x) = \begin{cases} 1, & x \in [0, \pi) \\ -1, & x \in [\pi, 2\pi), \end{cases}$$

and where we use the convention

$$\sum_{k=0}^{-1} (\cdot) = 0.$$

Then f and g are both measurable functions, and $f \geq 0$, $g \leq 0$. Let $E = \{f < \infty\}$, $F = \{g > -\infty\}$. It suffices to prove that either E or F has zero measure. Assume the contrary. Since they are both T -invariant sets, then, they both have full measure by ergodicity. Furthermore, the measures of the sets $E_\lambda := f^{-1}([0, \lambda))$, $F_\lambda := g^{-1}((-\lambda, 0])$ increase to one as $\lambda \rightarrow \infty$; thus there exists $\lambda > 0$ such that $m(E_\lambda \cap F_\lambda) > 0$. Let

$$E(N) = \bigcup_{k \geq N} T^k(E_\lambda \cap F_\lambda).$$

We have

$$\bigcup_{k \in \mathbb{Z}} T^k(E_\lambda \cap F_\lambda) = E(N) \cup [E(N-1) \setminus E(N)] \cup [E(N-2) \setminus E(N-1)] \dots$$

The measure of the left-hand-side is one by ergodicity, and the measure of each $E(k) \setminus E(k+1)$ is zero since $T(E(k)) = E(k+1) \subset E(k)$ and T is measure-preserving. Thus $mE(N) = 1$. Let $G = E(0)$.

Now, for $x \in E$ (resp. $x \in F$), we have

$$f(Tx) = f(x) - v'(x) \quad (\text{resp. } g(Tx) = g(x) - v'(x)).$$

Inductively, we obtain

$$\begin{aligned} f(T^n x) &= f(x) - \sum_{k=0}^{n-1} v'(T^k x) \\ g(T^n x) &= g(x) - \sum_{k=0}^{n-1} v'(T^k x) \end{aligned}$$

for $x \in E \cap F$. Given $y \in G$, write $y = T^n x$, $x \in E_\lambda \cap F_\lambda$, $n \geq 0$. Then

$$\begin{aligned} 0 \leq f(y) &= f(T^n x) \\ &= f(x) - \sum_{k=0}^{n-1} v'(T^k x) \\ &\leq f(x) - g(x) \\ &< 2\lambda. \end{aligned}$$

Similarly, we obtain $-2\lambda < g(y) \leq 0$ for $y \in G$. Since $mG = 1$, we have

$$f, g \in L^\infty(\mathbb{T}) \subset L^2(\mathbb{T}).$$

Furthermore, f solves the functional equation

$$v' = f - f \circ T.$$

However, taking the Fourier transform of this equation gives

$$\sum v'_n \delta_n = \sum (1 - e^{-2\pi i n \alpha}) f_n \delta_n,$$

or

$$f_n = \frac{v'_n}{1 - e^{-2\pi i n \alpha}}.$$

It is easy to calculate that $v'_n \sim 1/n$ for n odd, $v'_n = 0$ for n even. From the continued fraction expansion of α , we can find a sequence $\{p_k/q_k\}$ of rational numbers, with q_k odd and

$$\left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k^2}.$$

This implies that $|e^{2\pi i q_k \alpha} - 1| < \frac{1}{q_k}$. But then

$$|f_{q_k}| > (c/q_k)(q_k) = c$$

for some $c > 0$, contradicting $f \in L^2(\mathbb{T})$. □

REFERENCES

- [1] Avron, J., Simon, B.; Transient and recurrent spectrum, *J. Funct. Anal.* **43** (1981).
- [2] Combesure, M (1992): Recurrent versus diffusive quantum behavior for time dependent Hamiltonians, *Operator Theory: Advances and Applications.* **57**, Birkhauser
- [3] Fornæss, J. E.; Infinite-dimensional complex dynamics: quasiconjugacies, localization, and quantum chaos, *Disc. Cont. Dynam. Sys.*, Jan. 2000.
- [4] Knauf, A., Sinai, Y. G.; *Classical nonintegrability, Quantum Chaos*, Birkhauser (1997), Chap. 4.
- [5] Reed, M., Simon, B., *Methods of Modern Mathematical Physics, vol 3: Scattering Theory*, Academic Press (1978).

DEPT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, IL 60637, USA
E-mail address: `brendan@math.uchicago.edu`