

RESEARCH SUMMARY AND PROPOSAL

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ABSTRACT. My primary fields of interest are several complex variables and dynamical systems, beginning in my dissertation with a problem of local dynamics near non-hyperbolic fixed points, and continuing with problems of global dynamics in complex projective space, random iteration, attractors, and unitary dynamics on separable Hilbert spaces as an important case of infinite-dimensional complex dynamics.

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My immediate research plan is to continue to investigate the largely unexplored subject of infinite-dimensional complex dynamics, with a particular focus on the class of unitary self-maps of separable Hilbert space. The importance of this class stems from the fact that it contains the *quantizations* of finite-dimensional Hamiltonian systems, so that natural comparisons may be made between the infinite-dimensional systems and their finite-dimensional counterparts. For example, in [FW3] we study a unitary self-map of $L^2(\mathbb{R})$ that is naturally related to the Hénon map on \mathbb{R}^2 , and in [W4] a unitary self-map of $L^2(\mathbb{T})$ that is naturally related to a simple map resembling a random walk on the cylinder $\mathbb{T} \times \mathbb{R}$. A common theme in these papers is that dynamical information may be obtained from information about the spectrum of the unitary operator being iterated, and vice versa.

The direction of this work differs significantly from that of a large part of the mathematical physics literature in which dynamical terms such as “quantum chaos” are used. In this literature, one usually assumes that one is given a system with pure point spectrum (the wave group of a compact manifold, for instance), and then investigates problems relating to the distribution of eigenvalues, for instance, or the behavior of eigenfunctions as a certain parameter (the Planck constant) of the system goes to zero. From our perspective, however, systems with pure point spectrum are dynamically trivial—every element is quasi-periodic. Furthermore, for most systems that occur, spectral information is not so readily available.

As examples of the sort of result that can be obtained, we show in [FW3], via a dynamical argument, that for certain parameter ranges, a quantized Hénon map has purely continuous spectrum. This fact is equivalent to a dynamical statement describing a certain non-recurrence of orbits. These parameter ranges coincide with those for which the corresponding classical Hénon map has no bounded orbits. In [W4], we study a system depending on a parameter $\alpha \in \mathbb{T}$, and obtain spectral information about the system based on the Diophantine properties of α . In particular, we show that for almost all $\alpha \in \mathbb{T}$ the system has pure point spectrum (implying that every orbit is quasiperiodic), while its classical counterpart on the cylinder $\mathbb{T} \times \mathbb{R}$ has almost every orbit unbounded—a failure of the expected correspondence between the two systems. We also give a condition on α implying that the operator has purely continuous spectrum.

To describe the direction in which I see this work as going, I must give a small amount of background information. The following terminology is basic: if $f : M \rightarrow M$ is a holomorphic self-map of a complex manifold, then the largest open subset of M on which the family $\{f^n\}$ of iterates of f is *normal*—that is, satisfies the property that every sequence of members of the family has a convergent subsequence—is called the Fatou set. Its complement is called the Julia set. An alternate definition is obtained by taking the Julia set to be the closure of the set of repelling periodic points for f , and the Fatou set to be its complement. These definitions coincide when f is a rational function on the Riemann sphere, but generally not in higher dimension. Roughly speaking, the behavior of the dynamical system generated by f is considered predictable on the Fatou set, and chaotic on the Julia set. Some of the subtlety of the study of holomorphic dynamics in higher dimension stems from the existence of competing definitions of the Fatou and Julia sets.

Returning now to the infinite-dimensional setting, we confront a common but largely unexplored assumption in the study of unitary dynamical systems on Hilbert space—namely, that such systems cannot exhibit chaos, since the maps being iterated are linear. However, the infinite-dimensional setting reveals a degree of competition between different definitions of chaos, much as the jump from one to two dimensions revealed competition between different definitions of the Julia set. If chaos is measured by the rate of growth of derivatives, then certainly the dynamical system generated by any linear map is not chaotic. But a more natural definition of chaos, since it involves directly the question of predictability, is the failure of the family of iterates of a map to be normal. The next problem that I would like to solve is to characterize this definition of chaos in terms of the spectral properties of the map. (For instance, an easy first observation is that the family of iterates of the map is normal on the subspace of Hilbert space corresponding to its pure point spectrum, so that this subspace would not be considered to be in the chaotic set.) This would give spectral definitions of the Fatou and Julia sets for unitary operators on Hilbert space.

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This recent work continues a line of inquiry I have pursued since completing my Ph.D. dissertation in 1997: to study features of holomorphic dynamical systems that persist under perturbation of the map being iterated. My dissertation contained the following results. First, a local interpolation result for holomorphic automorphisms of \mathbb{C}^k , $k \geq 2$: Given any polynomial self-map of \mathbb{C}^k for which the origin is not critical, and any natural number N , we may find a holomorphic automorphism of \mathbb{C}^n tangent to the polynomial mapping to order N (this result clearly does not hold in dimension one). Second, a local result about germs near the origin of holomorphic self-maps of \mathbb{C}^2 that are tangent to the identity: there exist certain second-order invariants of the mapping, and a condition on those invariants that implies the existence of an open set of points attracted to the origin by forward iterates in the topology of uniform convergence on compact subsets. Furthermore, near the origin the map is conjugate on this set to translation. Combining the two results and extending some of the constructions, we prove that for the special case where the mapping is a global automorphism, the conjugation to translation extends to the entire basin of attraction, and the basin itself is biholomorphic but not equal to \mathbb{C}^2 .

This gives a new class of examples of so-called Fatou-Bieberbach domains—proper subdomains of \mathbf{C}^k that are biholomorphic to \mathbf{C}^k (clearly only possible when $k \geq 2$).

After completing this work, I became interested in global questions about iteration, especially for holomorphic self-maps of complex projective space \mathbf{P}^k , $k \geq 2$. Such maps are natural generalizations of the class of rational functions on the Riemann sphere, which has been the focus of most of the work on holomorphic dynamics in dimension one. The phenomena that particularly interested me were those that persisted under perturbation; and consequently, with Fornaess, I have worked on random iteration on complex projective space of dimension strictly greater than one, extending and reinterpreting some earlier one-dimensional results of Fornaess and Sibony. In [FW1], we consider a family $\{f_\lambda\}$ of holomorphic self-maps of \mathbf{P}^n , varying non-degenerately, in a certain sense, with the parameter $\lambda \in B(0, \delta) \subset \mathbf{C}^n$. We think of this family as consisting of perturbations of f_0 . Random iteration consists in choosing sequences

$$\Lambda := (\lambda_1, \lambda_2, \dots) \subset \prod_{i=1}^{\infty} B(0, \delta),$$

and studying the iterates

$$F(\Lambda, k) := f_{\lambda_k} \circ \dots \circ f_{\lambda_1}$$

with respect to some sort of probability measure on the space of sequences.

In the one-dimensional case, Fornaess and Sibony had previously shown that, given $z \in \mathbf{P}$, almost every sequence carries z to a neighborhood of the attracting periodic orbits for f_0 . If f_0 has no attracting periodic orbits, almost every sequence moves z ergodically over the entire sphere. The proof relied on the Sullivan Classification of Fatou components—a result which is not available in higher dimensions.

Fornaess and I reproved these results in a way which eliminates the need for the Sullivan theorem, and removes the dichotomy between the cases where f_0 has or does not have attracting orbits. Instead, it is shown that the result is implied by a fundamental property of *attractors*. This allows us to generalize the result to higher dimensions, and obtain the one-dimensional result by observing that the only attractors in this case are periodic orbits or the whole sphere. A quantitative version of the result provides finitely many limit measures, supported near the attractors for f_0 , such that average push-forwards of an arbitrary initial measure converge to a linear combination of these limit measures.

In another section of this work, we study average pull-backs of currents. In the classical case (that is, where we consider only constant sequences), successively pulling back closed, positive currents of various dimensions usually converges to a current whose support is some sort of Julia set for the map. Generally speaking, pulling back bottom-dimensional currents leads us to the complement of the set of normality, while pulling back top-dimensional currents leads us (approximately) to the closure of the set of repelling periodic points. These sets coincide in dimension one. We use the word “usually” above because there can be some initial currents which do not converge properly under the operation of normalized pullback. These currents are called “exceptional”, and they are rare in a sense (though there is as yet no satisfactory theory which makes that sense precise). In the random case, we show in the top- and bottom-dimensional cases that, for the operation of average normalized pullback, there is no exceptional set.

In the work described above, the notion of an *attractor* (in the sense of Ruelle, see for instance [R]) is particularly important, since for instance the long-term dynamics of perturbative systems takes place near the attractors for the perturbed map. Attractors are therefore a natural object of study, and it turns out that for holomorphic self-maps of complex projective space they have certain restrictive properties, which I have explored with different collaborators in [FW2] and [JW]. Among them are the following: an attractor for a map f is an attractor for any iterate f^k , attractors must either be periodic orbits or contain a nonconstant, entire image of the complex line, that they need not be algebraic sets, that a map may have at most one attractor that is not a periodic orbit, and that the complement of an attractor is pseudoconvex. Many questions remain open, including this one: can an attractor for a holomorphic self-map of complex projective space contain a repelling periodic point? The answer is “yes” if in the question we replace “attractor” with “attracting set” (see [FW2]; but attracting sets, while related to attractors and having many properties in common with them, also differ from them in many important respects: for instance, a single attracting set may contain many attractors. A related question is whether an attractor can contain an open set. I am continuing to work on these sorts of problems, as well as that of describing the behavior of germs of holomorphic maps near fixed points for more general conditions on the derivative than those considered in [W1].

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In closing, I would like to return again to the problems discussed in the first section; namely, those of iterating holomorphic self-maps of infinite-dimensional spaces. It seems to me that this is a natural, and even a necessary, direction for the field of holomorphic dynamical systems to take, and moreover one that will require entirely new tools and techniques. For example, one of the most important tools of holomorphic dynamics is the Arzela-Ascoli theorem, which relates the notions of equicontinuity and normality for families of functions. The usefulness of this theorem, however, is greatly diminished in the infinite-dimensional setting, since it requires that the closure of orbits be compact in order to give information. In $L^2(\mathbb{R})$, for instance, the unit ball is not compact. But if we restrict our attention to the very natural class of unitary operators, for instance (certainly natural from a physicist’s perspective), a new class of tools become available: those of operator theory. Therefore these relatively unexplored problems seem destined to provide a fertile ground for collaboration between complex analysts and operator theorists. Furthermore, certain of them, for instance the question of whether or not these systems can exhibit chaos, are fairly crying out to be answered. I look forward to shedding some light on them in the course of my next few years of research.

REFERENCES

- [CG] L. Carleson and T. W. Gamelin. *Complex Dynamics*. Springer-Verlag, 1993.
- [FW1] J. E. Fornæss and B. Weickert. Attractors in \mathbf{P}^2 . Math. Sci. Res. Inst. Publ. **37**, Cambridge Univ. Press, Cambridge, 1999.
- [FW2] J. E. Fornæss and B. Weickert. Random iteration in \mathbf{P}^k . *Ergod. Theory Dynam. Syst.* **20**, 4 (2000)
- [FW3] J. E. Fornæss and B. Weickert. A quantized Hénon map. *Disc. Cont. Dyn. Syst.* **6**, 3 (2000).
- [JW] M. Jonsson and B. Weickert. A nonalgebraic attractor in \mathbf{P}^2 . *Proc. Amer. Math. Soc.*, to appear.
- [R] D. Ruelle. *Elements of differentiable dynamics and bifurcation theory*. Academic Press, 1989.
- [W1] B. Weickert. Attracting basins for automorphisms of \mathbf{C}^2 . *Inv. Math.* **132**, 3 (1998).
- [W2] B. Weickert. Automorphisms of \mathbf{C}^n . Preprint.
- [W3] B. Weickert. Hyperbolicity of certain normal limit sets in \mathbf{P}^2 . Preprint.
- [W4] B. Weickert. Infinite-dimensional complex dynamics: a quantum random walk. Preprint.

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