1. First Order Condition for the General 2-input problem.

In general, for a 2-input production function, the first order conditions coming out of the direct profit maximization problem are

\[
\frac{\partial f(x_l, x_k)}{\partial x_l} = \frac{w_l}{p} \\
\frac{\partial f(x_l, x_k)}{\partial x_k} = \frac{w_k}{p}
\]

If you multiply both sides of these equations by \(p\), you get a more direct statement of our standard interpretation: The marginal revenue product of a factor must be equal to its own price at the profit-maximizing choice. Now let’s consider Cobb-Douglas technology.

2. Factor Demand for General 2-Input Cobb Douglas Production

If \(f(x_l, x_k) = C x_l^a x_k^b\) (where we assume that \(a + b < 1\), why?), then the first order conditions become

\[
a C x_l^{a-1} x_k^b = \frac{w_l}{p} \\
b C x_l^a x_k^{b-1} = \frac{w_k}{p}
\]

Using the first equation to solve for \(x_l^a\) in terms of \(x_k\), we get

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\[ x_i^{a-1} = \frac{w_i}{aCpx_k^b} \]

\[ \Leftrightarrow (x_i^{a-1})^{\frac{a}{a-1}} = \left(\frac{w_i}{aCpx_k^b}\right)^{\frac{a}{a-1}} \]

\[ \Leftrightarrow x_i^a = \left(\frac{w_i}{aCpx_k^b}\right)^{\frac{a}{a-1}} \]

Plug this into the second first order condition to get

\[ bC \left[ \left(\frac{w_i}{aCpx_k^b}\right)^{\frac{a}{a-1}} \right] x_k^{b-1} = \frac{w_k}{p} \]

divide by \( bC \) and factor out and collect the \( x_k \) factors on the LHS to get

\[ (x_k^{-b})^{\frac{a}{a-1}} x_k^{b-1} \left[ \left(\frac{w_i}{aCp}\right)^{\frac{a}{a-1}} \right] = \frac{w_k}{bCp} \]

Divide by \( \left(\frac{w_i}{aCp}\right)^{\frac{a}{a-1}} \) and combine the exponents on the \( x_k \) factors to get the following. \(^1\)

\[ x_k^{\left[\frac{a+b-1}{1-a}\right]} = \frac{w_k}{bCp} \left(\frac{w_i}{aCp}\right)^{\frac{1}{1-a}} \]

\[ \Leftrightarrow x_k^{\left[\frac{a+b-1}{1-a}\right]} = \frac{w_k}{bCp} \left(\frac{w_i}{aCp}\right)^{\frac{a}{1-a}} \]

Now raise both sides to the \(-1\) power to get

\[ x_k^{\left[\frac{1-a-b}{1-a}\right]} = \frac{bCp}{w_k} \left(\frac{aCp}{w_i}\right)^{\frac{a}{1-a}} \]

Raising both sides to the \(\frac{1-a}{1-a-b}\) power we get

\(^1\)Note that dividing both sides by \( \left(\frac{w_i}{aCp}\right)^{\frac{a}{1-a}} \) is the same thing as multiplying both sides by \( \left(\frac{w_i}{aCp}\right)^{\frac{1-a}{1-a}} \) - note the reverse sign on the exponent.
\[
x_k = \left[ \frac{bCp}{w_k} \left( \frac{aCp}{w_l} \right)^\frac{a}{1-a} \right]^{\frac{1-a}{1-a-b}} \\
= \left( \frac{bCp}{w_k} \right)^{\frac{1-a}{1-a-b}} \left( \frac{aCp}{w_l} \right)^{\frac{a}{1-a-b}} \\
= (Cp)^{\frac{1}{1-a-b}} \left( \frac{b}{w_k} \right)^{\frac{1-a}{1-a-b}} \left( \frac{a}{w_l} \right)^{\frac{a}{1-a-b}} \\
= (Cp)^{\frac{1}{1-a-b}} \left( \frac{b}{w_k} \right)^{\frac{1-a}{1-a-b}} \left( \frac{a}{w_l} \right)^{\frac{a}{1-a-b}} \\
\]

Simplify the expression for \( x_k \) and appealing to symmetry for the solution for \( x_l \) we have

\[
x_k(p, w_l, w_k) = \left[ Cp \left( \frac{b}{w_k} \right)^{1-a} \left( \frac{a}{w_l} \right)^a \right]^{\frac{1}{1-a-b}} \\
x_l(p, w_l, w_k) = \left[ Cp \left( \frac{a}{w_l} \right)^{1-b} \left( \frac{b}{w_k} \right)^b \right]^{\frac{1}{1-a-b}} \\
\]

3. Example

Suppose that \( f(x_l, x_k) = 30x_l^{\frac{2}{3}} x_k^{\frac{2}{3}} \), then the profit maximizing choices (factor demands) would be

\[
x_k(p, w_l, w_k) = \left[ 30p \left( \frac{2}{5w_k} \right)^{\frac{2}{3}} \left( \frac{2}{5w_l} \right)^\frac{2}{5} \right]^{\frac{5}{2}} \\
x_l(p, w_l, w_k) = \left[ 30p \left( \frac{2}{5w_l} \right)^{\frac{4}{3}} \left( \frac{2}{5w_k} \right)^\frac{2}{5} \right]^{\frac{5}{2}} \\
\]

which simplifies to
\[ x_k(p, w_l, w_k) = \left[ 12p \left( \frac{1}{w_k} \right)^{\frac{2}{5}} \left( \frac{1}{w_l} \right)^{\frac{2}{5}} \right]^5 \]
\[ x_l(p, w_l, w_k) = \left[ 12p \left( \frac{1}{w_l} \right)^{\frac{2}{5}} \left( \frac{1}{w_k} \right)^{\frac{2}{5}} \right]^5 \]

or if you prefer

\[ x_k(p, w_l, w_k) = \frac{12^5 p^5}{w_k^2 w_l^2} \]
\[ x_l(p, w_l, w_k) = \frac{12^5 p^5}{w_l^2 w_k^2} \]

4. Capital Intensiveness

Notice that the factor demands are decreasing in both input prices. However, they are more responsive to changes in their own price. For example, if the price of labor doubles demand labor fall by 8 times, while the demand for capital only falls by 4 times, so that we end up at a more capital intensive method whenever the price of labor increase. We can see this more compactly by taking the ratio

\[ \frac{x_k(p, w_l, w_k)}{x_l(p, w_l, w_k)} = \frac{\frac{12^5 p^5}{w_k^2 w_l^2}}{\frac{12^5 p^5}{w_l^2 w_k^2}} = \frac{w_l^2 w_k^2}{w_l^2 w_k^2} = \frac{w_l}{w_k} \]

Note the cleanliness of this result probably has something to do with the fact that we assumed \( a = b \) in this example, but it provides a nice benchmark in which the ratio of capital to labor (a measure of "capital intensiveness") is exactly the input price ratio. Doubling the wage paid to labor will double the ratio of capital to labor. (though demand for both inputs would decrease holding \( p \) constant).

However, the result is not very different when \( a \neq b \). We would just get

\[ \frac{x_k(p, w_l, w_k)}{x_l(p, w_l, w_k)} = \frac{bw_l}{aw_k} \]

How can we know this without actually trying to simplifying the ratio of \( \frac{x_k(p, w_l, w_k)}{x_l(p, w_l, w_k)} ?? \)
(Hint: appeal to the Cost-Minimization problem)